

Geometry of leaves

[GH 1]

Let F be a foliation on a closed 3-dim'l manifold M

A holonomy invariant measure is the given of Radon measures $\mu = \{\mu_x\}_{x \in M}$ in a family of transversals $\{Z_x\}_{x \in M}$ so that for $x, y \in L \in F$

and $\gamma: [0,1] \rightarrow L$ $\gamma(0)=x, \gamma(1)=y$ we have that $(H_\gamma)_* (\mu_x|_{U_x}) = \mu_y|_{H_\gamma(U_x)}$

where $H_\gamma: U_x \subseteq Z_x \rightarrow Z_y$ is the holonomy.

Given a holonomy invariant measure μ one can define its Euler characteristic as $\chi(\mu) = \int K d\text{leaf } d\mu \leq 0$ if F does not have spherical or projective plane leaves (very restrictive by Reeb stability)

(Plante) If F has some leaf with polynomial growth of volume $\Rightarrow$ it has a holonomy invariant measure. But many foliations do not even have one.

Theorem (Candel) Assume that $\forall \mu$ transverse inv. measure one has that $\chi(\mu) < 0$. Then, $\exists$ metric on leaves of F (leafwise smooth, globally continuous) such that every leaf has constant negative curvature $= -1$

Idea: Define the uniformization map (given the conformal structure induced by the Riemannian metric) and show that lack of continuity forces "explosion" which forces μ with $\chi(\mu) = 0$.

(one semicontinuity is easier and established by Ghys before, the other argument is more delicate, but actually works).

Rmk: If F is a foliation with transverse invariant measure with $\chi(\mu) = 0$ then one can "approximate" leaves by tori (either $\exists$ toral leaf, or $\exists$ sequence of incompressible tori approaching the support of μ).

Some foliations do not even have $\hbar$ invariant measures.

[GH2]

(First: Comment on regularity; given $\mathcal{F}$ continuous foliation, $\exists$ smooth structure on M such that leaves of $\mathcal{F}$ become C^∞ and C^∞ -jets vary continuously _{to leaves} with the point. (Calegari))

[Proposition: Let $\mathcal{F}$ be the weak stable foliation of a (topological) Anosov flow, then $\mathcal{F}$ does not have $\hbar$ invariant measures.]

Proof: Fix a recurrent point for ϕ_t the flow, then the measure must be atomic, but there are no compact leaves. $\square$

Something more delicate is the following:

[Thm (Fenley) Let $\phi_t: M \rightarrow M$ be an Anosov flow ^(smooth) on a closed 3-mfd then leaves of $\mathcal{F}^{ws}$ are Gromov hyperbolic and orbits of $\tilde{\phi}_t$ in $\tilde{M}$ are quasi-geodesics in their corresponding leaves of $\tilde{\mathcal{F}}^{ws}$.]

Rmk: This was extended by Butler with a similar proof (though cannot use Candel) to higher dimensions. And recently, with Sergio & Katie we proved this in higher dimensions for topological Anosov flows with a more elementary proof. (Hopefully I can briefly explain later).

I want first to describe a result we produced with Barthelmie & Sergio that gives a sort of "normal" form to topological Anosov flows.

Def: Let $L \cong \mathbb{H}^2$ a geodesic fan G_L on L is a foliation by geodesics on L such that $\exists \xi \in S_\infty^1(L)$ (funnel point) such that $\forall \eta \neq \xi$ there is a leaf of G_L from η to ξ .



Theorem (BFP) Let $\varphi_t: M \rightarrow M$ be a topological Anosov flow

GH 3

in closed 3-manifold. Then, φ_t is orbit equivalent to another TAF $\psi_t: M \rightarrow M$ such that $\mathcal{F}^{ws}$ is leafwise smooth with constant negative curvature leaves and $\tilde{\psi}_t$ orbits on $L \in \tilde{\mathcal{F}}^{ws}$ is unit speed flow on a geodesic fan foliation on $L \cong \mathbb{H}^2$.

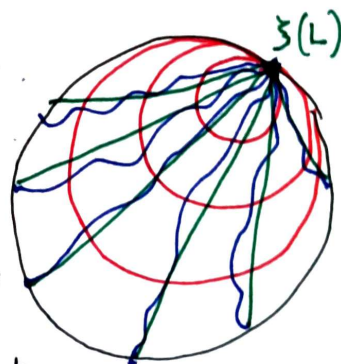
Proof: Use Calegari to get leafwise smooth leaves and Candel to get constant curvature leaves. Now, by (FMP) it is $\tilde{\varphi}_t$ leafwise quasi-geodesic, with common endpoint

Consider $\tilde{\psi}_t$ to be geodesic fan with same endpoint, it is well defined (unit speed)

and $\pi_1(M)$ -equivariant $\Rightarrow$ descends to M .

To produce an orbit equivalence we do as follows:

Each $x \in \tilde{M}$ is first mapped to $H_0(x)$ the closest point in the corresponding geodesic (same leaf of $\tilde{\mathcal{F}}^{ws}$ same endpoints as $\{\tilde{\varphi}_t(x)\}$) — if not unique, take barycenter.



$H_0(x)$ may fail to be continuous, but it is $\pi_1(M)$ -equivariant. & bd dist from id.

It induces continuous bijection on orbits, but maybe not monotonic w.r. to orbits.

What we do is integrate large pieces in order to make it continuous & monotonic in leaves — if integration is done equivariant, obtain homeo. □

In the next, I will try to explain briefly how to get that leaves are Gromov hyperbolic, and orbits quasi-geodesics.

This will be based on work in progress with Sergio and Katie.

(mention no need for dim 3 & Butler's work)

The key is to use a criterion due to Masur-Minsky which is very convenient if you already have a good guess on which are the (quasi)-geodesics in your space: GH4

Theorem (Masur-Minsky) Let X be a geodesic metric space ^{so that} ~~and~~ there are constants $a, b, c, d > 0$ and a family of paths $\{\gamma_\alpha\}_{\alpha \in \mathcal{A}}$ such that:

(1) $\forall \gamma_\alpha: I \rightarrow X$ in the family, it has the (a, b, c) -contracting property: i.e. $\exists \pi: X \rightarrow I$ s.t.:

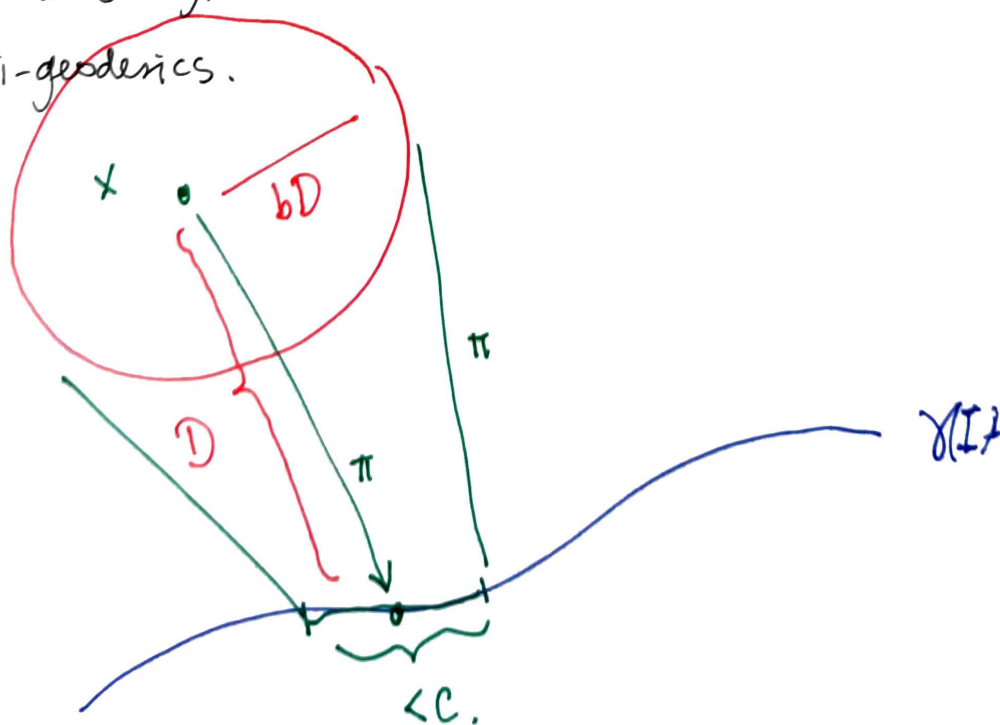
(i) $\forall t \in I \quad \text{diam}(\gamma[t, \pi(\gamma_+(t))]) < c$

(ii) $\forall x, y \in X \quad d(x, y) > 1 \Rightarrow \text{diam}(\gamma[\pi(x), \pi(y)]) < c$

(iii) If $d(x, \gamma(\pi(x))) \geq a$ & $d(x, y) \leq b d(x, \gamma(\pi(x)))$
 $\Rightarrow \text{diam}(\gamma[\pi(x), \pi(y)]) < c.$

(2) The family is d -coarsely transitive (i.e.) $\forall x, y / d(x, y) > d$
 $\exists \gamma$ in collection joining x and y .

Then, X is δ -Hyperbolic and the curves in the collection are uniform quasi-geodesics.



For a topological Anosov flow (in any dimension, but can think in dimension 3 for this talk) one can first "discretize" the universal cover of leaves (uniformly) by choosing a "nice" cover by flow boxes

GH5

Intersection of flow boxes with universal cover of the leaf^{of $\mathcal{F}^ws$} gives a locally finite cover of L which we consider as vertices ^{$\mathcal{V}$} of a graph

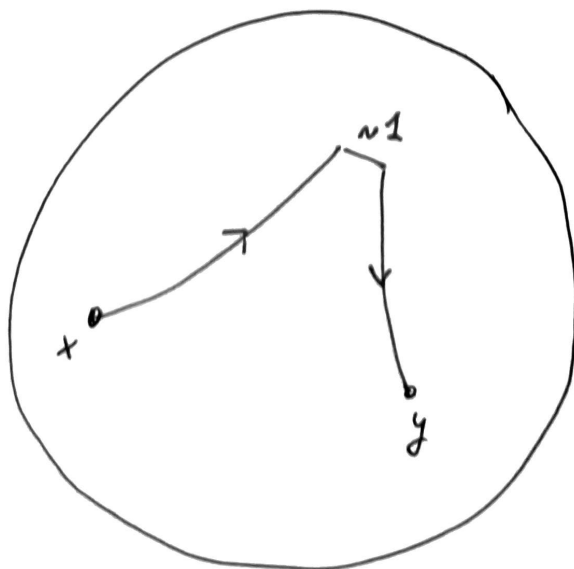
For a box $B \in \mathcal{V}$ we consider the graph $G_{\mathcal{V}}^N$ to be the set of the first N boxes in $\mathcal{V}$ intersected by flowing B forward.

Prop: $\exists N_0 > 0$ so that if $N > N_0 \Rightarrow G_{\mathcal{V}}^N$ is connected and QI to the leaf L . (uniform constants)

Now one defines the following family of paths in $G_{\mathcal{V}}^N$:

FP: {traces in $G_{\mathcal{V}}^N$ of ^(finite) forward orbits of the flow φ_t / L }

AP: {concatenations of path in FP, length 1 path, another path in FP going backwards}



one can verify that this family works.

Rmk: With some work one can show ∂L is a ~~q~~ (topological) sphere.