Seminar Chicago PH & Transverse foliations in dim 3

(j.w. S. Fenley)

M closed 3-manifold

$f : M \to M$ a partially hyperbolic diffeo : $TM = E^s \oplus E^c \oplus E^u$ Df-invariant

(contr. → E^s, in between → E^c, expanded → E^u)

FACTS: . E^s, E^u integrate uniquely into 1D foliations by lines.
. Are always Hölder cont., but typically not better. E^c can fail to be integrable.

GOAL: Describe the topology of these systems, and search for dynamical applications.

Corollary (of our work) Let $f : M \to M$ be a volume preserving C^{1+} PH diffeo on M with non solvable fund. group. Then f is ergodic.

(Conjectured by Hertz-Hertz-Ures)
(generic-open dense already known HHU, Burns-Wilkinson)

To describe the topology first mention examples.

Examples: → Algebraic (afine in Homogeneous mfds)
- → linear tori affine maps (3 ≠ real eigenv)
- → automorph. nilmfds
- → translations in SOL
- → diagonal in $\widetilde{SL_2}/\Gamma$

} Anosov flows

→ Anosov flows : Rich theory in dim 3.

Let $\varphi_t : M \to M$ be Anosov flow and $\tau : M \to \mathbb{R}_{>0}$ smooth so that $f(x) = \varphi_{\tau(x)}(x)$ is diffeo $\Rightarrow$ f is PH. (exercise)

→ discretized Anosov flows.

→ ~~Cone~~ Cone field criteria (local to global principle) $\Rightarrow$ open property so deformations...

Conjecture (Pujals '01, Bonatti-Wilkinson '05) Up to finite cover & iterate every transitive PH diffeo belongs to one of these classes.

(Need to precise what deformation means: HPS leaf conjugacy is one possibility)

Remarks:

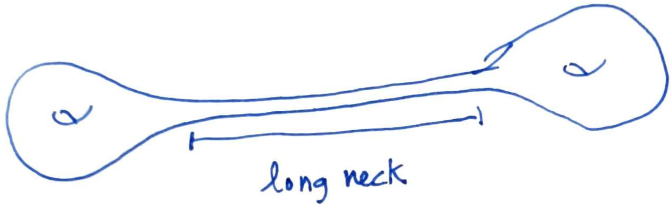
→ Brin, Burago, Ivanov, Parwani, Hammerlindl, P...

The conjecture is true if $\pi_1(M)$ is (virtually) solvable.

→ Bonatti, Gogolev, P. : False if $M = T^1S$. (and others)

Thm (BG, Hammerlindl, P.) $\forall\, \rho \in MCG(S)$ $\exists f: T^1S \circlearrowleft$ (ergodic) PH diffeo so that f induces ρ in the base.

idea: take



So Dehn twist "→" id in C^1-topology (even if "large in C^0")

so $D\rho_{\pi} \circ \varphi_T$ with large T is PH ... ▨

→ True in Seifert if $f \sim id$ or in Hyperbolic 3-mfds if $\exists$ center foliation (Barthelme, Fenley, Frankel, P.)

Motivated the following definition:

easy example BW: Take $\varphi_t: M \to M$ AF and lift to $\hat{M} \to M$ finite cov $\Rightarrow \eta \circ \hat{\varphi}_t$ where η is deck is CAF

Def: (Barthelme, Fenley, P.) $f: M \to M$ partially hyperbolic diffeo is a collapsed Anosov flow if $\exists \{\Phi^t\}_t$ Anosov flow in M and $\beta: M \to M$ a self orbit equivalence of Φ (homeo of M permuting the orbits of Φ) so that f is semiconjugate to β. ($\exists\, h: M \to M$ continuous $\sim id$ s.t. $f \circ h = h \circ \beta$.) (Keep in mind: • If f is CAF $\Rightarrow$ M admits AF) • h maps ~~center curves~~ orbits of Φ to center curves)

Question: Is every transitive ph on closed 3-mfd whose fundamental group has exponential growth a collapsed Anosov flow?

(Note: every self orbit equivalence class may be realized: Bowden-Mann)

Thm (j.w. Fenley) Yes! (Rmk: no need for finite lift or i

Note: One can study dynamics of collapsed Anosov flows independently, and for instance this allows to deduce the ergodicity claim.

In the rest of the talk I want to explain how the result "reduces" to a statement about pairs of foliations in 3-manifolds which somewhat came as a surprise to us.

FIRST STEP: Burago-Ivanov: Under orientability assumptions they constructed branching foliations W^{cs} and W^{cu} (f-inv) tangent to $E^{cs} = E^s \oplus E^c$, $E^{cu} = E^c \oplus E^u$ resp.

These branching foliations are collections of immersed planes which in $\tilde{M}$ (lift to properly embedded in) and cannot traverse one another (but may merge).

Also showed that one can "blow up" to true foliations $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ "almost" invariant and "almost" tangent to the bundles.

Application: First topological obstructions for PH dynamics (e.g. S^3 does not support). Essentially they show that these branching foliations are "Reebless".

SECOND STEP: Assume now that $\mathcal{F}^{cs}_\varepsilon$ and $\mathcal{F}^{cu}_\varepsilon$ are by Gromov hyp. leaves: i.e. $\forall L \in \tilde{\mathcal{F}}^\sigma_\varepsilon$ in $\tilde{M}$ the induced metric is coarsely negatively curved (Candel's thm, Danny's book). Upshot: This always holds if $\pi_1(M)$ exponential growth & f is (chain)-transitive.

Thm (Calegari) If $\mathcal{F}^{cs}_{\varepsilon}$ and $\mathcal{F}^{cu}_{\varepsilon}$ are <u>minimal</u> (every ") and they intersect in a $\mathcal{G}^c$ 1D foliation which is "<u>leafwise quasi-geodesic</u>", then, $\mathcal{G}^c$ is an (topologically) Anosov flow.

↳ in $\tilde{M}$ each $l \in \tilde{\mathcal{G}}$ is a QG in its corresponding $\tilde{\mathcal{F}}^{cs}_{\varepsilon}$, $\tilde{\mathcal{F}}^{cu}_{\varepsilon}$ leaf.

We use this to show: (we also remove "minimal")

Thm (j.w. Barthelme & Fenley) If $\mathcal{G}^c$ is leafwise quasi-geodesic $\Rightarrow$ f is a collapsed Anosov flow.

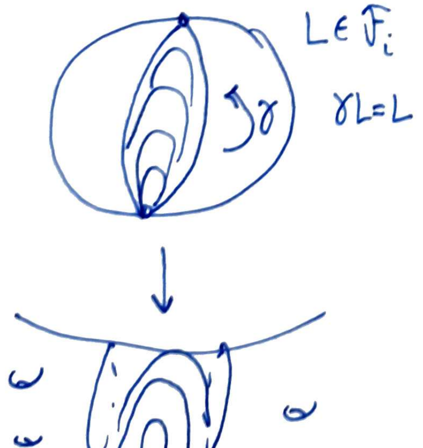
THIRD STEP: Transverse pairs of foliations.

Theorem (j.w. Fenley) Let $\mathcal{F}_1, \mathcal{F}_2$ be two $\pitchfork$ foliations by GH leaves in a closed 3-manifold M and let $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$.

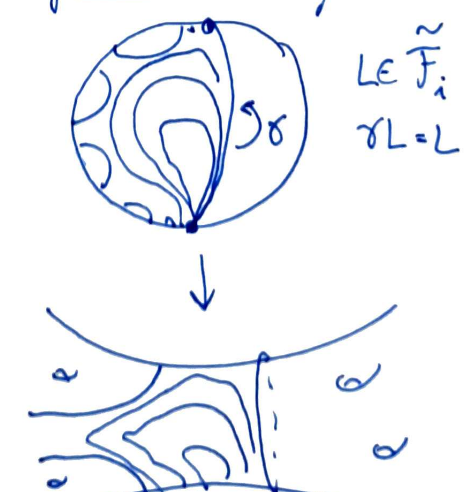
<u>Then</u>: 1) either $\mathcal{G}$ is leafwise QG, or,

2) $\mathcal{G}$ contains generalized Reeb surfaces in either $\mathcal{F}_1$ or $\mathcal{F}_2$ leaves.

<u>Reeb surface</u>:



<u>generalized Reeb surface</u>:



FOURT STEP: UNIQUENESS STATEMENTS (to deal with finite covers & iterates)

Cartoon of the proof:

(Levitt, ... 70's 80?) Let S be a closed surface (genus $g \geq 2$) and let Λ be a closed lamination on S: then, either Λ is by quasi-geodesics, or, there is a pair of closed leaves of Λ so that in $\tilde{\Lambda}$ non separated (like Reeb annulus)

If time: Explain this. Else continue.

In 3D separate in 2 cases: $\rightarrow$ injective developing map ($\forall L \in \tilde{\mathcal{F}}_1$ & $E \in \tilde{\mathcal{F}}_2$, $L \cap E$ connected)

$\rightarrow$ non inj: ($\exists L \in \tilde{\mathcal{F}}_1$ & $E \in \tilde{\mathcal{F}}_2$, $L \cap E$ has 2 c.c.)

First case: Show that one can "push" $\mathcal{G}$ to nearby leaves (else, get some splitting) and try some strategy.

SECOND CASE: Show this produces Reeb surfaces.