Foliations seminar

Transverse foliations by GH leaves. Part 1 (j.w. S Fenley)

Thm (j.w. Fenley) Let $\mathcal{F}_1, \mathcal{F}_2$ be $\pitchfork$ foliations in closed M by GH leaves and $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ 1D foliation. Then:

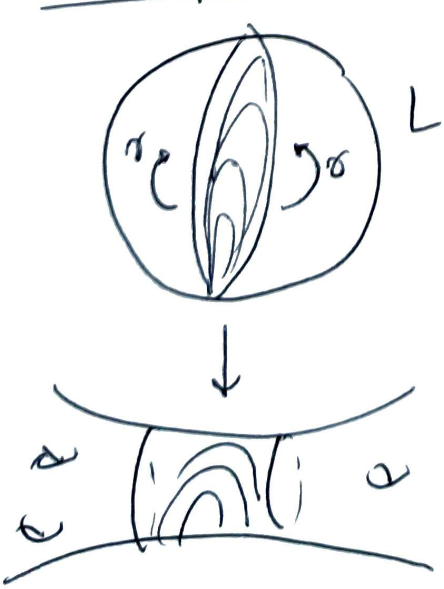
1) either $\mathcal{G}$ is leafwise QG, or
2) $\mathcal{G}$ has a generalized Reeb surface.

Some comments:

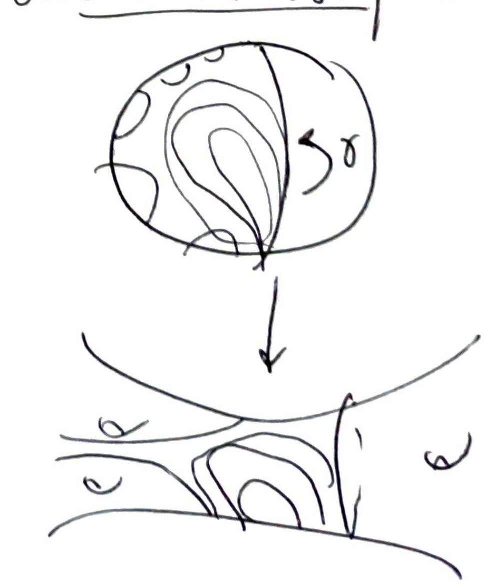
1 – Leafwise QG means that in $\widetilde{M}$, if $L \in \widetilde{\mathcal{F}}_i$ (so L is QI to $\mathbb{H}^2$) then every $\ell \in \widetilde{\mathcal{G}}|_L$ is a QG (uniform constants)

Recently (in SLMath) with Barthelme, Fenley & Mann we showed that this implies that $\mathcal{F}_1, \mathcal{F}_2$ are blow ups of weak stable/unstable of Anosov flow and $\mathcal{G}$ can be blown down to AF.

2 – Reeb surface



Generalized Reeb surface



Statement splits in 2 parts (Assume everything orientable & oriented)

Case A (Sergio will explain) Injective developping map

Means: $\forall L \in \widetilde{\mathcal{F}}_1$, $E \in \widetilde{\mathcal{F}}_2$ we have that $L \cap E$ is connected
(can be empty)

Thm: If $\mathcal{F}_1, \mathcal{F}_2 \pitchfork$ by GH leaves and inj dev map, then, either
Leafwise QG or Gen. Reeb surface.

Comments: → both can happen (even if both minimal & M atoroidal)
→ when $\mathcal{F}_1, \mathcal{F}_2$ are minimal, we only know examples with both foliation being Anosov foliations.
→ we do not know examples with gen Reeb but no Reeb surf.
→ If $\mathcal{F}_1$ or $\mathcal{F}_2$ is $\mathbb{R}$-cov ⇒ this case one gets leafwise QG.

CASE B (This TALK) Non Injective developping map.

Theorem If $\mathcal{F}_1 \pitchfork \mathcal{F}_2$ do not have injective developping map
⇒ $\mathcal{G}$ has Reeb annuli in both foliations.

Comments: → No need for GH leaves (not even Reebless)
→ No examples (Reebless) in M atoroidal
→ When leaves are GH, all examples are Anosov foliations
→ In T^3's we understand all examples → Matsumoto–Tsuboi.

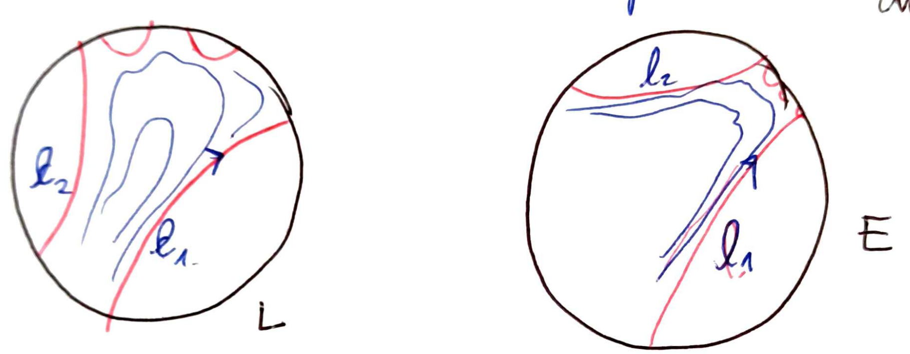
Outline:

→ Examples (Hardop, Matsumoto-Tsuboi)

→ Doubly non separated leaves:

Prop: If $\mathcal{F}_1, \mathcal{F}_2$ have non ~~so~~ inj dev. map, then $\exists L \in \widetilde{\mathcal{F}}_1$ and $E \in \widetilde{\mathcal{F}}_2$ so that $\exists \ell_1, \ell_2 \subset L \cap E$ leaves of $\widetilde{\mathcal{G}}$ which are non separated in both leaf spaces (+ in same direction)

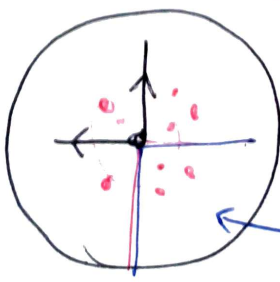
doubly non separated intersection



→ Main result (by contradiction)

Thm: If L and E have doubly non separated intersection, then $L \cap E$ project to closed curves (bounding Reeb annuli in both)

Idea: by contradiction, assume not, then, project to M and non sep ray accumulates somewhere. Non separation gives expanding holonomy for $\mathcal{G}$ in a cuadrant. Key use the other intersection to produce some limit and a contradiction.



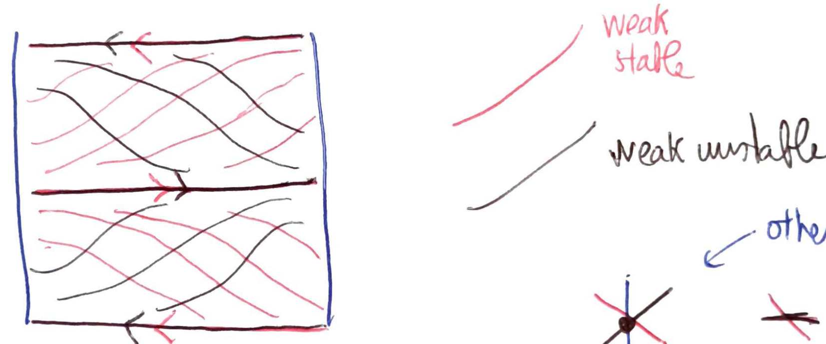
looks like easier quadrant. The others, more work.

MATSUMOTO - TSUBOI EXAMPLES

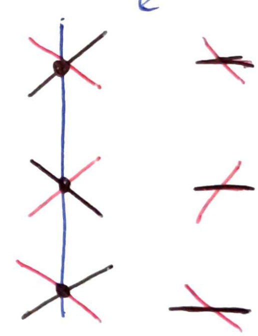
(Others: Hordop, bi-contact structures, Massoni ....)

Idea: produce examples in $\mathbb{T}^2 \times [0,1]$ so that intersection with $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{1\}$ are identical and like "Birkhoff" Tori (So, one can glue easily in Anosov flows with such tori)

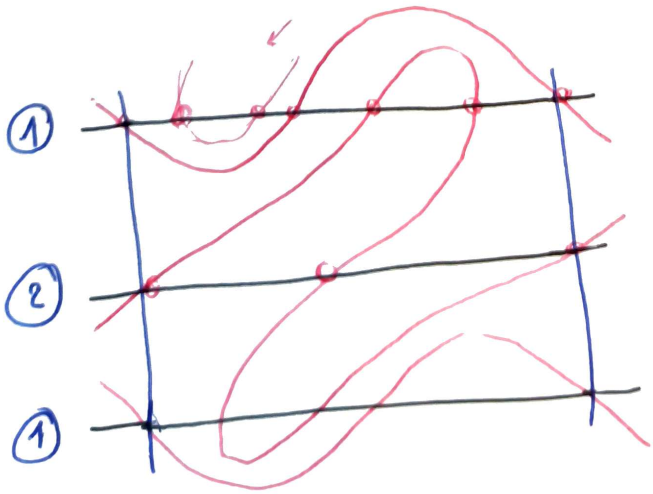
Birkhoff torus:



other view point:

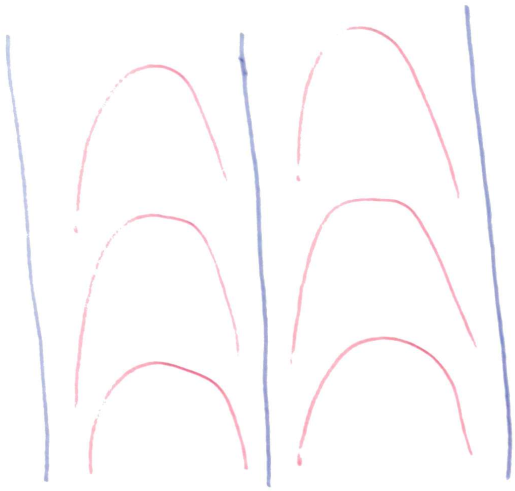


We are going to cut along this torus and put

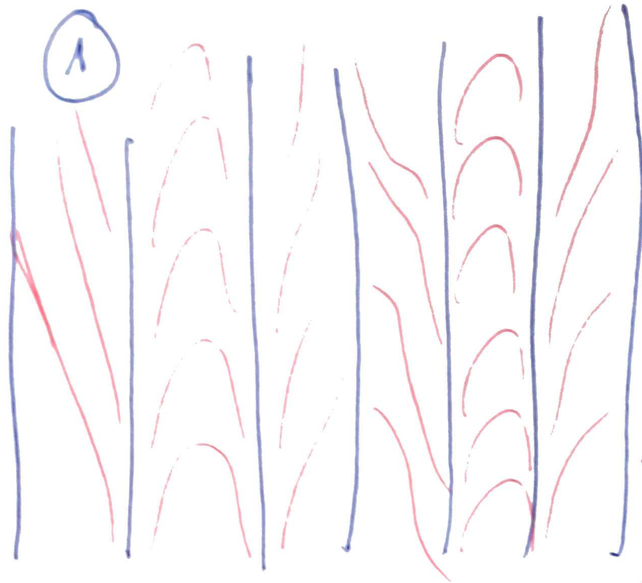


Annuli

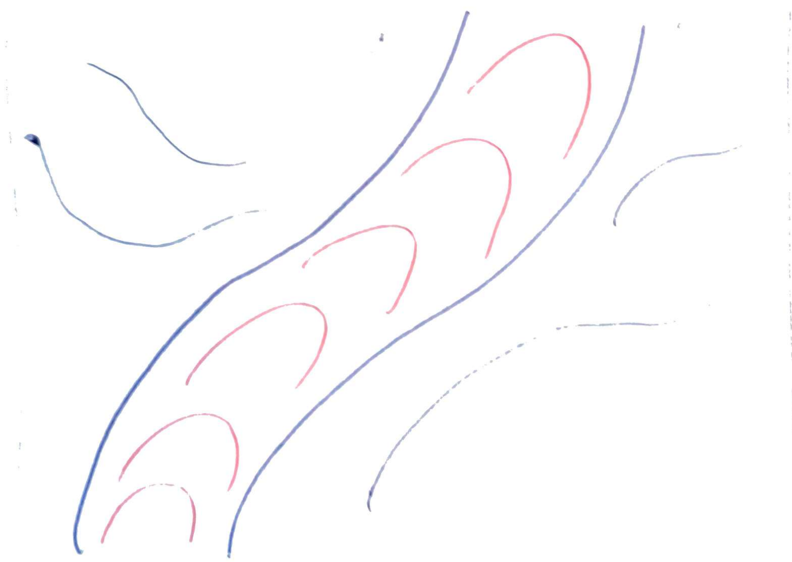
2



1



Interpolation in between :

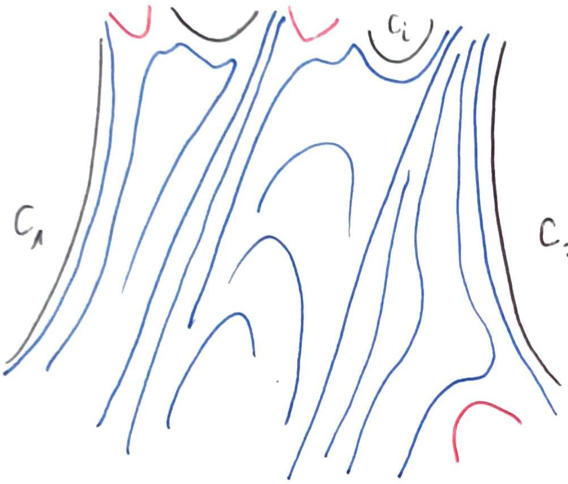


Double non separation (can skip)

(important step. joint also with T. Barbot)

Let $L_0 \in \tilde{\mathcal{F}}_1$, $E_0 \in \tilde{\mathcal{F}}_2$ intersecting in more than 1 connected component.

L_0:



Let $c_1, c_2 \subseteq L_0 \cap E_0$

$[c_1, c_2] = \{ \ell \in \tilde{\mathcal{G}}|_{L_0} \,/\, \ell$ separates c_1 from $c_2 \}$

$\Rightarrow [c_1, c_2] = \bigcup_{i=1}^{n} [x_i, y_i]$ where

$x_1 = c_1$ and $y_n = c_2$.

(intervals can be singleton)

Key lemma For some $i \in \{1, \dots, n-1\}$ the points y_i, x_{i+1} belong to the same leaf of $\tilde{\mathcal{F}}_2$.

Proof: (can be skipped) $\mathcal{P} = \{ (u,v) \in (\tilde{\mathcal{G}}|_{L_0})^2$ in $[c_1, c_2]$ which belong to same leaf of $\tilde{\mathcal{F}}_2 \}$

Properties: $\to$ if $(u,v) \in \mathcal{P} \Rightarrow [u,v] \subseteq [c_1, c_2]$

$\to$ if $(u_n, v_n) \in \mathcal{P}$ nested seq (i.e. $[u_{n+1}, v_{n+1}] \subseteq [u_n, v_n]$) $\Rightarrow \bigcap [u_n, v_n] = [u_\infty, v_\infty] \in \mathcal{P}$.

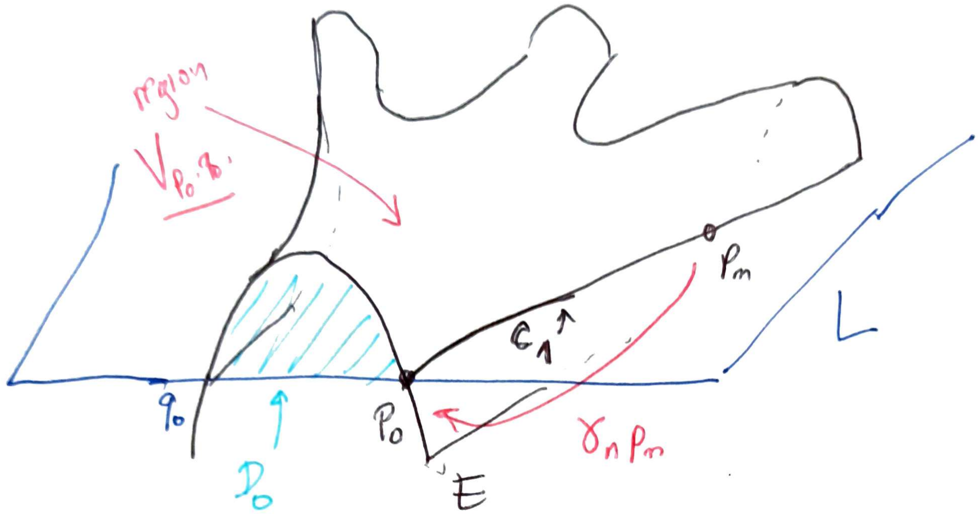
Take (u,v) minimal. and let $E \in \tilde{\mathcal{F}}_2$ s.t. $u, v \in E$

If $u \in [x_i, y_i]$ then, E_i (leaf of $\tilde{\mathcal{F}}_2$ through y_i) then if $E_i \neq E$ then separates u from v (by minimality, cannot cut L_0 again between u and v).

$\boxtimes$

MAIN THEOREM (Everything orientable, oriented)

Let $L \in \tilde{\mathcal{F}}_1$ and $E \in \tilde{\mathcal{F}}_2$ with doubly non separated intersection.

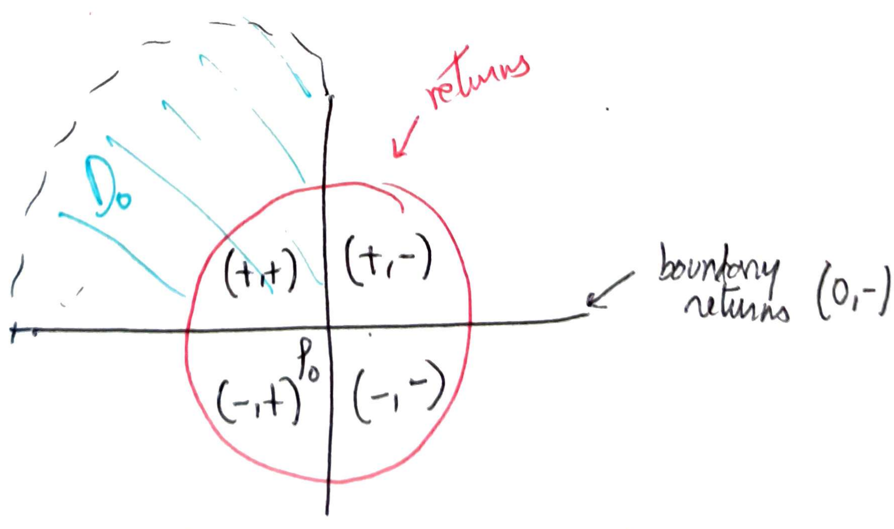


Every leaf in V_{p_0,q_0}, escapes (~~cuts~~ if $E' \in \tilde{\mathcal{F}}_2$ cuts V_{p_0,q_0} $\Rightarrow$ (E' 'below' E) $E' \cap L$ compact interval)

Assume by contradiction that c_1 does not project to closed.

Then: $\exists\, p_n \to \infty$ in c_1 and $\gamma_n \in \pi_1(M)$ such that $\gamma_n p_n \to p_\infty$ (we can assume all ~~p~~ $\gamma_n p_n$ in small bifoliated box)

4 cuadrants:



Expanding holonomy in the "positive" side when flowing forward.

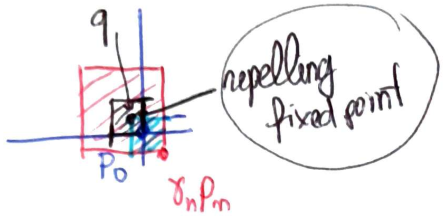
Simplification: Get existence of closed $\mathcal{G}$ leaf. $(+,+)$ & $(-,-)$ T2

(instead of c_1 projects to closed $\Rightarrow$ Reeb annuli)

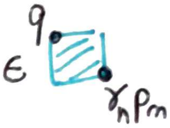
Under this simplification, cases $(-,-)$ and $(+,+)$ (including returns in the boundaries) are very simple:

Case $(-,-)$:

(or $(0,-),(-,0)$)



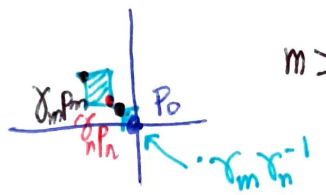
(Rmk: $\gamma_n q = q$ and $\forall z \in$



backward flow spirals to the orbit of q when proyected to M.)

Case $(+,+)$:

(or $(0,+),(+,0)$)

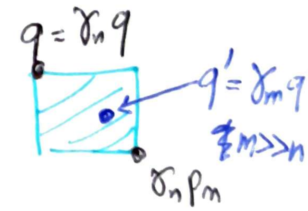


$m >> n$.

Find returns in $(-,-)$ position with respect to $\gamma_n p_m$ $\rightsquigarrow$ put in previous situation.

To get contradiction (and not just periodic orbit)

Note that in $(-,-)$ one can produce points in



contradicts that those points spiral to q.

Similar in $(+,+)$ a bit more work.

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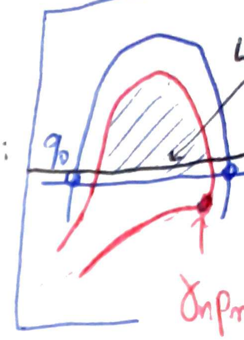
MIXED CASE $\rightarrow$ Difficult case: We will use the 3D picture more.

Consider γ_n, p_m "deep" return. $(-,+)$



Prop (A serrated set) $\exists\, S_n$ closed subset of $\overline{V_{p_0,q_0}}$ such that:

1) the interiors of $\gamma_n^k S_n$ are pairwise disjoint.

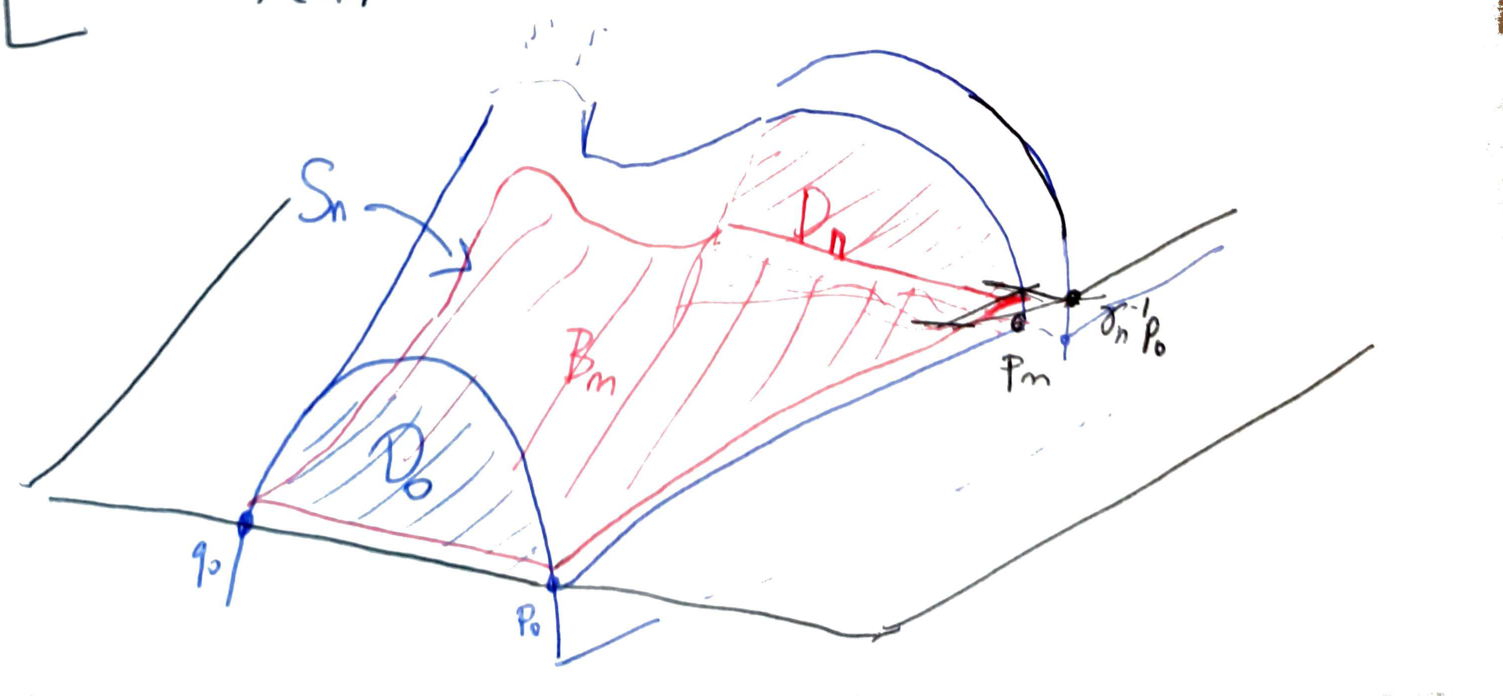
2) the interior of the projection of S_n to $\widetilde{M}/\langle\gamma_n\rangle$ is a solid torus and $\mathcal{S}_m = \bigcup_{k\in\mathbb{Z}} \gamma_n^k S_n$ is the lift of the proyection.

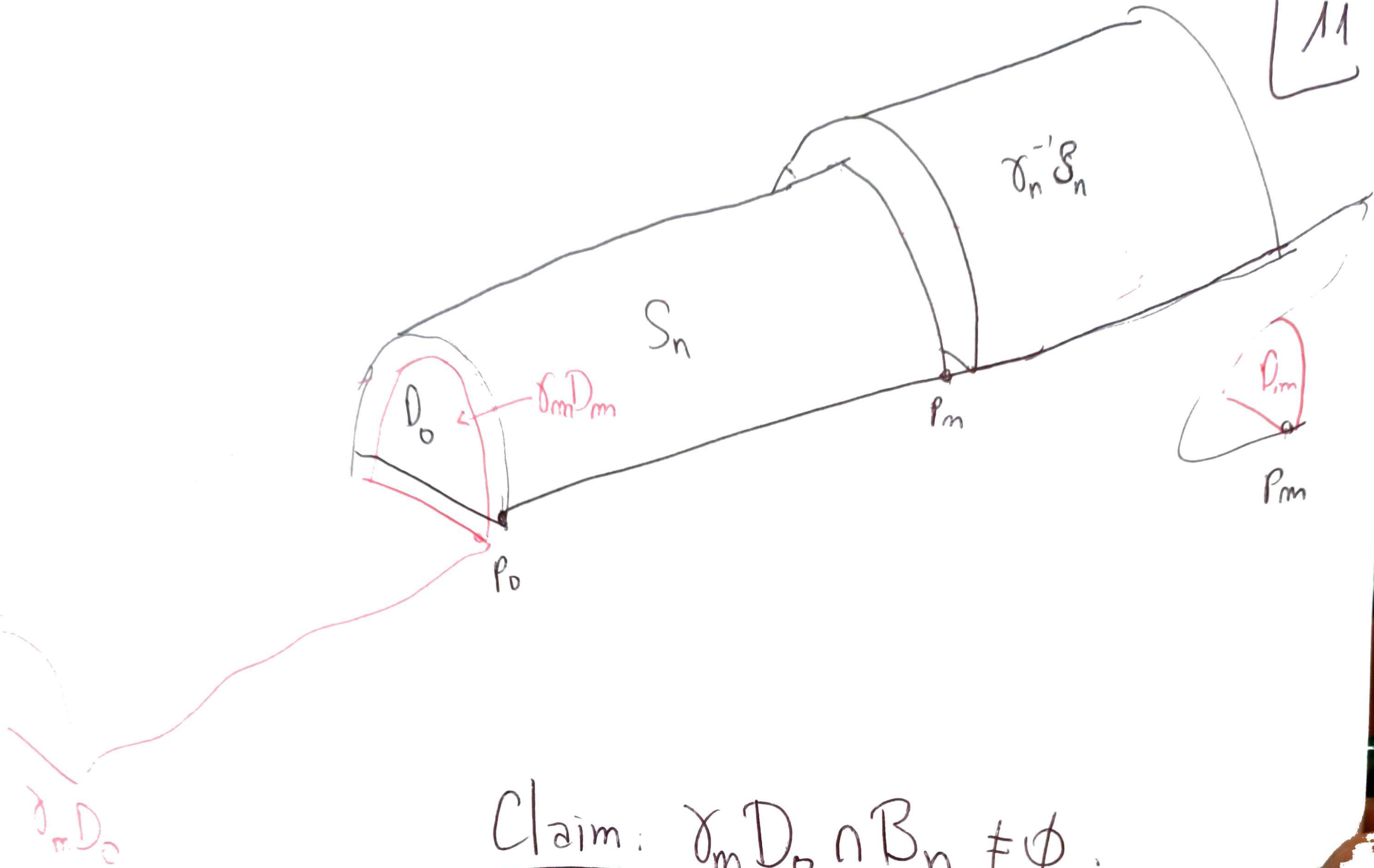
3) $\partial S_n = D_0 \cup D_n \cup \{$subsets of L & E bounded by $\partial D_0, \partial D_m$ and some arcs in $L \cap E\}$

disk with one bd in L and one in E and p_m in corner.

4) If $L_n = \gamma_n L_n$ is the lowest fixed leaf $\Longrightarrow$

$L_n \cap \mathcal{S}_n = B_m$ is a band whose projection to $\widetilde{M}/\langle\gamma_n\rangle$ is a compact annulus.





Claim: $\gamma_m D_0 \cap B_n \neq \emptyset$.

Proof: It cannot escape because it escapes forward, so cannot escape backwards

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Conclusion: If $\pi_n : \tilde{M} \to \tilde{M}/\langle \gamma_n \rangle$ then for infinitely many m, m' one has $\pi_n(\gamma_m D_0) = \pi_n(\gamma_{m'} D_0)$

so the returns are all homotopic, but this contradicts the $\uparrow$ expansion.