Anosov Flows in dimension 4 (j.w. S. Fenley & K. Mann)

Let X be a C^1-vector field generating flow $\varphi_t : N \to N$ where N is a closed manifold.

φ_t is Anosov if $D\varphi_t$ preserves $TN = E^s \oplus \mathbb{R}X \oplus E^u$ and contracts vectors in E^s / expands vectors in E^u.

(Anosov) Structurally stable & ergodic if preserve volume.

Examples: → Geodesic flows in negative curvature.

→ suspensions of Anosov diffeos.

In 60's Palis & Smale proposed conjectures to understand structurally stable systems (Mañé, Palis, Hayashi-) and in Smale ICM ('66) the problem of classifying Anosov systems was raised.

($\varphi_1 \sim \varphi_2$ if $\exists\, h : N_1 \to N_2$ homeo sending orbits to orbits – ORBIT EQUIVALENCE)

— by S.S. at most countably many classes in given mfd. —

This problem is wide open even for suspensions (Katok – Five more resistent problems in dynamics)

However, in codimension one:

[(Franks-Newhouse '69) $f : M \to M$ Anosov diffeo in codim 1 is conjugate to linear automorphism of $\mathbb{T}^d$.

Verjovsky (1970) Studied codimension one Anosov flows showing transitivity when $\dim N \geq 4$ and conjectured: Codim 1-Anosov flows in $\dim M \geq 4$ are orbit equivalent to suspensions.

[Mention: dim 3 different – surgeries / Recently BBGRH surgeries in higher dim, but $\dim E^s = \dim E^u$]

Thm (j.w. Fenley & Mann) Let M be a hyperbolic 3-manifold ($M = \mathbb{H}^3/\Gamma$) admitting a quasi-geodesic flow Ψ (e.g. if M is a surface bundle over S^1)
Then, $\exists N_\Psi$, a circle bundle over M (depending on Ψ) which admits a smooth (codim 1) Anosov flow.

Moreover, if M admits an Anosov flow, then Ψ can be chosen so that $N_\Psi \simeq M \times S^1$ ($\Rightarrow N$ admits ∞'ly many distinct AF up to OE)

Rmks: • The flows are C^∞, but weak-stable (codim 1) foliations are less regular. (c.f. work of Ghys, Simic, Asaoka...)
- Every hyp. 3-mfd admits finite cover with a QG flow (Agol)
- $\exists$ hyp 3-mfds with Anosov flows (Goodman, ...)
- N is S^1-bundle over $M \Rightarrow \pi_1(N)$ not solvable ($\Rightarrow$ no suspension)
- Higher dimensional examples unknown because of use of Cannon-Thurston maps.

§ CANNON-THURSTON MAPS

M closed hyperbolic (enough $K<0$) mfd of dimension $d \geq 2$
$\Gamma = \pi_1(M)$ acts by isometries of $\tilde{M} (\simeq \mathbb{H}^d)$ and action extends to $\partial \tilde{M} \simeq S^{d-1}$. minimal action

Def: A Cannon-Thurston map is a pair (f, ρ) where:
- $\rho : \pi_1(M) \to \mathrm{Homeo}(S^1)$ is a minimal action.
- $f : S^1 \to \partial \tilde{M}$ equivariant map: $(\gamma f(p) = f(\rho(\gamma) p))$

Examples: • If $M = \mathbb{H}^2/\Gamma$ hyperbolic surface $\to \partial \mathbb{H}^2 \simeq S^1$ and identity is a CT map.
- If $M = S \times [0,1]/\varphi \xrightarrow{PA} \exists \xi : \partial \tilde{S} \to \partial \mathbb{H}^3$, $\partial \tilde{S} \simeq S^1$, constructed by CT.
- In general, for $M = \mathbb{H}^3/\Gamma$ with QG-flow $\Psi_t : M \circlearrowleft$, a pair exists (FRANKEL, FENLEY..)

§ CONSTRUCTION:

Let (f,ρ) be a Cannon-Thurston map.

Define $N_\rho = \tilde{M} \times S^1 / \Gamma$ where $\gamma \cdot (x,p) = (\gamma . x, \rho(\gamma)(p))$

So N_ρ is a $(d+1)$-dimensional manifold, a circle bundle over M.

There is a d-dimensional "horizontal" foliation $\tilde{\mathcal{F}}_h = \{\tilde{M} \times \{p\}\}_{p \in S^1} \subseteq \hat{N}_\rho \simeq \tilde{M} \times S^1$ descending to a foliation $\mathcal{F}_h \pitchfork$ to fibers.

GEODESIC FAN: Fix $L \simeq \tilde{M}$ (isometric) and $\xi \in \partial L \simeq \partial \tilde{M}$ define the (oriented) 1D-foliation $\mathcal{G}_\xi$ in L as the foliation by geodesics in L with one endpoint in ξ and leaves pointing there:

Leaves of $\mathcal{G}_\xi \simeq \partial L \setminus \{\xi\}$.

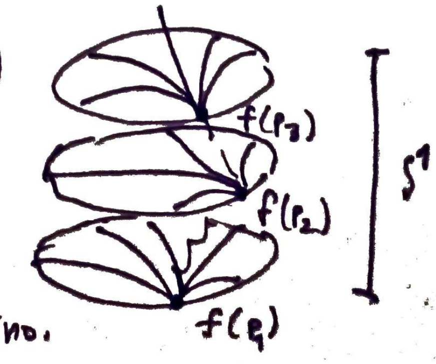
Consider now the 1D foliation (subfoliating $\tilde{\mathcal{F}}_h$) of $\hat{N}_\rho = \tilde{M} \times S^1$ given by $\mathcal{G}|_{\tilde{M} \times \{p\}} = \mathcal{G}_{f(p)}$.

This is a continuous foliation which descends to N_ρ (leafwise smooth, globally continuous)

Good idea to follow construction in case M hyp. surface (one gets geodesic flow in T^1M)

In general, one thinks as $\mathcal{G}$ inducing a flow in N_ρ which is "blow up" of geodesic flow on T^1M (changing one sphere in Hopf coordinates by a circle using the map $f: S^1 \to \partial \tilde{M}$).

Prop: The flow induced by $\mathcal{G}$ in N_ρ is expansive and preserves a (topological) foliation $\mathcal{F}_v \pitchfork \mathcal{F}_h$ on which orbits converge in past & expand in future.



⟵ Calegari / Pujals-Sambarino.

§ SMOOTHING STEPS

① Choose a smooth structure on N_ρ (Tholozan's trick $\leftrightarrow$ Related to Cawley's work)

② Estimate holonomy $\pitchfork$ to $\mathcal{F}_h$ of the $\mathcal{G}$ foliation.

③ Make leafwise deformation to produce a C^1-Anosov vector field. (Asaoka's trick)

To put smooth structure on N_ρ we will consider a smooth structure on S^1 such that ρ acts by diffeomorphisms. The choice will simultaneously handle steps ① & ②.

Idea: $f: S^1 \to \partial\tilde{M}$ and $\Gamma = \pi_1(M)$ acts on $\partial\tilde{M}$ preserving natural measure (Patterson-Sullivan)

if $\tilde{M} \simeq \mathbb{H}^d$, get Lebegue measure.

Consider μ measure on S^1 given by $f_* \mu = Leb$

(need to either assume f is 1-1 in full Leb measure subset of $\partial\tilde{M}$, or use that because flow is expansive $\Rightarrow$ f is uniformly finite to 1.)

Because ρ is minimal, then μ is atom free and full support $\Rightarrow$ gives a smooth structure on S^1, and using f it is easy to compute derivative of $\rho(\gamma)$ by

$$\rho(\gamma)'(\xi) = \frac{d(\gamma_* \mu)}{d\mu}(\xi) = \underbrace{\frac{d\,\gamma_* Leb}{d\,Leb}(\xi)}_{\text{Known}}$$

[OBTAIN: ρ is by C^1 (no more than C^1) diffeos in this structure + holonomy along $\mathcal{G}$ leaves expands uniformly $\pitchfork$ to $\mathcal{F}_h$]

Finally, to do ③ we produce $X_k \to X_0$ such that:

- 1) $X_k \to X_0$ in C^0
- 2) X_k is C^1 and tg to $\mathcal{F}_h$
- 3) $X_k \to X_0$ leafwise C^1.

(vect field tg to $\mathcal{G}$)

This completes the proof with some conefield computations.

$\blacksquare$