Partial hyperbolicity:

$f: M \to M$ a diffeomorphism of closed manifold (compact w.o. boundary)

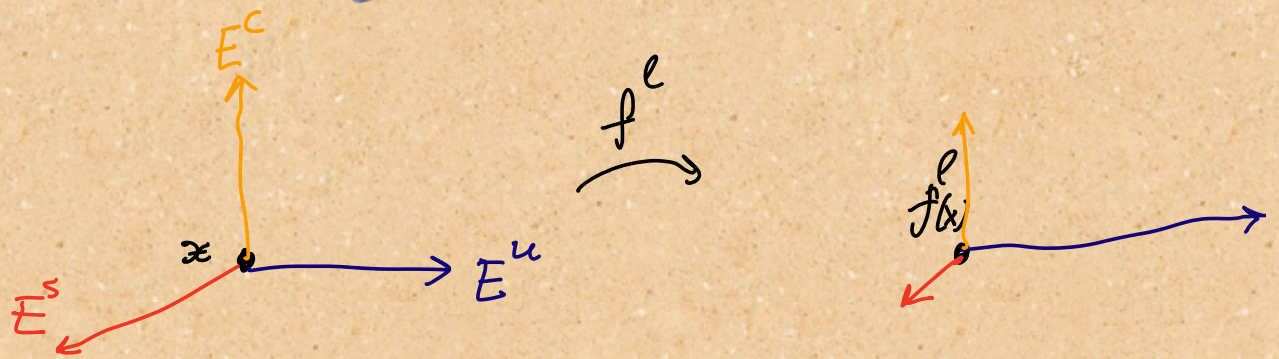
is PARTIALLY HYPERBOLIC if $\exists\, TM = E^s \oplus E^c \oplus E^u$ (i.e. $\forall x \in M$, $T_xM = E^s_x \oplus E^c_x \oplus E^u_x$)

non trivial splitting of tangent space which is Df-invariant and

$\exists \ell > 0$, $\forall x \in M$, and unit vectors $v^s \in E^s_x$, $v^c \in E^c_x$ and $v^u \in E^u_x$

one has: $\|Df^\ell v^s\| < \frac{1}{2} \min\{1, \|Df^\ell v^c\|\}$

$\|Df^\ell v^u\| > 2 \max\{1, \|Df^\ell v^c\|\}$



Notion introduced in 70's by Hirsch-Pugh-Shub. (Normal hyperb)

Brin-Pesin (Ergodicity)

Mañé (Related notion of DOMINATED SPLITTING) (Structural stability & robust transitive)

→ Among other things they showed local structural stability of normally hyperbolic foliations.

In the 90's the notion resurrected particularly thanks to the seminal work of Grayson-Pugh-Shub and subsequent Pugh-Shub conjectures about STABLE ERGODICITY.

(notions mentioned by Mike)

I will concentrate on the dissipative case

[$f: M \to M$ is a diffeomorphism but not a priori volume preserving.

Side note: Dynamics of non conservative systems was launched by Smale in 60's and Mike was part of the development of hyperbolic dynamics then.

(Paper: Homology & dynamical systems, Shub-Sullivan, Topology '75)

Also related with Shub's Entropy conjecture

General idea: Conley's fundamental theorem of dynamical systems gives a "sort of" Morse function interacting with dynamics.

So "all is left" is to understand the pieces of this decomposition.

very hard!

(quite successfull in hyperbolic)



§ CONTEXT M closed 3D manifold.

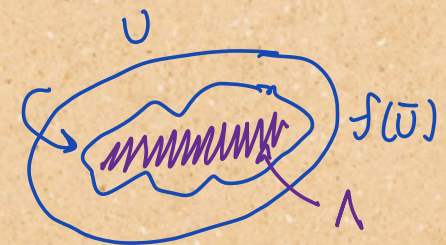
$f: M \to M$ partially hyperbolic diffeo

Problem: Understand the structure of pieces.

Particularly important: Atractors (or quasi-Atractors)

$\exists U$ open set s.t. $f(\bar{U}) \subseteq U$

and $\Lambda = \bigcap_{n \geq 0} f^n(U)$ is transitive ← dyn. indecomposable.

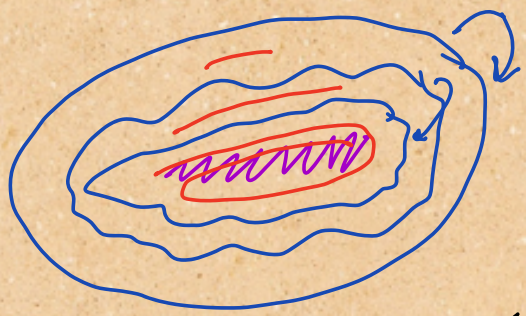


Sample result:

Thm (j.w. Abdenur & Crovisier) Among PH diffeos in dim 3 for which the conley function is trivial, those which are transitive contain a C^1 open and dense set.

Key fact (Hirsch-Pugh-Shub) E^s and E^u are tangent to foliations $\mathcal{F}^s, \mathcal{F}^u$ (strong stable/unstable foliation)

Remark: If Λ is an attractor $\Rightarrow$ Λ is $\mathcal{F}^u$-saturated



GOAL: Understand minimal un-saturated sets.

(and relate to #, uniqueness of attractors/transit.)

Some results:

Thm (j.w. Crovisier, Sambarino) Among partially hyp. diffeos in $\dim M = 3$ those with finitely many u-saturated sets contains a C^1-open and dense set.

(⇒ finitely many attractors)

For some classes of systems, this can be improved:

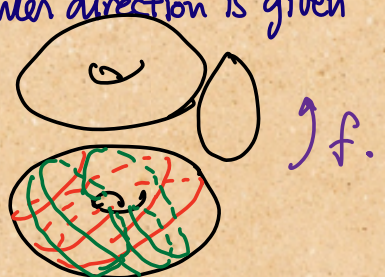
Thm: (j.w. Hammerlindl) For PH systems in nilmanifolds there is a unique minimal u-saturated set.

Thm (Guelman-Martindich) For a large class of PH systems - called discretised Anosov flows - which contain perturbations of most Anosov flows, there is a unique minimal u-saturated set.

Counterexample: Take $f : \mathbb{T}^3 = \mathbb{T}^2 \times S^1 \circlearrowleft$ given by

$f(x,t) = (Ax, t+\alpha)$ and $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ acting on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

- f is partially hyperbolic: A is hyperbolic matrix (trace 4, det 1) so $\exists\, E^s_A, E^u_A$ in $\mathbb{T}^2$ and center direction is given by tangent to circle direction



g_f.

Minimal u-saturated sets: $\mathbb{T}^2 \times \{t\}$.

In partially hyperbolic setting, already characterizing those manifolds (even in dim 3) which are indecomposable pieces is very challenging (In hyperbolic setting, Anosov diffeos in dimension 3 only exist in $\mathbb{T}^3$)

Question: (Non conservative version of Pugh-Shub's conjecture??)
Are there isotopy classes of maps of 3-manifolds where for every partially hyperbolic diffeomorphism in such isotopy class, the dynamics is indecomposable?

This question is also related with minimal u-saturated sets.
Note that in many cases the answer is NO (Bonatti-Guelman; Yi Shi)

Based on some work in progress with A. Avila, S. Crovisier, A. Eskin, A. Wilkinson & Z. Zhang

it is possible to show:

Thm: Let $f: \mathbb{T}^3 \to \mathbb{T}^3$ partially hyperbolic isotopic to a linear Anosov diffeomorphism (e.g. $f_*: \pi_1(\mathbb{T}^3) \circlearrowleft$ is $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$) then f is chain recurrent. (In particular those which have a dense orbit contain a C^1 open and dense subset.)

(Note: There are many isotopy classes of maps where this question makes sense and which are open to understanding.)

I will try to convey one insight and one technical idea.

§Insight: Understanding the topology of f can help.

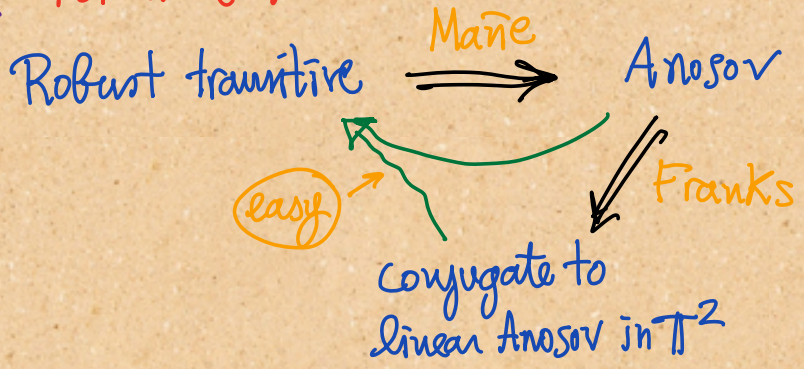
f is partially hyperbolic. What does 'its topology' mean?

Facts: . E^s and E^u integrate to foliations (f-invariant)

. E^c is one dimensional but only continuous, so, a priori not integrable.

Can one still understand the topology of curves tangent to E^c and how they are permuted by f?

Other aspect: For $\dim(M)=2$.



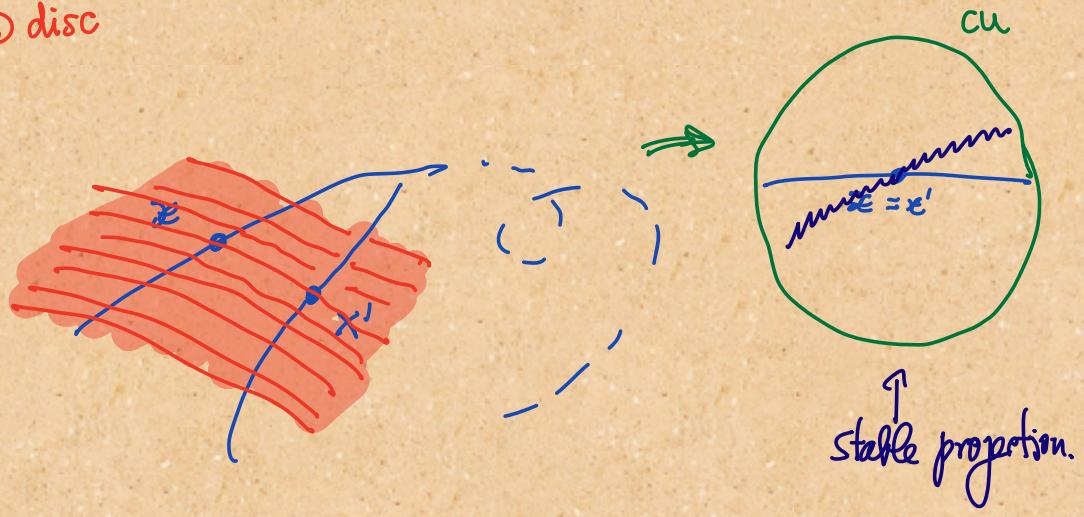
§ Technical Idea: Transversality to understand minimal u-saturated sets.

(a dissipative analog of accessibility ??)

↑ in CPS we have shown it is C^1 open & dense

Def: We say the u-foliation is s-transverse if $\exists L > 0$ so that for every segment $I \subseteq \mathcal{F}^u$ of length $> L$ we have that there are points $x, x' \in I$ so that $x' \in W^{ss}_{loc}(x)$ and such that $W^{uu}_{loc}(x')$ intersects

$\underbrace{W^{ss}_{loc}(W^{uu}_{loc}(x))}_{\text{2D disc}}$ topologically transversally.



Proposition: Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be partially hyperbolic isotopic to Anosov, then, either $E^s \oplus E^u$ is jointly integrable or the u-foliation is s-transverse.

Proof: Use semiconjugacy (Franks) to reduce to the following problem:

Given a curve in the plane and a minimal action of $\mathbb{Z}^3$ by translations, then:

→ either the curve is a (linear) segment, or

→ there is a translation with a (topologically) transverse intersection.

