# Partial hyperbolicity & transverse foliations

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$f: M \to M$ is partially hyperbolic if $TM = E^s \oplus E^c \oplus E^u$

$Df$-invariant dominated & $E^s$ contracting, $E^u$ expanding.

Models: → Affine maps on quotients of Lie groups ($\mathbb{T}^3$, nilmanifolds, solvmflds, $T^1S$ ...)

→ Anosov flows & self orbit equivalences ($\varphi_t: M \to M$ anosov flow
$\beta: M \to M$ SOE if it is homeo sending orbits to orbit.

(Note: Bowden-Massoni, all models are realized)

**Theorem** (j.w. S. Fenley & A. Hammerlindl)

Let $f: M \to M$ partially hyperbolic on closed 3-mfd $M$.

Then, $f$ is semiconjugate to a model. (VERSION OF PUJALS CONJECTURE, see Bonatti-Wilkinson)

Rmks: Previously we proved this with Fenley assuming $f$ chain transitive.

- Corollary: every PH volume preserving in $M$ with non solvable fund group is ergodic (HHU's conjecture)
- Self orbit equivalences of Anosov flows as models is motivated by examples with Bonatti, Gogolev, Hammerlindl ... and named "Collapsed Anosov flows" with Barthélme & Fenley (also previous work with Frankel)
- When $\pi_1(M)$ is (virtually) solvable → precise description beyond semiconjugacy (j.w. Hammerlindl), takes into account examples by Hertz, Hertz, Ures.
- No assumption on orientability & extends Burago-Ivanov to get branching foliations in general. Fun consequence: if $\pi_1(M)$ non solvable → $E^c$ bundle is orientable.

The main strategy of the proof consists in reducing it to the understanding of pairs of transverse foliations in 3-mfds.

For simplicity, we will assume $M$ orientable & bundles $E^s, E^c, E^u$ or ($Df$ preserves their orientation) and ignore the work needed to remove this assumption. The following is not historical structure of proof.

**First step** (Burago-Ivanov) There exist $f$-inv branching foliations $\mathcal{F}^{cs}$ & $\mathcal{F}^{cu}$ tangent respectively to $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

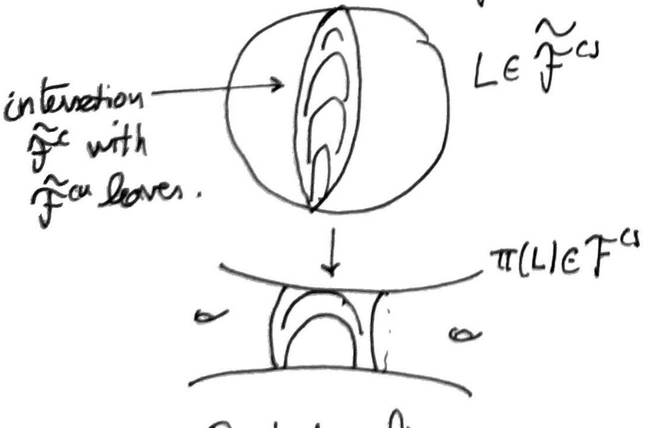
Moreover, these can be "blown up" to (Reebless) foliations $\mathcal{F}^{cs}_\varepsilon$, $\mathcal{F}^{cu}_\varepsilon$ converging (as well as tangent spaces) to $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$.

**Why Reebless**: If, say, $\mathcal{F}^{cs}_\varepsilon$ contains a Reeb component

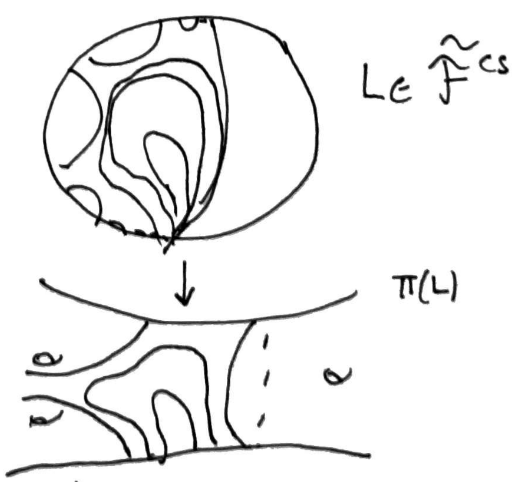
then, the strong unstable foliation $\mathcal{W}^u$ (tg to $E^u$) contains a closed leaf: Contradiction!

**Second Step**: New structures incompatible with PH foliations.

Generalized Reeb surfaces:




Reeb Annuli




Reeb Crown.

These structures also produce closed leaves for $\pitchfork$ 1D foliations, so incompatible with PH by same reason.

**Third Step** Geometry of leaves ~~criteria for Models~~.

**Thm** (Sullivan, Gromov, Candel) "Most" foliations in 3-manifolds are by Gromov hyperbolic leaves. In such case, $\exists$ metric in $M$ (leafwise smoth, & continuous) so that every leaf is of curvature $-1$.

When $f$ is chain transitive it is easy to see that $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$ are by GH leaves. In general, we prove: (THIS IS THE MAIN & MOST DIFFICULT THM.)

**Thm** (j.w. Fenley & Hammerlindl) If $\pi_1(M)$ not solvable and $\mathcal{F}_1 \pitchfork \mathcal{F}_2$ transverse foliations so that $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ does not have generalized Reeb surfaces (without TORI), then ~~leaves of~~ $\mathcal{F}_1, \mathcal{F}_2$ are by GH leaves and leafspace of $\tilde{\mathcal{G}}$ (i.e. $\tilde{M}/\tilde{\mathcal{G}}$) is Hausdorff.

When $\pi_1(M)$ is solvable one can do similar things, but will be ignored in this talk.

**Fourth Step** Criteria for Models

**Thm** (j.w. Fenley) Let $\mathcal{F}_1, \mathcal{F}_2$ be transverse foliations by GH leaves and $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ the intersection foliation. Then, $\tilde{\mathcal{G}}$ has Hsdff leafspace if and only if $\mathcal{G}$ is leafwise quasi-geodesic (unless $M$ has solvable fund group...)

When $\mathcal{F}$ is a foliation by GH leaves & $\mathcal{G}$ a 1D subfoliation we say it is leafwise QG if $\forall \ell \in \tilde{\mathcal{G}} \subseteq L \in \tilde{\mathcal{F}}$ we have that $\exists C > 0$

$$d_\ell(x,y) \leq C\, d_L(x,y) + C. \quad \text{(a posteriori } C \text{ is indep of } \ell\text{)}$$

**Thm** (j.w. Barthelme & Fenley) Let $f: M \circlearrowleft$ be a(n orientable) PH diffeo ~~...~~ so that $\mathcal{F}^{cs}_{\varepsilon}, \mathcal{F}^{cu}_{\varepsilon}$ are by GH leaves and $\mathcal{F}^{c}_{\varepsilon} = \mathcal{F}^{cs}_{\varepsilon} \cap \mathcal{F}^{cu}_{\varepsilon}$ ~~are~~ is leafwise QG $\Longrightarrow$ $f$ is a (strong) collapsed Anosov flow (in part, semiconj to SOE of Anosov flow)

**Rmk**: Recently (j.w. Barthelme, Fenley & Mann) we showed that if $\mathcal{F}_1 \pitchfork \mathcal{F}_2$ by GH leaves and $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ is leafwise QG then $\mathcal{G}$ is orbit foliation of blow up of Anosov flow.

Both results rely on some beautiful argument due to Calegari

**Last step**: Remove orientability assumption (Key: uniqueness of branching foliations).

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**Rest of the talk**: Ideas on the proof of the following:

**Thm** (j.w. Fenley & Hammerlindl) Let $\mathcal{F}_1, \mathcal{F}_2$ be transverse foliations ~~...~~ without toral leaves. Then, if $\widetilde{\mathcal{G}} = \widetilde{\mathcal{F}}_1 \cap \widetilde{\mathcal{F}}_2$ has non Hausdorff leaf space $\Longrightarrow$ it contains a generalized Reeb surface.

**Rmk**: Work in progress (j.w. Fenley, Tsang), examples where Reeb crowns are necessary.

**Corollary**: If $M$ non solvable fund group & $\mathcal{F}_1 \pitchfork \mathcal{F}_2$ transverse foliations without torus leaves $\Longrightarrow$ $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ has a closed leaf.

<u>Key insight</u> (traces back to j.w.w. Barbot & Fenley)

One should separate in two possibilities:

→ <u>injective developing map</u> (IDM): $\forall\, L \in \widetilde{\mathcal{F}}_1,\ E \in \widetilde{\mathcal{F}}_2$ we have $L \cap E$ connected (maybe empty)

→ <u>non injective developing map</u>: $\exists\, L \in \widetilde{\mathcal{F}}_1, E \in \widetilde{\mathcal{F}}_2$ such that $L \cap E$ has at least two connected components.

## IDM: In this case we show:

<u>Theorem</u> Let $L \in \widetilde{\mathcal{F}}_1$ and $\ell \in \widetilde{\mathcal{G}}/_L$ leaf non separated from some leaf $\ell' \in \widetilde{\mathcal{G}}/_L$ then the proyection of the ray of $\ell$ in the non separated side accumulates in a closed leaf of $\mathcal{G}$ which is boundary of generalized Reeb surface in $\mathcal{F}_1$.


The main idea is to use the injective developing map to "push" the behaviour at infinity to nearby leaves and treat this case in a very similar way one could treat a lamination in a compact surface.

<u>Non IDM</u>: Here, we use the following remark we made in a paper with <u>Barbot & Fenley</u>: if $L' \in \widetilde{\mathcal{F}}_1$ and $E' \in \widetilde{\mathcal{F}}_2$ intersect in more than one connected component, then, $\exists\, L \in \widetilde{\mathcal{F}}_1$ & $E \in \widetilde{\mathcal{F}}_2$ which are "<u>doubly non separated</u>" i.e. if $\ell, \ell' \in L \cap E$ then $\ell$ is non separated from $\ell'$ both in $\widetilde{\mathcal{G}}/_L$ and $\widetilde{\mathcal{G}}/_E$.

We show:

Theorem: Let $L \in \tilde{\mathcal{F}}_1, E \in \tilde{\mathcal{F}}_2$ be doubly non separated leaves then their intersection projects to closed curves which bound a Reeb annuli in both the projection of $L$ and $E$.

Main idea: by contradiction, if curves do not project in circles then they accumulate somewhere. Use non separation to "track" returns and show they are all "freely homotopic". This gives a contradiction because non separation forces expanding holonomy in one quadrant.