Partial hyperbolicity on hyperbolic 3-mfds

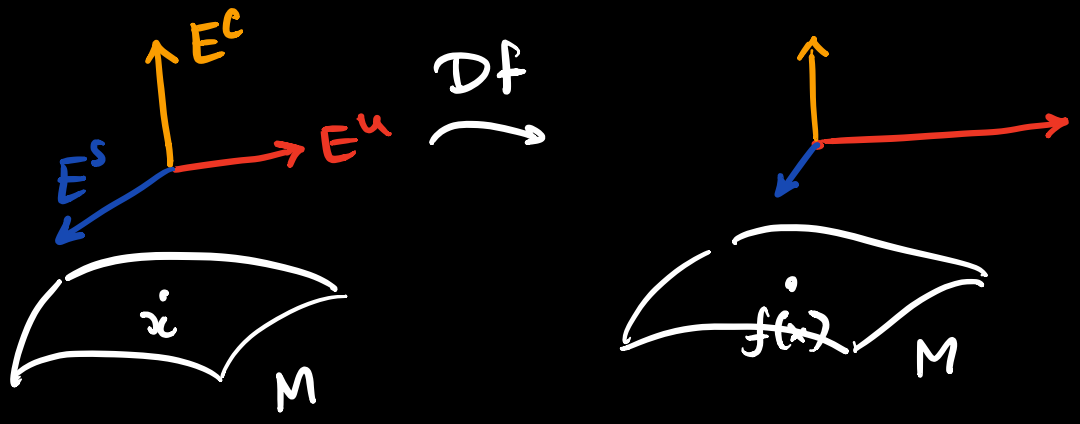
M closed 3-manifold and $f: M \to M$ a partially hyperbolic diffeomorphism.

$$TM = E^s \oplus E^c \oplus E^u \leftarrow \text{1 dimn'l bundles}$$

Df-invariant + $\exists\, \ell > 0$ s.t.

$$\|Df^{\ell}/_{E^s}\| \leq \frac{1}{2} \min\{1, \|Df^{\ell}/_{E^c}\|\}$$

$$\|Df^{\ell}/_{E^u}\| \geq 2 \max\{1, \|Df^{\ell}/_{E^c}\|\}$$



GOAL: Classify these objects.

Why $\dim M = 3$? Rich enough context + Not impossible a priori.

Examples:

- Algebraic
  - automorphisms of tori + nilmanifolds
    - → DA
    - → skew prod.
  - algebraic Anosov flows (solvmfds, geodesic flow surf. $\chi = -1$)
- time one maps of Anosov flows
  - → algebraic Anosov flows
  - → surgery constructions (including many introduced by Christian & collaborators)

(2005) Bonatti-Wilkinson Formalized a proposal of Pujals suggesting that above examples could account for ALL examples. (transitive) [comment: assumptions on dyn of E^c. BZ]

In view of HPS work, a natural notion of equivalence implicit in BW work (formalized by Hammerlindl) is that of leaf conjugacy ← Not relevant what is now, but requires "dynamical coherence"

Comment: Non transitive case seemed "more flexible" and was first to have examples not in the list: HHU in T^3 (non DC) & solvmfds, Bonatti-Parwani-P. other mfds (D.C.)

3 types of problem:

1) Topological obstructions: Which manifolds & isotopy classes of maps admit.

2) Integrability of E^c: Does E^c integrate into an f-inv foliation (dynamical coherence)

3) Classification modulo E^c: Compare to some "model dynamics"

Important breakthrough: (also with participation of Christian!)
Brin-Burago-Ivanov introduced the concept of branching foliations which allows for concrete problems in the direction of 2) and allowed the first results in direction of 1) (e.g. S^3 does not admit)

(Comment: Further work allowed for complete classif when $\pi_1(M)$ is (virtually) solvable, see survey Hammerlindl-P. ETDS (2018))

New examples:

Thm (Christian, Gogolev, Hammerlindl, ?.)
For certain Anosov flows $\phi_t : M \to M$ $\exists$ diffeomorphisms $\varphi : M \to M$ not isotopic to the identity such that for large t one has that $f_t = \varphi \circ \phi_t$ is partially hyperbolic.
In particular, for $M = T^1S$, one can take, $\forall\, \rho \in MCG(S)$ a diffeomorphism $\varphi \sim D\rho$ of T^1S.

How do these examples interact with the questions:

1) Top. obstructions: The manifolds admit Anosov flows (so no new mfds), but new isotopy classes.

2) Integrability: Some of the examples do not admit a foliation tangent to E^c. (But in general it is open even for the specific examples!)

3) Classification modulo E^c: A priori, these examples represent a big challenge to this point.
We show they are PH by using the cone criteria, but this gives little information about their structure.

GENERATION OF MODELS: (joint work w. Barthelme & Fenley)

Def: A partially hyperbolic diffeomorphism $f: M \to M$ is said to be a collapsed Anosov flow if

$\exists$ $\phi_t: M \to M$ Anosov flow

$\beta: M \to M$ a self orbit equivalence of ϕ_t

$h: M \to M$ continuous $\sim$ id

such that: 1) $f \circ h = h \circ \beta$

2) h maps orbits of ϕ_t to curves tangent to E^c.

Thm (with Barthelme & Fenley) This is an open & closed (full connected components) class among PH diffeos which contains all known examples of PH in mfds with $\pi_1(M)$ non solvable.

Main point: This gives models and "solves" the meaning of classification modulo E^c (even if E^c not integrable)

Questions: • Is every (transitive) p.h. in M with $\pi_1(M)$ not solvable a collapsed Anosov flow?

• Is every self orbit equivalence realized by a collapsed Anosov flow? (the examples above give a partial answer which is complete for $M = T^1S$)

SOME RESULTS IN THIS DIRECTION

Thm (with Barthelme, Fenley, Frankel)

Let $f: M \to M$ be partially hyperbolic homotopic to identity which is transitive and $\tilde{f}: \tilde{M} \to \tilde{M}$ a lift to universal cover s.t. $\tilde{f}$ commutes with $\pi_1(M)$ and let $\mathcal{F}^{cs}$ the branching foliation of Burago-Ivanov

(finite cover if it exists)

then: 1) either every leaf of $\tilde{\mathcal{F}}^{cs}$ is fixed by $\tilde{f}$, or

2) $\tilde{\mathcal{F}}^{cs}$ is uniform (leafs pairwise bounded Hsdff distance) and $\tilde{f}$ is a translation on its leaf space ($\simeq \mathbb{R}$)

Comments: i) If M is hyperbolic or Seifert there is no need to assume f is transitive

ii) If M is hyperbolic or Seifert then, if 1) holds for $\mathcal{F}^{cs}$, then it also holds for $\mathcal{F}^{cu}$

and in this case we can show f is a (special kind) of collapsed Anosov flow which we call discretised Anosov flow.

iii) If M is Seifert, then an iterate of f verifies 1)

Recently, we used this result to prove

Thm (with Fenley) If $f: M \to M$ is a partially hyperbolic diffeomorphism of a hyperbolic 3-mfd M then f is a collapsed Anosov flow.

[Corollary: If M hyperbolic 3-mfd admits a PH diffeo $\Rightarrow$ it admits an Anosov flow.]

Another fun consequence of techniques: If $f: M \to M$ of M hyperb. preserves a normally hyperbolic foliation $\Rightarrow$ it is plaque expansive.