ERGODICITY OF CERTAIN PH DIFFEOS in DIM 3 (j.w. with S. Fenley)

$f: M \to M$ volume preserving diffeo

Broad question: When is f ergodic/mixing with respect to volume? (i.e. if $A \subseteq M$ f-invariant measurable $\Rightarrow \mathrm{vol}(A)$ or $\mathrm{vol}(M \setminus A) = 0$)

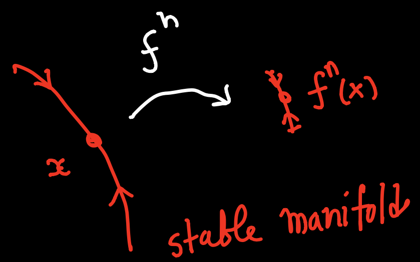
Note: This is very subtle question: depends on regularity

for instance:
- Oxtoby-Ulam: Generic volume preserving homeos are ergodic
- KAM: $\exists$ open subsets of $\mathrm{Diff}^3_{\mathrm{vol}}(M)$ with no ergodic diffeos.

Theorem (Anosov-Sinai) Anosov systems are stably ergodic.

Hopf argument, + absolute continuity of foliations

Forward Birkhoff averages are constant along stable manifolds



PUGH-SHUB'S CONJECTURES

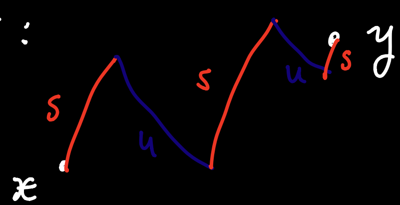
Most partially hyperbolic systems are stably ergodic.

PH: $f : M \to M$ is partially hyperbolic if $TM = E^s \oplus E^c \oplus E^u$ is Df-invariant and $\exists\, \ell > 0$ s.t. $\forall\, v^\sigma \in E^\sigma(x)$ $(\sigma = s, c, u)$ unit vectors

$$\Rightarrow \|Df^\ell v^s\| < \tfrac{1}{2} < 2 < \|Df^\ell v^u\|$$

$$\& \quad 2\|Df^\ell v^s\| < \|Df^\ell v^c\| < \tfrac{1}{2}\|Df^\ell v^u\|$$

Idea: Such f have foliations W^s, W^u f-inv. tangent to E^s & E^u, so one can try to apply Hopf's argument provided $E^s \oplus E^u$ is accesible ($\forall\, x, y$ $\exists$ path, piecewise C^1 & tangent to $E^s \cup E^u$:



This splits the problem in 2:

1) Show accesibility is abundant

2) Show accesibility $\Rightarrow$ ergodicity.

(Strategy in seminal work of Grayson-Pugh-Shub, Wilkinson, ...)

About accesibility $\Rightarrow$ ergodicity

turns out to be a very subtle problem. However, in some cases it has been solved.

Notably: Burns-Wilkinson: PH + "center bunching" + accesible $\Big\}\Rightarrow$ ergodic

(actually, they get K-system, stronger than mixing)

center bunching = technical hypothesis. Always true if $\dim E^c = 1$.

About abundance of accesibility.

Very sensible to choice of topology in $\mathrm{Diff}_{vol}(M)$

$\boxed{\text{In } C^1}$ ← contains open dense: Dolgopyat-Wilkinson

(in C^1, more recently Avila-Crovisier-Wilkinson established the full version of Pugh-Shub conjecture)

$\boxed{\text{In } C^r,\ r>1}$ ← when $\dim E^c = 1$: Hertz-Hertz-Ures

(this gives full PS in this context)

← when $\dim E^c > 1$: Some progress, but still mild. Recent advances by Avila-Viana, Horita-Sambarino, Leguil-Zhang, Leguil-Piñeyrua...

How about showing ergodicity without perturbation?

In ~ 2006, Hertz-Hertz-Ures suggested (conjectured) that maybe ALL partially hyperbolic diffeos in dim $M = 3$ except some trivial counterexamples should be ergodic.

Much progress:

1- Hertz-Hertz-Ures : Nilmanifolds.

2- Hammerlindl : Solvmanifolds.

3- Hammerlindl-Hertz-Ures & indep Fenley-P $\Bigg\} \Rightarrow$ some isotopy classes in Seifert mfds.

4 - Fenley-P: $\Longrightarrow$ Hyperbolic manifolds and other classes.

Today I want to explain a recent result with S. Fenley in the direction of this problem, but with a somewhat different taste.

I will define a class of partially hyperbolic diffeos in dimension 3 that we call "collapsed Anosov flows"

Thm (w. Fenley) Every volume preserving collapsed Anosov flow (not suspension) is ergodic.

(in fact, accesible)

What is a "collapsed Anosov flow" (cf. introduced in a paper joint with Barthelmé & Fenley) ← Rmk: Open & closed class + contains all known examples in dim 3

Def: $f: M \to M$ partially hyperbolic diffeomorphism is a collapsed Anosov flow if ↳ ($\dim E^c = 1$)

$\exists\, \phi_t : M \to M$ (topological) Anosov flow

$\exists\, \beta : M \to M$ self orbit equivalence of ϕ_t

$\exists\, h : M \to M$ continuous $\sim id$ s.t.

1) $f \circ h = h \circ \beta$

2) h maps orbit of ϕ_t to curves tangent to E^c.

ϕ_t is an Anosov flow if $\phi_1 : M \to M$ is a partially hyperbolic diffeomorphism and $E^c = \frac{\partial}{\partial t} \phi_t \Big|_{t=0}$.

$\beta : M \to M$ is a self orbit equivalence if it is a homeomorphism mapping orbits of ϕ_t to orbits of ϕ_t; more precisely, for every $x \in M$, $\exists\, s_x : \mathbb{R} \to \mathbb{R}$ increasing homeo such that

$$\beta(\phi_t(x)) = \phi_{s_x(t)}(\beta(x))$$

Some ideas:

If $f: M \to M$ is not accesible in $\dim M = 3$ it means that there are some places where $E^s \oplus E^u$ is tangent to a surface.

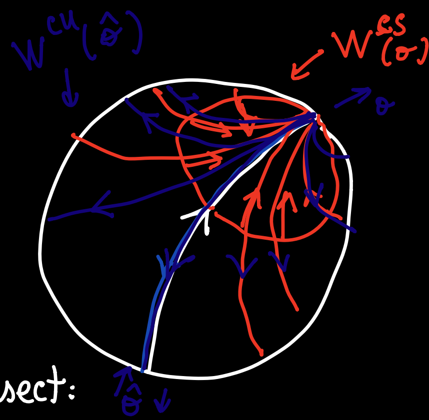
In fact: (HHU) if f not accesible then $\exists \Lambda \subseteq M$ lamination by surfaces tangent to $E^s \oplus E^u$.
When f is volume preserving, complementary regions of Λ can be completed to a foliation with nice propeties

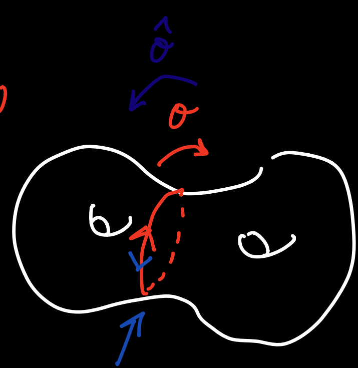
↳ to simplify keeping idea we will assume that Λ is in fact a foliation $\mathcal{F}^{su}$ tangent to $E^s \oplus E^u$.

Key point: If $\phi_t: M \to M$ is an Anosov flow and not a suspension, then, $\exists$ periodic orbits "flowing in different" direction.

freely homotopic.

E.g: Geodesic flow





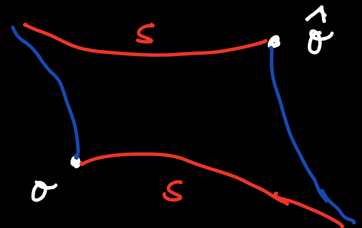
Stable mfd of one and unstable manifold of other "almost" intersect:

↳ Key word: LOZENGE

In general, when lifted to universal cover $\tilde{M}$, any Anosov flow ϕ_t, lifts to $\tilde{\phi}_t$ and verifies that the ORBIT SPACE (i.e. $\tilde{M}/_\sim$ where $x \sim y$ if $\exists t / \tilde{\phi}_t(x) = y$) is a topological plane (i.e. homeomorphic to $\mathbb{R}^2$)

bifoliated ($\tilde{W}^{cs}$ and $\tilde{W}^{cu}$ induce 1D-foliations)

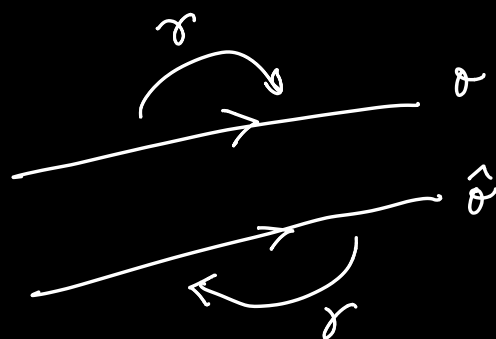
LOZENGE



Both o and $\hat{o}$ fixed by deck transf γ

We call: o & $\hat{o}$ the "corners of the lozenge"

but



Main technical point in our proof: Let $f: M \to M$ be a collapsed anosov flow for ϕ_t, β and h. Assume moreover that f is not accessible (and $\mathcal{F}^{su}$ is foliation tangent to $E^s \oplus E^u$)

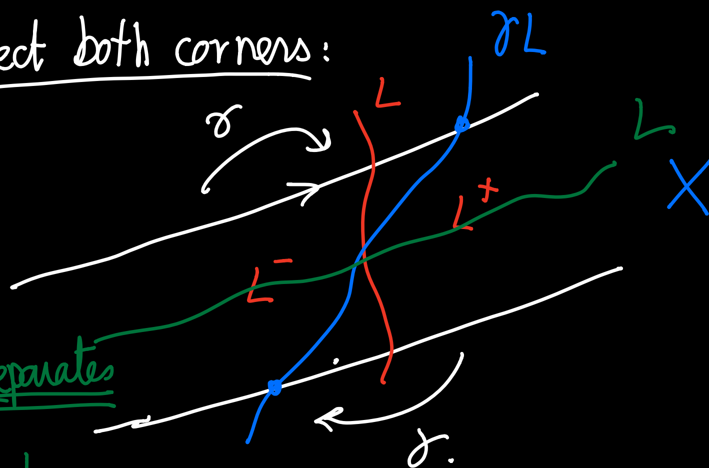
Proposition For every lozenge of ϕ_t fixed by deck $\gamma \in \pi_1(M)$ $\exists$ γ-invariant leaf $L \in \tilde{\mathcal{F}}^{su}$ which intersects h(interior Lozenge) but does not intersect both corners.

This allows to construct with some work a circle tangent to E^s or E^u (a contradiction)

Very sketchy cartoon of proof

1) Cannot intersect both corners:

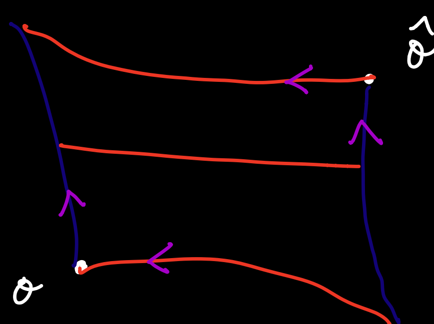
$\boxed{\text{2D cartoon}}$



$\Rightarrow$ there is a leaf L which separates the corners $\Rightarrow$ γ-invariant.

2) Intersects everything: Since h maps orbits of ϕ_t to curves tangent to E^c, L is $\pitchfork$ to h(orbits)

In particular set of orbits which intersect is open.



Idea: show that if α is orbit of ϕ_t s.t. $h(\alpha)$ is in boundary of what L intersects $\Rightarrow$ stable or unstable is there.

Since it is open & connected one can play with these simple dynamics to conclude.