FREE GROUPS QUASI ISOMETRIC IN $SL_3(\mathbb{R})$

(j.w. L. Carvajales & P. Lessa)

Theorem (Sullivan) Let $\rho: \Gamma \to PSL_2(\mathbb{C})$ be a structurally stable representation (i.e. robustly faithful), then, it is either rigid or convex-cocompact.

The proof is based on a technique in holomorphic dynamics (holomorphic motions) (Mañé-Sad-Sullivan) and essentially says: if $\exists$ holomorphic one parameter family of deformations (i.e. not rigid) $\Longrightarrow$ one can promote structural stability to "Hyperbolicity" (expansion property at ∞) which implies convex cocompactness.

Related:

Thm (Mañé) Let $f: M \to M$ be a structurally stable diffeomorphism (C^1-perturbations of f are topologically conjugate) $\Longrightarrow$ f is hyperbolic (Axiom A).

The proof of the latter result is more "linear algebraic" in nature it has 3 steps: (Assume for simplicity Periodic orbits are dense)

1) (Easy) Structurally stable $\Rightarrow$ gaps in eigenvalues of periodic points.

2) (Hard) Gap in eigenvalues $\Longrightarrow$ dominated splitting

3) (Hard) $\left.\begin{array}{l}\text{Dominated splitting}\\ +\text{ Structural stability}\end{array}\right\} \Longrightarrow$ hyperbolicity.

<u>General question for discrete groups:</u>

Problem: Let $\rho : \Gamma \to G$ be robustly faithful: is it either rigid or Anosov?

Very general and requires good definition of "rigid". (small number of deformations)

Many recent examples due to Tsouvalas show that the problem is subtle, so, to find (Danciger-Guéritaud-Kassel-Lee-Marquis)

a sensible question, I propose:

Question: Let $\rho : \mathbb{F}_k \longrightarrow SL_3\mathbb{R}$ be a Robustly quasi-isometric representation, is it Anosov?

Remark: If one looks at $\rho : \mathbb{F}_k \to SL_2\mathbb{R}^{k_1} \times SL_2\mathbb{C}^{k_2}$ then

robustly faithful $\Rightarrow$ Anosov (convex cocompact in some coordinate)

but open for $\rho : \mathbb{F}_2 \to SO(1,n)$ $n \geq 4$.

One approach could be to try to extend Sullivan's:

Thm (Martin-Baillon) Assuming some plurisharmonicity of Lyapunov exponents one can show that robustly faithful to $PSL_d(\mathbb{C})$ is equivalent to Anosov.

Notion of proximal stability, not immediate to check

<u>Remark</u>: ~~[illegible]~~ Sullivan's result does not imply the case of $SL_2(\mathbb{R})$.

<u>Thm</u> (elementary) Let $\rho : \mathbb{F}_k \to SL_2\mathbb{R}$ robustly faithful $\to$ Anosov (convex-cocompact or Schottky).

comment on Avila-Bochi Yoccoz for semigroups.

<u>Pf</u>: $\overline{\rho(\mathbb{F}_k)}$ can be discrete or everything (because other closures are solvable)

If everything, since trace of a word is non-constant polynomial one can produce torsion by perturbation $\Rightarrow$ kill injectivity.

If discrete, then enough to show that every element is hyperbolic (quotient is a closed surface, finite type Σ, so either compact convex core or there is a parabolic).

If it is not, then parabolic is either generator or product of commutators. One can compute the trace polynomial for those words and does ~~[illegible]~~ not have local minima at 2.

$\blacksquare$

Still, unclear how to approach the problem in $SL_3(\mathbb{R})$ using Sullivan's approach. So, we can try Mañé's. Problem: the easy part is hard. <u>Good news</u> the hard part is easy:

<u>Thm</u> (j.w. Kassel) Let ~~[illegible]~~ $\rho : \Gamma \to SL_d(\mathbb{R})$ where Γ hyp. group and there is an i-gap on eigenvalues $\Rightarrow$ ρ is i-Anosov.

Question: Let $\rho : \mathbb{F}_k \to SL_3\mathbb{R}$ quasi-isometric representation

(i.e. $\|\rho(\gamma)\| > Ce^{\lambda|\gamma|}$ for some $C>0$, $\lambda>0$.)

if ρ is not Anosov is it possible to perturb ρ so that it is no longer QI? so that it is no longer discrete? no longer faithful? (Rmk: RoF. faith $\Rightarrow$ discrete)

A.Glutsyuk.

Split in 2 problems:

$\boxed{P1}$ Can one make a perturbation to have an element γ so that $\lambda_1(\rho(\gamma)) = \lambda_2(\rho(\gamma))$? (touches the walls)

$\boxed{P2}$ Let ρ be a QI representation so that it has an element γ with $\lambda_1(\rho(\gamma)) = \lambda_2(\rho(\gamma))$, is it possible to perturb in order to break QI? discreteness? faithfulness?

P1: Very little progress... Some examples (not QI), unclear to which extent they exist.

P2: Also not much progress, but I want to present some perturbation technique and some applications/examples.

Prop (jw. CL) Let $\Gamma = \mathbb{F}_k$ $(k \geq 2)$ and $\rho : \Gamma \to SL_3\mathbb{R}$ and assume $\exists\, a, b \subseteq$ generating set of Γ. (can be completed to free generators) with the following properties:

(1) $\exists\, m, n \neq 0$ such that $w := a^m b^n$ verifies that

$P_0 = \mathrm{Ker}(\rho(w^q) - \mu)$ is 2-dim for some $q \neq 0$ and $\mu \neq \pm 1$

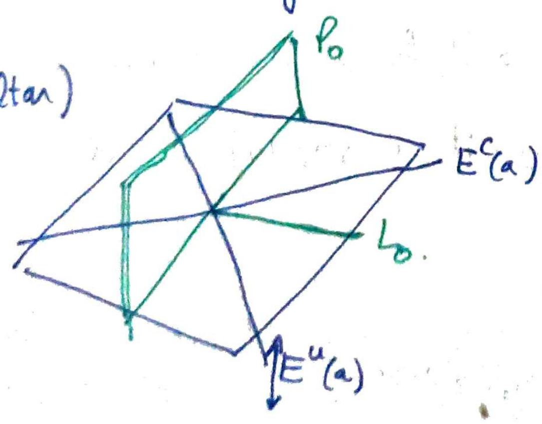
(Moreover, the subspace $L_0 = \mathrm{Ker}(\rho(w^q) - \mu^{-2})$ is 1D)

(2) $\rho(a)$ and $\rho(b)$ are loxodromic (i.e. 3 $\neq$ real eigenv) and

L_0 is contained in either $E^{cu}(\rho(a))$ (attracting plane of $\rho(a)$) or in $E^{cs}(\rho(a))$ (repelling plane)

Then, $\exists\, \rho_t : \Gamma \to SL_3\mathbb{R}$ continuous path with $\rho = \rho_0$ and $t_k \to 0$ such that $\forall k$ the element $\rho_{t_k}([w^q, a^{p_k} w^q a^{-p_k}])$ is unipotent for some choice of $p_k \neq 0$.

Idea (Saltar)



(without loss of generality we assume this general position; except $L_0 \subseteq E^{cu}(a)$)

We consider $\rho_t(a)$ so that $E^u(\rho_t(a))$ $\pitchfork$ to $E^{cu}(\rho_0(a))$

and $\rho_t(b)$ so that $\rho_t(w) = \rho(w)$

We will get that both $\rho_{t_0}(w^q)$ and $\rho_{t_0}(a^p w^q a^{-p})$ for large p preserve same plane and act as identity there $\longrightarrow$ $\underbrace{\rho_{t_0}(a^p) P_0}_{\text{chosen so that it contains } L_0 \leftarrow \text{pres by } w}$ for large p

$\Longrightarrow \rho_{t_0}([w^q, a^p w^q a^{-p}])$ is unipotent.

because w [illegible] $\subseteq P_0$ — Also preserves $\rho(a^p) P_0 \cap P_0$

Some applications:

1) Lahn showed that $\exists$ non-Anosov QI images of $\mathbb{F}_2$ in $SL_3(\mathbb{R})$ which are reducible:

We called them "derived from Barbot"

idea:

$$\rho(\gamma) = \begin{pmatrix} e^{-\frac{1}{2}\varphi(\gamma)} \cdot \rho_p(\gamma) & \kappa(\gamma) \\ 0 & \pm e^{\varphi(\gamma)} \end{pmatrix}$$

where: $\rho_p : \Gamma \to SL_2^{\pm}\mathbb{R}$ is Schottky (QI)
$\varphi : \Gamma \to \mathbb{R}$ morphism
$\kappa : \Gamma \to \mathbb{R}^2$ morphism.

Lahn describes for which φ the repr. is Anosov but it is always QI.

Thm (j.w. Carvajales–Lessa) If $\rho : \Gamma \to SL_3\mathbb{R}$ is a reducible suspension (i.e. preserves some plane P) then it is robustly QI iff it is Anosov

(While QI $\Leftrightarrow$ derived from Barbot)

NOTE: We don't know if non Anosov DfB can be rob. discrete (we produce unipotents)

2) We construct examples of QI free groups which are Zariski dense and touch the walls. (rational eigenvalues)

We show (only in some cases) that one can perturb to kill QI (in this case, likely kill discrete too but in specific examples)

Thm $\exists\, \rho : \mathbb{F}_k \to SL_3\mathbb{R}$ $k \geq 2$ which are QI and have elements with $\lambda_2 = \lambda_3$. Can be perturbed to minimal action in flags. If $k \geq 3$ can be done so that not Rob. QI.