On a conjecture of Darmon–Rotger in the adjoint CM case

Seminario Latinoamericano de Teoría de Números

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Outline

Kato classes

Generalised Kato classes

Main result

Ideas from the proof



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- E/\mathbb{Q} elliptic curve of conductor N.
- $p \nmid 2N$ good ordinary prime for E.
- $V_{\rho}E := (\varprojlim_n E[\rho^n]) \otimes \mathbb{Q}_{\rho}$

Theorem (Kato's ERL)

There is a class $\kappa_p^{ ext{Kato}} \in \mathrm{H}^1(\mathbb{Q}, V_p E)$ such that

$$\exp^*_{\mathrm{BK}}(\mathrm{Loc}_{\rho}(\kappa_{
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where $\operatorname{Loc}_p : \operatorname{H}^1(\mathbb{Q}, V_p E) \to \operatorname{H}^1(\mathbb{Q}_p, V_p E)$ is the restriction maps at p, and

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is Bloch–Kato's dual exponential map.

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Let $\operatorname{Sel}(\mathbb{Q}, V_p E) \subset \operatorname{H}^1(\mathbb{Q}, V_p E)$ be the *p*-adic Selmer group of *E*:

 $0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p \to \operatorname{Sel}(\mathbb{Q}, V_p E) \to (\varprojlim \operatorname{III}(E/\mathbb{Q})[p^n]) \otimes \mathbb{Q}_p \to 0.$

By Kato's ERL, $L(E, 1) = 0 \iff \kappa_p^{\text{Kato}} \in \text{Sel}(\mathbb{Q}, V_p E).$

Conjecture (Perrin-Riou, 1993)

Suppose that L(E, 1) = 0. Then the following are equivalent:

(1) $\kappa_p^{\text{Kato}} \neq 0$.

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Moreover, in that case $\log_E(\kappa_p^{\text{Kato}}) = \log_E(P)^2 \pmod{\mathbb{Q}^{\times}}$, where $P \in E(\mathbb{Q}) \otimes \mathbb{Q}$ is any generator.



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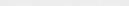
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Theorem (Darmon-Rotger's ERL)

There is a generalised Kato class

 $\kappa_{\gamma,\delta}(f,g,h) \in \mathrm{H}^1(\mathbb{Q},V_{\mathrm{fgh}}),$

where $V_{fgh} = V_f \otimes V_g \otimes V_h$, such that

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Running hypotheses:

• $\alpha_g \neq \beta_g$ and $\alpha_h \neq \beta_h$, so we have **four** *a priori distinct*

 $\kappa_{\gamma,\delta}(f,g,h) \in \mathrm{H}^1(\mathbb{Q},V_{\mathrm{fgh}}).$

• $gcd(N_f, N_g N_h) = 1$, so $\epsilon(f \otimes g \otimes h) = +1$.

Conjecture (Darmon-Rotger)

Suppose that $L(1, f \otimes g \otimes h) = 0$. Then the following are equivalent:

The classes κ_{γ,δ}(f, g, h) span a non-trivial subspace of Sel(Q, V_{fgh}).
 dim_{Q₀}Sel(Q, V_{fgh}) = 2.

- The conjecture does not predict that the classes κ_{γ,δ}(f, g, h) span the entire Sel(Q, V_{fgh}).
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• Suppose now that f corresponds to an elliptic curve E/\mathbb{Q} , and $h = g^*$. Then

 $V_{fgg^*} \cong V_p E \oplus (V_p E \otimes \mathrm{ad}^0 V_g)$

and $L(s, f \otimes g \otimes g^*) = L(E, s) \cdot L(E, \operatorname{ad}^0 V_g, s).$

• Let $\kappa_{\gamma,\delta} \in \mathrm{H}^1(\mathbb{Q}, V_p E)$ be the image of $\kappa_{\gamma,\delta}(f, g, g^*)$ under the projection

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- Conjecture (Darmon–Rotger, rank (2,0) adjoint case) Suppose that L(E, 1) = 0 and has sign +1, and that $L(E, ad^0V_g, 1) \neq 0$. Then the following are equivalent:
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Outline

Kato classes

Generalised Kato classes

Main result

Ideas from the proof

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University of California Santa Barbara

We consider the adjoint CM case:

• K imaginary quadratic field of discriminant prime to N in which

 $(p) = p\overline{p}$ splits.

g = θ_ψ = ∑_a ψ(a)q^{Na}, with ψ a ray class character of K of conductor prime of Np.

Then we have the four generalised Kato classes

$$\kappa_{\alpha,\alpha^{-1}}, \ \kappa_{\alpha,\beta^{-1}}, \ \kappa_{\beta,\alpha^{-1}}, \ \kappa_{\beta,\beta^{-1}} \in \mathrm{H}^{1}(\mathbb{Q}, V_{\rho}E),$$

where $\alpha = \psi(\overline{\mathfrak{p}})$ and $\beta = \psi(\mathfrak{p})$. **Note:** In this case,

$L(E, \mathrm{ad}^0 V_p(g), s) = L(E^K, s) \cdot L(E/K, \chi, s)$

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Let N^- be the largest factor of N divisible only by primes inert in K. Theorem (C.-Hsieh)

Suppose that L(E,1) = 0 and has sign +1, $L(E^{K},1) \cdot L(E/K,\chi,1) \neq 0$, and that:

- *E*[*p*] is absolutely irreducible as *G*_Q-module.
- N⁻ is the squarefree product of an odd number of primes.
- E[p] is ramified at every prime $\ell | N^-$.

Then $\kappa_{\alpha,\beta^{-1}} = \kappa_{\beta,\alpha^{-1}} = 0$ and the following hold:

$$\begin{aligned} \kappa_{\alpha,\alpha^{-1}} \neq 0 &\implies \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_p E) = 2, \\ \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_p E) = 2 \\ \mathrm{Sel}(\mathbb{Q}, V_p E) \neq \mathrm{ker}(\mathrm{Loc}_p) \end{aligned} \qquad \Longrightarrow \quad \kappa_{\alpha,\alpha^{-1}} \neq 0, \end{aligned}$$

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- E[p] is absolutely irreducible as $G_{\mathbb{Q}}$ -module.
- N⁻ is the squarefree product of an odd number of primes.
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Then $\kappa_{\alpha,\beta^{-1}} = \kappa_{\beta,\alpha^{-1}} = 0$ and the following hold:

$$\begin{split} \kappa_{\alpha,\alpha^{-1}} \neq 0 & \Longrightarrow & \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_p E) = 2, \\ \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_p E) = 2 \\ \mathrm{Sel}(\mathbb{Q}, V_p E) \neq \mathrm{ker}(\mathrm{Loc}_p) \end{split} \implies & \kappa_{\alpha,\alpha^{-1}} \neq 0, \end{split}$$

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Remarks

- The condition Sel(Q, V_pE) ≠ ker(Loc_p) is automatic if #E(Q) = ∞ or #III(E/Q)[p[∞]] < ∞.
- Our proof also shows if $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 2$ and $\#\mathrm{III}(E/\mathbb{Q})[p^{\infty}] < \infty$, then

$$\kappa_{\alpha,\alpha^{-1}} = \log_E(Q) \cdot P - \log_E(P) \cdot Q \pmod{\mathbb{Q}_p^{\times}},$$

where (P, Q) is any basis of $E(\mathbb{Q}) \otimes \mathbb{Q}$.

• Thus $\kappa_{\alpha,\alpha^{-1}}$ generates (over \mathbb{Q}_p) the image of the line

$$\langle P \land Q := P \otimes Q - Q \otimes P \rangle$$

under the map $\bigwedge^2 E(\mathbb{Q}) \otimes \mathbb{Q} \to E(\mathbb{Q}) \otimes \mathbb{Q}_p = \operatorname{Sel}(\mathbb{Q}, V_p E)$ induced by \log_{E^*} .

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Main result: rank (2,0) adjoint CM case

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Outline

Kato classes

Generalised Kato classes

Main result

Ideas from the proof



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• Let
$$\Lambda := \mathbb{Z}_p[[T]]$$
, $u := (1+p)$, and

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the weight k specialization map $\Lambda \to \overline{\mathbb{Q}}_p$.

• $\underline{g}, \underline{g}^* \in \Lambda[[q]]$ Hida families through the *p*-stabilizations $g_{\alpha}, g_{\alpha^{-1}}^*$:

$$\underline{g}|_{\mathcal{T}=0} = g_{\alpha}, \quad \underline{g}^*|_{\mathcal{T}=0} = g^*_{\alpha^{-1}}$$

(i.e., weight 1 specializations).

• By construction,

$$\kappa_{\alpha,\alpha^{-1}} := \kappa_{f\underline{gg}^*}(T)|_{T=0},$$

where $\kappa_{fgg^*}(\mathcal{T})\in\mathrm{H}^1(\mathbb{Q},V_{
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Theorem (C.–Hsieh)

There is a generalised Coleman power series map

 $\mathcal{L}_{p}:\mathrm{H}^{1}(\mathbb{Q},V_{p}E\otimes V_{\underline{g}\underline{g}^{*}})\longrightarrow\Lambda$

such that

$$\mathcal{L}_{\rho}(\kappa_{f\underline{gg}^{*}}(T)) \doteq \theta_{\rho}^{\mathrm{BD}}(T). \tag{(4)}$$

- Here A becomes the anti-cyclotomic variable for K.
- To prove the main result, we compute the *leading term* of (♠) at T = 0

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$\kappa_{\alpha,\alpha^{-1}}$ and *p*-adic *L*-functions

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• By the theory of anticyclotomic *derived p*-adic heights, there is a filtration

$$\operatorname{Sel}(K, V_{\rho}E) = S_{\rho}^{(1)} \supseteq S_{\rho}^{(2)} \supseteq \cdots \supseteq S_{\rho}^{(r)} \supseteq \cdots \supseteq S_{\rho}^{(\infty)} = 0$$

and a sequence of skew-symmetric (resp. symmetric) pairings for even r (resp. odd r)

$$h_p^{(r)}:S_p^{(r)} imes S_p^{(r)}\longrightarrow \mathbb{Q}_p$$

with $h_p^{(1)} = Mazur-Tate$ pairing, and $ker(h_p^{(r)}) = S_p^{(r+1)}$.

- The τ -eigenspaces of Sel $(K, V_p E)$ are isotropic for $h_p^{(1)}$, since $h_p^{(1)}(x^{\tau}, y^{\tau}) = h_p^{(1)}(x, y)^{\tau} = -h_p^{(1)}(x, y)$.
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- The τ -eigenspaces of $\operatorname{Sel}(K, V_{\rho}E)$ are isotropic for $h_{\rho}^{(1)}$, since $h_{\rho}^{(1)}(x^{\tau}, y^{\tau}) = h_{\rho}^{(1)}(x, y)^{\tau} = -h_{\rho}^{(1)}(x, y)$.
- This is in sharp contrast with the cyclotomic *p*-adic height pairing, which should be non-degenerate.

Building on the formula

$$\mathcal{L}_{p}(\kappa_{f\underline{g}\underline{g}^{*}}(T))) = \theta_{p}^{\mathrm{BD}}(T) \qquad (\clubsuit)$$

we show:

Theorem (C.-Hsieh) Let $\mathfrak{r} := \operatorname{ord}_{\mathcal{T}} \theta_{p}^{\operatorname{BD}}(\mathcal{T})$. Then $\kappa_{\alpha,\alpha^{-1}} \in S_{p}^{(\mathfrak{r})}$, and $h_{n}^{(\mathfrak{r})}(\kappa_{\alpha,\alpha^{-1}}, \chi) \doteq \left(\frac{d}{d\tau}\right)^{\mathfrak{r}} \theta_{n}^{\operatorname{BD}}(\mathcal{T}) = -\log_{\mathfrak{r}}$

for all $x \in S_p^{(r)}$.

Note: For the ("underived") cyclotomic *p*-adic height and first derivatives, such formula was proved by Rubin in the mid 1990s.

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$$h_{\rho}^{(\mathfrak{r})}(\kappa_{\alpha,\alpha^{-1}},x) \doteq \left(\frac{d}{dT}\right)^{\mathfrak{r}} \theta_{\rho}^{\mathrm{BD}}(T)\Big|_{T=0} \cdot \log_{E}(x)$$

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Leading term formula

Building on the formula

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for all $x \in S_p^{(\mathfrak{r})}$.

Note: For the ("underived") cyclotomic *p*-adic height and first derivatives, such formula was proved by Rubin in the mid 1990s.

• For the proof that $\kappa_{\alpha,\alpha^{-1}} \neq 0$, under our hypotheses we have:

 $\dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_p E) = 2, \quad \dim_{\mathbb{Q}_p} \mathrm{Sel}(\mathbb{Q}, V_p E^K) = 0,$

and the filtration reduces to

$$Sel(\mathbb{Q}, V_{\rho}E) = S_{\rho}^{(1)} = S_{\rho}^{(2)} = \dots = S_{\rho}^{(r)} \supset S_{\rho}^{(r+1)} = \dots = S_{\rho}^{(\infty)} = 0$$

for some $r \ge 2$.

• Skinner–Urban's divisibility in IMC implies $\mathfrak{r} \leq r$, so the above gives

$$S_{\rho}^{(\mathfrak{r})} = \mathrm{Sel}(\mathbb{Q}, V_{\rho}E).$$

- If Sel(Q, V_pE) ≠ ker(Loc_p), the proof that κ_{α,α⁻¹} ≠ 0 then follows from our generalised Rubin formula.
- The proof that $\kappa_{\alpha,\alpha^{-1}} \neq 0 \implies \dim_{\mathbb{Q}_p} \operatorname{Sel}(\mathbb{Q}, V_p E) = 2$ is similar.

• For the proof that $\kappa_{lpha, lpha^{-1}}
eq 0$, under our hypotheses we have:

 $\dim_{\mathbb{Q}_{\rho}}\mathrm{Sel}(\mathbb{Q}, V_{\rho}E) = 2, \quad \dim_{\mathbb{Q}_{\rho}}\mathrm{Sel}(\mathbb{Q}, V_{\rho}E^{\kappa}) = 0,$

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- If $\operatorname{Sel}(\mathbb{Q}, V_p E) \neq \ker(\operatorname{Loc}_p)$, the proof that $\kappa_{\alpha,\alpha^{-1}} \neq 0$ then follows from our generalised Rubin formula.
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First instance of non-vanishing: Gross's example

• The elliptic curve E/\mathbb{Q} with $\operatorname{rank}_{\mathbb{Z}}E(\mathbb{Q}) = 2$ of smallest conductor:

$$E = 389a1 : y^2 + y = x^3 + x^2 - 2x.$$

- Take $K = \mathbb{Q}(\sqrt{-2})$ and p = 11. Then:
 - p is ordinary for E, and splits in K.
 - E[p] is irreducible, and ramified at $N^- = 389$.
 - $L(E^{K}, 1) \neq 0.$
 - Can find infinitely many χ with $L(E/K, \chi, 1) \neq 0$ by Vatsal.
- We numerically check

$\theta_p^{\mathrm{BD}}(\mathcal{T}) \equiv -\mathcal{T}^2 + 58\mathcal{T}^3 + \cdots \pmod{(p^2, \mathcal{T}^p)}.$

 dim_{Q_p}Sel(Q, V_pE) = 2 by Bertolini–Darmon's divisibility in IMC, and κ_{α,α⁻¹} ≠ 0 by our main result.

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Main result

Gracias por vuestra atención!

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University of California Santa Barbara