COARSE DYNAMICS & PARTIAL HYPERBOLICITY

Members Seminar IAS
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GENERAL GOAL: Describe asymptotic behaviour of orbits of a dynamical system $f: M \to M$ using information that can be obtained in finitely many iterates (and posibly with some imprecition).

In general very hard problem. Point: In the hyperbolic/partially hyperbolic regime many things can be said; at least in low dimensions.

Some examples (Geodesic flows in curvature -1 surfaces)

- Geometrically is the solution of a differential equation in T^1S.
  Its time one map is an example of partially hyperbolic diffeo;
  its flow is an example of an Anosov flow (hyperbolic)
- Algebraically; when curvature is constant -1, it is a translation on a quotient of $PSL_2\mathbb{R}$.

Definition: $f: M \to M$ diffeo of closed 3-mfd is partially hyperbolic if $TM = E^s \oplus E^c \oplus E^u$ continuous Df-invariant splitting into 1-dim bundles so that

$\exists \ell > 0$ so that: $\|Df^\ell/_{E^s}\| < \frac{1}{2}\min\{1, \|Df^\ell/_{E^c}\|\}$

$\|Df^\ell/_{E^u}\| > 2\max\{1, \|Df^\ell/_{E^c}\|\}$

Comments: 1) examples include various algebraic examples and time 1-maps of Anosov flows, but it is also a C^1-open property (and detectable in finitely many iterates via cone-fields.

2) partially hyperbolic diffeos "describe" certain dynamical phenomena which are "robust", so makes sense to think "backards" and try to induce dynamics from the property.

3) Also serve to model "random dynamics" in dimension $\dim E^c$, since one can "morally" think of the bundles E^s and E^u to be "fast" variables.

(indeed, it is well known that E^s & E^u integrate to f-inv. foliations, so that their tangent dynamics is actually reflected in M.)

PROBLEMS: → Topological characterization: Which manifolds/isotopy classes support partially hyperbolic systems? How do they compare to other known classes of examples? How do they look like? (Note: Anosov flows in $\dim M = 3$ are far from classified; so in general the classification is looked up to this.)

→ Dynamical characteristics: What should one expect about their long term behaviour?

To simplify statements I will restrict to a class of 3-mfds. (This implies ignoring a lot of the history of the problem)

M closed 3-mfd is said to be hyperbolic if $M \cong \mathbb{H}^3/\Gamma$ ← cocompact is $\mathrm{Isom}(\mathbb{H}^3) \simeq PSL_2\mathbb{C}$.

Thm A (with BFF) If M is a hyperbolic 3-mfd which admits a PH diffeo → it admits an Anosov flow.

Thm B (with Fenley) If $f: M \to M$ is volume preserving $C^{1+\kappa}$ partially hyperbolic diffeo on a hyperbolic 3-mfd $\Rightarrow$ f is a K-system ($\Rightarrow$ ergodic, mixing)

Goal of the rest of the talk:
- Explain the "classification structure" leading to thm A
- Give a proof of a statement related to this result that uses "coarse" dynamics to produce some structure.

Some structure:

Fact: E^s and E^u integrate uniquely into f-invariant 1-dim foliations.

In general, E^c may not integrate, but 1-dim, so $\exists$ integral curves.
We want characterization up to the center direction.

(Note: When $E^s \oplus E^c = E^{cs}$ and $E^c \oplus E^u = E^{cu}$ integrate to f-invariant foliations we say that f is dynamically coherent. Always the case for Anosov flows or algebraic examples, but not always the case and not "detectable" in finitely many iterates.

Burago-Ivanov: Important technical work to have "initial" 2D-structure will be hidden in this talk, but crucial for proofs.)

Definition: $f: M \to M$ partially hyperbolic is a collapsed Anosov flow if $\exists\, \phi_t: M \to M$ Anosov flow; $\beta: M \to M$ a self orbit equivalence of ϕ and $h: M \to M$ continuous homotopic to identity so that:

(1) h maps orbits of ϕ_t injectively onto C^1-curves tangent to E^c
(also weak stable/unstable to C^1-surfaces tg to E^{cs}, E^{cu}, but this is technical)

(2) $f \circ h = \beta \circ h$.

Examples: → Discretised Anosov flows: h homeomorphism; $\beta(x) = \phi_{t(x)}(x)$

" $f(x) = \phi_{t(x)}(x)$ "

→ Recent examples: (w. Bonatti-Gogolev-Hammerlindl) where h cannot be homeo and β permutes orbits.

Thm (BFFP) $f: M \to M$ partially hyperbolic on hyperbolic 3-mfd M
→ f is a collapsed Anosov flow.
Moreover, if f is dynamically coherent $\Rightarrow$ f is discretised Anosov flow.

↳ Recall: $\exists\, W^{cs}, W^{cu}$ f-inv. foliations tg to E^{cs}, E^{cu}.

I want to explain one argument that appears in the proof in the simplest possible setting:

Thm (BFFP) Let $M = T^1S$ and $f: M \to M$ partially hyperbolic diffeo inducing pseudo Anosov class in S. (surface of genus $g \geq 2$).
Then f cannot be dynamically coherent.

Note: → In BGHP we construct examples in this isotopy class.

→ pseudo Anosov means that does not preserve finite family of closed curves up to isotopy (detected in 1-iterate)

Thurston ↔ Coarse dynamics
Handel...

PROOF BY CONTRADICTION (unnecessary but simplifies since one can assume $\exists$ foliation)

We assume $\exists$ foliation W^{cs} f-invariant & tangent to E^{cs}.

[Prop. (j.w. Hammerlindl & Shannon) W^{cs} is horizontal (up to isotopy of fibers ~~is~~ every leaf is uniformly $\pitchfork$ fibers)]

idea: No Reeb components (else, closed W^u-leaf)

If foliation were $C^2 \Rightarrow$ immediate from Thurston's theris.

Otherwise, need to take care of vertical laminations; it can be done. ∎

We lift to $\hat{M} = T^1\tilde{S}$ $\left(= \widetilde{T^1S}/_{\langle \text{Fibers} \rangle} \simeq \mathbb{D}^2 \times S^1\right)$

↑ hyperbolic disk.

we put coordinates so that leaves of $\hat{W}^{cs}$ are $\mathbb{D}^2 \times \{t\}$ with $t \in S^1$

so that deck transformations ($\approx \pi_1(S)$) act as $\gamma \cdot (x,t) = (\gamma \cdot x, \rho(x)(t))$

where $\rho : \pi_1(S) \to \text{Homeo}_+(S^1)$ representation.

[**Key Proposition** $\exists$ $\hat{f}$ lift of f to $\hat{M}$ and a compact $\hat{f}$-invariant set $T_{\hat{f}} \subseteq \hat{M}$ intersecting every leaf of $\hat{W}^{cs}$.]

This is enough to conclude: $\hat{f}$ acts on the space of leaves since it preserves the foliation, it has the form: $\hat{f}(x,t) = (F_0(x,t), F_1(t))$

where F_1 is a circle homeomorphism.

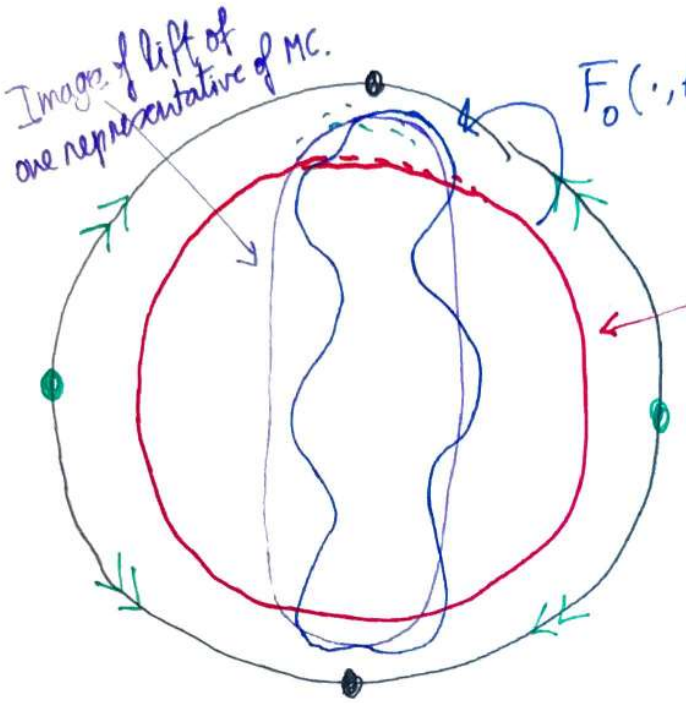
Since $\exists$ $T_{\hat{f}}$ compact invariant set, one sees that F_1 must expand everywhere

so ~~this~~ this gives a contradiction.

[Warning: The compact set is crucial, otherwise points can "move" in the direction where leaves "separate". This is what happens i.e. when one lifts the geodesic flow.]

Proof of Key proposition: (coarse dynamics.)

Key point is that $\exists$ lift of f which looks like:



Iterating forward one obtains connected set from "side to side" that remains in the disk for all backward iterates.

Symmetric argument gives compact connected set in the other direction which remains in the disk for forward iterates.

The intersection produces $T_{\tilde{f}}$.

∎

Comments: → Hyperbolic 3-Mfds also have some "pseudo Anosov" dynamics.

→ Other regimes need to be analised, same techniques do not work.