Charla IHP SH property for minimal unstable laminations and applications to uniqueness of u-Gibbs

M closed 3-manifold.

$f: M \to M$ is a partially hyperbolic diffeomorphism if $TM = E^s \oplus E^c \oplus E^u$ is a Df-invariant splitting such that $\exists\, \ell > 0$ so that $\forall x$ if $v^\sigma \in E^\sigma_{(x)}$ unit vectors ($\sigma = s, c, u$)

then
$$2\|Df^\ell v^s\| \leq \min\{1, \|Df^\ell v^c\|\} \leq \max\{1, \|Df^\ell v^c\|\} \leq \frac{1}{2}\|Df^\ell v^u\|$$

~~Consequences~~ [Fact: E^s and E^u integrate into f-invariant foliations W^s, W^u]

This has some analogy with unipotent groups, thus, it makes sense to classify: $\to$ $\overline{W^{s/u}(x)}$ for general $x \in M$.

$\to$ invariant measures. Rmk: No a priori natural parametrization of W^s, W^u $\to$ transverse inv. measures. $\to$ Gibbs-u-states.

[μ which is f-invariant is a Gibbs-u-state if when desintegrated along W^u-leaves is riemannian volume. (abs. cont.)]

– (eq. in normal form coordinates is Lebesgue.) –

[**Theorem** (j.w. Avila, Crovisier, Eskin, Wilkinson, Zhang)
There is an open and dense set $\mathcal{U} \subseteq PH^1_{vol}(M)$ so that for every $f \in \mathcal{U} \cap C^{1+\alpha}$ both W^u and W^s are minimal. Moreover, for every $f \in \mathcal{U} \cap C^\infty$ volume is the unique Gibbs-u-state (also Gibbs-s-state)]

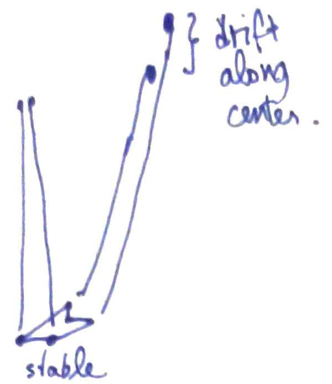
Consequences in understanding of statistics of such systems.
(Dolgopyat, ...). This is combination of several works in progress between the authors.

I will present part of the topological proof, involving the concept of SH for unstable laminations introduced by Pujals & Sambarino.

General strategy:

- Ratner: polinomial drift (many times where drift is attained)
- Benoist-Quint, Eskin-Mirzhakani: exponential drift (only one time, so harder to choose the points)

Idea:



So, important:

→ some kind of transversality.

→ some expansion on center.

Transversality:

→ some cases it is "automatic" (Anosov difeos in $\mathbb{T}^3$ not jointly integrable)

→ others, it is a generic property to be non jointly integrable (harder to get than accesibility, but still abundant. C^1 - Crov.-P.-Sambarino. C^r - likely to work in dim 3 using ideas of EPZ... + ACW..)

Expansion:

→ Katz, BEFRH ask for positive center exponents.

→ related notion of uniform expansion (ELindenstrauss..)

Here, for the topological version (developed by ACW) we use the notion of SH introduced by Pujals-Sambarino.

Def: Let $f \in PH(M)$ and let Λ be an f-minimal ~~set~~ u-saturated set (i.e. Λ is compact f-inv, non empty and W^u-saturated and minimal with respect to that) we say that Λ has the SH property if $\exists N>0, \varepsilon>0$ so that for every unstable arc $I \subset \Lambda$ of length $>\varepsilon$ $\exists I' \subseteq I$ subarc so that:

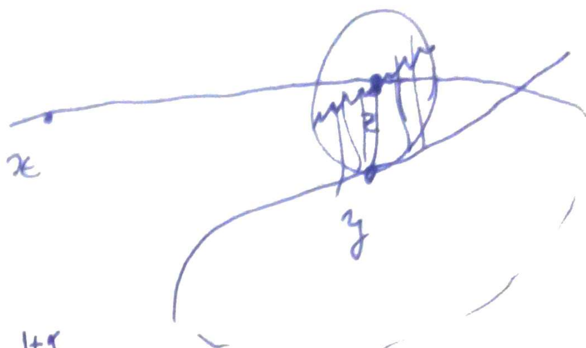
1) $f^N(I')$ has length $>\varepsilon$

2) $\forall x \in I'$ and $v \in E^c(fx) \setminus \{0\}$ one has $\|D_x f^N v\| > 2\|v\|$.

- Also, I will give a precise definition for transversality. – (Abdenur, Crovisier, Pujals, …)

Def: ~~…~~ A ph. diffeo $f: M \to M$ is <u>s-transversal</u> if $\exists L > 0$ so that $\forall x \in M$ there is points $y, z \in W^u_L(x)$ so that $z \in W^s_{loc}(y)$ and such that $\Pi^{ss}_{y,z}(W^u_{loc}(y))$ intersects $W^u_{loc}(z)$ transversally in a cu-disc around z

Can restrict to laminations.
If all minimal laminations are transverse $\Rightarrow$ transverse



<u>Theorem</u> (ACW) Let f be a volume preserving $C^{1+\alpha}$ PH diffeo in M and Λ be a minimal W^u-saturated set which is s-transverse and verifies SH property $\Longrightarrow$ ~~…~~ W^u is minimal.

What I want to explain is some ideas on why:

<u>Theorem</u> (j.w. Crovisier) There is an open and dense set $\mathcal{U}$ of $PH^1_{vop}(M)$ so that ~~…~~ if $f \in \mathcal{U}$ either f is Anosov with E^c contracting uniformly <u>or</u> every minimal u-saturated set is $\pitchfork$ and has SH property.

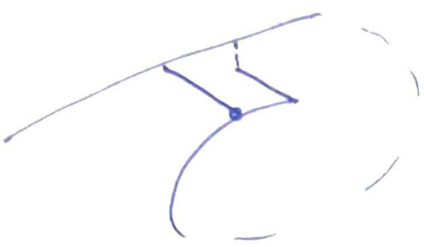
- Combined with ACW gives minimality. –
- Combined with EPZ (ref Rosa's talk) + Katz gives uniqueness of u-Gibbs states –

<u>Note</u>: • In Anosov case, we get minimality without pert.
• Volume preserving can be removed and there is dichotomy, since one can have proper attractors.

Some ideas on the proof:

· We will consider the minimal u-saturated sets for f.

After a small perturbation, we can assume non of them is jointly integrable



and so, each such minimal set occupies some space $\Rightarrow$ there are finitely many

(previous work j.w. Crovisier, Sambarino)

We want to study each minimal set. We can assume that E^c is not uniformly contracting (in that case, f is Anosov and W^u is the full unstable $\Rightarrow$ minimal)

GOAL: Show that each minimal set intersects a hyperbolic set with E^c expanding. (Such sets $\exists$ because f is not Anosov).

We discuss depending on the transverse geometry of the minimal sets.

Case 1: Take a cs-disk $D \pitchfork$ to Λ and assume that $D \cap \Lambda$ has a non trivial connected set.

In this case we can argue as follows,

(a) if $\mathscr{C} \subseteq D \cap \Lambda$ connected is contained in $W^s_{loc}(x)$, then pushing to nearby $\pitchfork$ disc by unstable holonomy, this is no longer the case. Then we can assume [$\mathscr{C}$ not contained in local stable]

(b) if f acting in $\mathscr{C}$ contracts along the center we can find a periodic unstable in Λ.

Without loss of generality we can work with ~~f~~ f being C^1-generic
(our conclusion will be open in $\mathrm{Diff}^1(M)$)
and by standard arguments (connecting lemmas, semicontinuity, ...)

Thm: Let f be a C^1-generic difeo in $PH^1_{vol}(M)$ and Λ a minimal u-saturated set $\rightarrow$ if Λ has a periodic point $\Rightarrow$ $\Lambda = M$.

if $\Lambda = M$ we are done (we get both SH and $\pitchfork$ since Λ intersects a blender with E^c expanding and not JI $\Rightarrow$ $\pitchfork$ in minimal case).

(c) Now we can assume some expansion along E^c (topological) enough to iterate until $\mathcal{C}$ "tilts" to a center curve

$\Rightarrow$ Λ contains a cu-disk.

Since f is volume preserving, this implies $\Lambda = M$. and we are done again.

Case 2 $\forall$ disc $D \pitchfork$ to Λ the intersection is totally disconnected.

In this case we use the fact that we can cover Λ by transversals not intersecting the boundaries to produce a "section" for the dynamics:

Prop: $\exists K$ compact invariant set $\subseteq \Lambda$ such that
1) for each $x \in K$ we have $W^{uu}(x) \cap K = \{x\}$
2) $\forall y \in \Lambda$ we have that $W^{uu}(y) \cap W^s_{loc}(K) \neq \{0\}$.

The first condition implies (Bonatti-Crovisier) that $K \subseteq$ locally inv surface tangent to $E^s \oplus E^c$ at K

And this allows to apply a result by Pujals-Sambarino to deduce that either K is a sink ($\Rightarrow \Lambda$ attractor) (again we use f is C^1-generic)

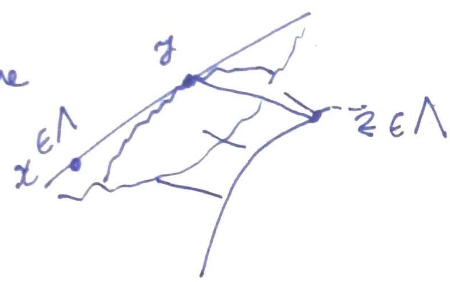
or K verifies E^c is uniformly expanding in K.

This allows to obtain the SH property.

To get robustness we need to show the transversality. (this is also important to get continuity of minimal sets, but the argument is independent)

We need to exclude that:

each time



projection is on one side.

Note that by entropy argument, many strong stables intersect Λ in many points. This, assuming non-transversality allows to "order" unstable leaves of Λ and get "boundary leaves".

This ordering helps produce a periodic unstable leaf which we said before was not possible.

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