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GOAL OF THE TALK:

Present some results on which topology & dynamics interact.

orbit of $x \in M$ is $\{f^n(x)\}_{n\in\mathbb{Z}}$

Smooth dynamics: Study of asymptotic behaviour of points by $f: M \to M$ diffeomorphism of (closed) smooth manifold M

$f^n = \underbrace{f \circ \dots \circ f}_{n \text{ times}}$

(some)

Goals/Questions:

Describe "most" orbits of "most" dynamical systems (i.e. of "most" $f: M \to M$)

Several approaches: → make some assumptions on f and deduce consequences.

→ Show that f can be "perturbed" to enjoy some conditions.

Main difficulties:

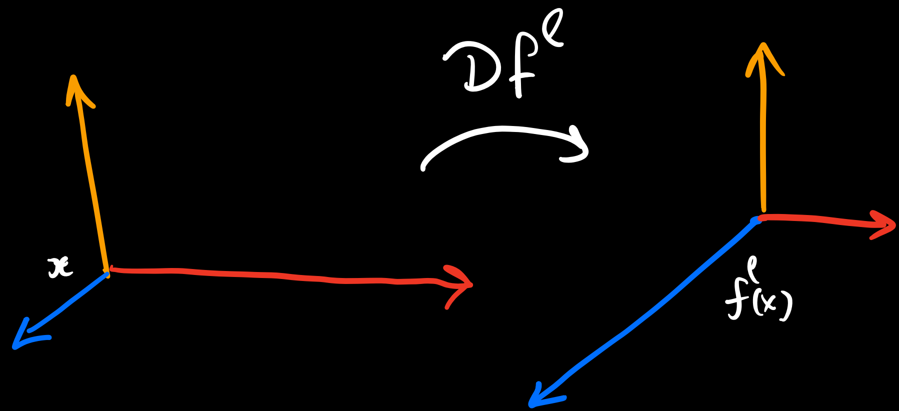
Either it is hard to check assumption for specific systems, or to check that systems belong to these "large sets" which are well described.

I will concentrate in a specific class of dynamics in low dimensions to illustrate some possible formulations of more precise questions.

Def: $f: M \to M$ is **partially hyperbolic** if $\exists$ Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ so that $\exists \ell > 0, \forall v^\sigma \in E^\sigma$ unit vectors $(\sigma = s, c, u)$ we have:

$$\|Df^{\ell} v^s\| < \frac{1}{2} \min\{1, \|Df^{\ell} v^c\|\}$$

$$\|Df^{\ell} v^u\| > 2 \max\{1, \|Df^{\ell} v^c\|\}$$



Why is this a meaningful definition?

* open property, can be detected with finitely many iterates.
* it is implied by several "robust dynamical behaviour"
* Things can be said about such systems.
* Many relevant examples (algebraic, geometric, hybrid...)

From now on, we restrict to low dimensions

So: M = closed 3-manifold

↳ (compact, no boundary)

<u>Why?</u> Smallest dimension where this is really meaningful.

(endomorphisms)

(<u>False</u>: Makes sense in dim $M = 1, 2$, but the kind of question I will address is already well understood there: Franks, Mañé, ...)

<u>In fact</u>: I will further restrict to

hyperbolic 3-manifolds $= \Gamma \backslash \mathbb{H}^3$

(i.e. admit metric of constant negative curvature)

→ "Most closed 3-manifolds" are hyperbolic. ←

(<u>Real reason</u>: Allows me to take some historical shortcuts and state some results which are more in "final" form & show the spirit of the goals we follow)

Theorem A (joint with S. Fenley)

If a hyperbolic 3-manifold admits a partially hyperbolic diffeomorphism $\Longrightarrow$ it admits an Anosov flow.

flow on M whose time 1-map is partially hyp.

Theorem B (joint with S. Fenley)

Let $f: M \to M$ be a volume preserving partially hyperbolic diffeomorphism of a closed hyperbolic 3-manifold $\Longrightarrow$ f is ergodic

(in fact K-system, mixing)

every finite measurable partition has positive metric entropy

$\forall A, B \subseteq M$ measurable

$$\frac{Vol(A \cap f^{-n}(B))}{Vol(M)} \to \frac{Vol(A)Vol(B)}{Vol(M)^2}$$

("asymptotic independence")

every f-inv. measurable set has 0 or full volume.

$\exists$ results in other classes of 3-manifolds of similar taste. The proofs so far depend on geometry/topology of M to some extent.

Rest of the talk: Explain main difficulties and main strategies.

I will concentrate on Theorem A:

$\exists f: M \to M$ partially hyperbolic

$\Rightarrow \exists \phi_t: M \to M$ Anosov flow.

($\Leftarrow$) is obvious!

* Topological obstruction:

Some hyperbolic 3-mfds are known not to admit Anosov flows. (Lots of theory, but still lots of problems)

* $\phi_t: M \to M$ is Anosov flow

means $\exists \; TM = E^s \oplus \underbrace{\mathbb{R}\frac{\partial \phi_t}{\partial t}\Big|_{t=0}}_{=E^c} \oplus E^u$

Classical theory gives E^s, E^u integrate uniquely.

(this is true also for partially hyperbolic)

For Anosov flow $\phi_t : M \to M$ the fact that center direction integrates is immediate; however this may fail for partially hyperbolic systems.

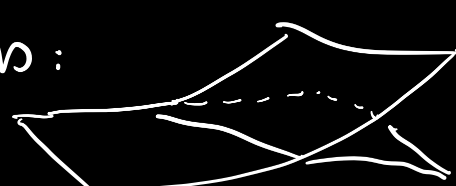
So: → Anosov flows in 3-manifolds naturally carry codimension 1-foliations
(a lot of developed theory)

→ Partially hyperbolic systems may not have f-invariant codimension 1 foliations.

curves tangent to E^c may merge

Weaker structure (Burago-Ivanov)

Branching foliations:
of codimension 1.



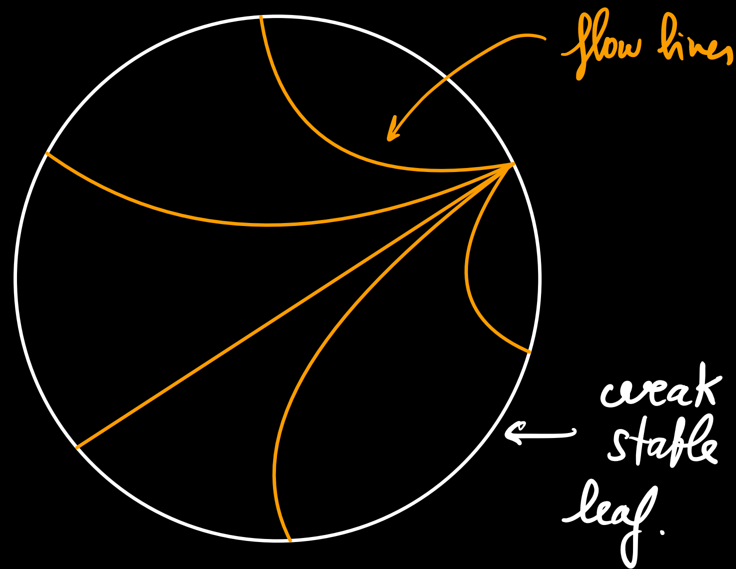
If we lift $\phi_t : M \to M$ Anosov flow to universal cover $\Rightarrow$ all curves tangent to E^c are fixed by $\tilde{\phi}_t$.

This can also fail for $f : M \to M$ partially hyperbolic.

[What is the strategy to build an Anosov flow on M from a partially hyperbolic system?]

Key point: If $\phi_t : M \to M$ is an Anosov flow on 3-mfd M $\Rightarrow$ each leaf of $\mathcal{F}^{ws}$ (weak stable foliation) can be uniformized to hyperbolic plane $\mathbb{H}^2$

and flow lines foliate such leaves as "geodesic fans"



When M is a hyperbolic 3-manifold we are able to produce this structure on leaves of the "branching" foliation tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$.

We call such partially hyperbolic systems quasi-geodesic partially hyperbolic which under some conditions we show are related to another class we call

Collapsed Anosov flows which in particular imply the manifold supports an Anosov flow.

Partial hyperbolicity & 3-manifold topology

Notices of the AMS (2021)