Transverse foliations and partially hyperbolic diffeomorphisms

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(joint work with Sergio Fenley)

Recently, using properties of foliations and transverse foliations in 3-manifolds, with Sergio Fenley we were able to obtain the following result:

Theorem 0.1. *Let $f : M \to M$ be a (chain-)transitive partially hyperbolic diffeomorphism on a closed 3-manifold with fundamental group of exponential growth. Then, f is a collapsed Anosov flow.*

In a nutshell this means that M admits an Anosov flow φ_t and up to some continuous surjective map $h : M \to M$ the diffeomorphism f behaves as a self-orbit equivalence β of φ_t. That is, there is a homeomorphism $\beta : M \to M$ mapping oriented orbits of φ_t to oriented orbits of φ_t so that $f \circ h = h \circ \beta$. As a consequences of the techniques used in the proof and our result, we deduce that all definitions given in [BFP] of collapsed Anosov flows coincide and we refer the reader to that paper for more properties of such diffeomorphisms. This notion extends the notion of *discretized Anosov flows* appearing in the original Pujals' conjecture (see [BW, BFP, Mar]) to account for new examples (see [BGP, BGHP]). The classification is complete, since every possible (at least if orientable) collapsed Anosov flow can be realised as follows from a recent result of Bowden and Massoni. Manifolds with smaller fundamental group had already been dealt with [HP].

It is to be emphasized that a non-trivial consequence of our result is that a 3-manifold supporting a chain-transitive partially hyperbolic diffeomorphism must support a transitive Anosov flow. We refer the reader to [BW, CHHU, HP, BGP, BFP] for more context on this problem. The assumption of chain-transitivity is not crucial, it is used to ensure that leaves of some foliations appearing in the proof are by Gromov hyperbolic leaves, a generic assumption thanks to Candel's uniformization theorem. In many cases, such as when f is absolutely partially hyperbolic, dynamically coherent, homotopic to the identity, it is not hard to see that the assumption is verified. It also holds unconditionally on hyperbolic and Seifert 3-manifolds [HP_2].

As a consequence of the previous result we also prove a conjecture due to Hertz-Hertz-Ures ([CHHU]), see [FP_2]:

Theorem 0.2. *Let $f : M \to M$ be a conservative C^{1+} partially hyperbolic diffeomorphism in a closed 3-manifold whose fundamental group is not (virtually) solvable. Then, f is ergodic (and in fact a K-system).*

The proof of Theorem 0.1 builds on the existence of branching foliations [BI] and on a strategy first devised in the classification of partially hyperbolic diffeomorphisms of hyperbolic 3-manifolds ([BFFP_1, BFFP_2, FP]) that requires showing that center curves are quasi-geodesics in their corresponding weak-stable and weak-unstable (branching) leaves. While the proof in the hyperbolic manifold case achieves this through a detailed analysis of curves inside the leaves, with a crucial and continued used of the dynamics of f and properties of the strong stable and strong unstable foliations, the new strategy, which allows for a more general result, relies on a different approach initiated in [FP_3, FP_4, BaFP].

In our work, we try to understand the geometry of the flow defined by two transverse foliations and to obtain, assuming that the flowlines are not quasi-geodesic in their corresponding leaves, that there must be some structure incompatible with partial hyperbolicity. Our main result is then a completely general result about transverse foliations in 3-manifolds (related to some questions in [Th]):

Theorem 0.3. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse foliations with Gromov hyperbolic leaves in a closed 3-manifold M. Then, if $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ is the intersected foliation, it follows that either leaves of $\widetilde{\mathcal{G}}$ are quasi-geodesic in their corresponding $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ leaves, or, the foliation $\mathcal{G}$ contains a* generalized Reeb surface*.*

A generalized Reeb surface is a geometric object that can appear in a leaf of $\mathcal{F}_1$ or $\mathcal{F}_2$ which is foliated by $\mathcal{G}$ and has some particular geometric properties that are incompatible with the foliations coming from a partially hyperbolic diffeomorphism. Instead of defining properly the object, we close the summary with a corollary of the previous result:

Corollary 0.4. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse foliations by Gromov hyperbolic leaves and let $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ the intersected foliation. Then, $\mathcal{G}$ contains a closed leaf (a circle).*

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