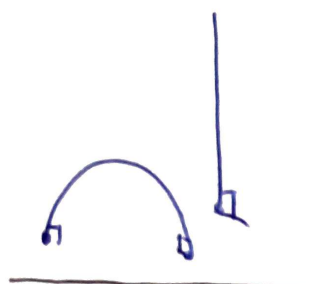


Examples (& non-examples) of QG flows

QG: 1

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 : z > 0\} \quad \text{metric: } \langle v, w \rangle_{(x, y, z)} = \frac{\langle v, w \rangle_{\text{Euc}}}{z^2}$$

$$\Rightarrow \langle v, w \rangle_{(x, y, z)} = \frac{\langle v, w \rangle_{\text{Euc}}}{z^2}$$



$\mathbb{H}^3$

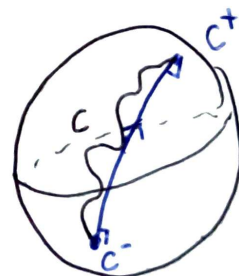
$$\partial \mathbb{H}^3 \simeq \mathbb{C} \cup \{\infty\} \supset \text{PSL}_2(\mathbb{C}) \simeq \text{Isom}(\mathbb{H}^3)$$

$\hat{\mathbb{C}}$: sphere de Riemann.

geodesics in $\mathbb{H}^3$
parts of $\neq$ points in $\hat{\mathbb{C}} \simeq \partial \mathbb{H}^3$

Def: Let $c: \mathbb{R} \rightarrow \mathbb{H}^3$ a curve in $\mathbb{H}^3$, we say it is a quasi-geodesic if $\exists K > 0$ such that $\frac{1}{K}|t-s| - K \leq d_{\mathbb{H}^3}(c(t), c(s)) \leq K|t-s| + K$

Morse Lemma: $\forall K > 0, \exists r := r(K)$ such that if $c: \mathbb{R} \rightarrow \mathbb{H}^3$ is a K -quasi geodesic, then, $\exists c^- \neq c^+ \in \partial \mathbb{H}^3$ so that the Hsdf distance between the curve $c(\mathbb{R})$ and the geodesic η joining c^- and c^+ is less than r .



Let M be a closed hyperbolic 3-mfd i.e. compact mfd w/o boundary such that $M = \mathbb{H}^3 / \Gamma$ with $\Gamma < \text{Isom}(\mathbb{H}^3)$.

(Perelman: equivalent to aspherical + atoroidal)
 $\pi_2(M) = \{0\} \quad \mathbb{Z}^2 \not\subset \pi_1(M)$

Def: A flow $\phi_t: M \rightarrow M$ is quasi-geodesic if $\forall \tilde{x} \in \tilde{M} \simeq \mathbb{H}^3$ the curve $t \mapsto \tilde{\phi}_t(\tilde{x})$ is a quasi-geodesic.

Calegari: $\rightarrow \exists \kappa > 0$ so that all orbits are κ -QG.

Q6 2

Fenley-Moshen: $\rightarrow \phi_t$ is QG $\iff \exists$ ^{continuous} maps $e^\pm: \mathbb{H}^3 \rightarrow \partial \mathbb{H}^3$
 so that $e^+(x) \neq e^-(x) \forall x \in \mathbb{H}^3$ and

$$e^\pm(x) = \lim_{t \rightarrow \pm\infty} \tilde{\phi}_t(x)$$

Rmk: stable under C^0 -perturbations ~~more~~ (local to global)

Examples: • stable under reparametrizations.

$\rightarrow$ (Cannon-Thurston, Ziegler): Let $M = S \times [0,1] / \sim$

where $(x,1) \sim (\varphi(x),0)$

Consider $\phi_t: M \hookrightarrow$ the suspension flow for φ

for some $\varphi: S \rightarrow S$
pseudo-Anosov $\simeq \overline{M}_{\text{atoroidal}}$

(i.e. $\frac{\partial \phi_t}{\partial t} \Big|_{t=0}$ is $(0,1)$)

then, ϕ_t is quasi geodesic:

proof: the Hsdlf distance between 2 consecutive lifts of $S \times \{0\}$ to $\mathbb{H}^3$ is bounded below by some constant $R > 0$

then, $d_{\mathbb{H}^3}(\tilde{\phi}_t(x), \tilde{\phi}_s(x)) > R \cdot ||t-s|| - R$

On the other hand, if $X = \frac{\partial \tilde{\phi}_t}{\partial t} \Big|_{t=0}$ then $||X||$ is bounded by R'

so $d_{\mathbb{H}^3}(\tilde{\phi}_t(x), \tilde{\phi}_s(x)) \leq \text{length}(\tilde{\phi}_{[s,t]}(x)) \leq R' |t-s|$. $\square$

Generalization: (Thurston, Calegari, Fenley) A foliation $\mathcal{F}$ of a closed 3 mfd is called $\mathbb{R}$ -covered if leafspace of $\tilde{\mathcal{F}}$ is $\mathbb{R}$. A flow $\phi_t: M \rightarrow M$ is said to be regulating if $\forall \tilde{x} \in \tilde{M}$, the orbit $\tilde{\phi}_t(x)$ intersects every leaf of $\tilde{\mathcal{F}}$.

Thm (Calegari, Fenley) if $\mathcal{F}$ is $\mathbb{R}$ -covered in hyperbolic $M \rightarrow \exists$ regulating QG pseudo Anosov flow to $\mathcal{F}$. (in general, non suspension) $\leftarrow$ NO Perfect fits

... (with ... without perfect lift)

Q65

Q63

Connection with geodesic flow on T^1M

Prop (Morse lemma) Given a q.g. flow $\phi_t: M \rightarrow M$ on a ^{closed} hyperbolic 3-mfd there exists $\Lambda_\phi \subseteq T^1M$ ^{compact} invariant under geodesic flow $g_t: T^1M \rightarrow T^1M$ such that ϕ_t and $g_t|_{\Lambda_\phi}$ are "semi-orbit equivalent", i.e. $\exists h: M \rightarrow T^1M$ continuous map such that $\text{proj} \circ h \sim \text{id}$ ($\text{proj}: T^1M \rightarrow M$) and $h(M) = \Lambda_\phi$ such that h maps orbits of ϕ_t to orbits of g_t .

Rmk: If $H: \tilde{M} \rightarrow T^1\tilde{M}$ is lift to universal cover, can be chosen so that $d(x, \text{proj}(H(x))) < R \quad \forall x \in \tilde{M} \simeq \mathbb{H}^3$.

Idea: First define $H_0: \tilde{M} \rightarrow T^1\tilde{M}$ such that

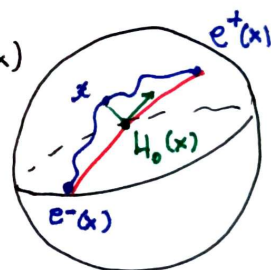
$H_0(x)$ is the point in the geodesic from $e^-(x)$ to $e^+(x)$

"closest" to x (since $H_0(x) \in T^1\tilde{M}$ we consider also the

vector tangent to the geodesic at that point)

(if more than one, take barycenter)

H_0 is equivariant: $H_0(\gamma \cdot x) = \gamma \cdot H_0(x) \quad \forall x \in \tilde{M}, \gamma \in \pi_1(M)$



but maybe not continuous. To make it continuous, note that by Morse-lemma $d(x, H_0(x)) < R$ (abuse of notation, since $H_0(x) \in T^1M$) so one can integrate the choice in long segment of orbit. This will provide the desired (averaging technique)

map H whose projection to M gives h and $h(M) = \Lambda_\phi$. □

Corollary: If ϕ_t is (pseudo)Anosov and QG $\Rightarrow$ can only have finitely many freely homotopic orbits. (because pairwise distance $> \epsilon > 0$)

Thm (Fenley '94) Any R -covered Anosov flow in hyperbolic 3-mfd cannot be QG.

(Constructions: Fenley, Foulon-Hasselblatt, Bowden-Mann, Bonatti-Iakovoglou)

Examples with perfect fits and others

QG 4

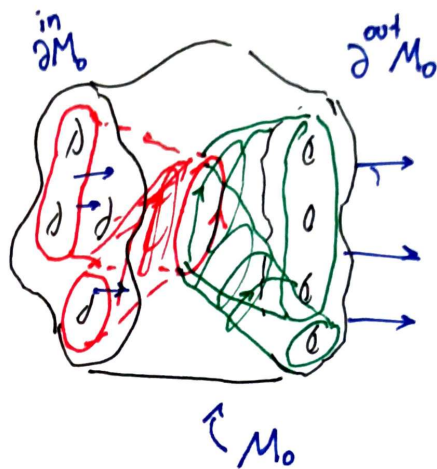
Gabai-Mosher (have some similarity with Bonatti-Langevin)

§ Finite depth foliations: (Depth one): Compact leaf, all other leaves spiral towards the compact leaf.

First idea: produce a "plug" such that

with vector field X inward in $\partial^{\text{in}} M_0$
outward in $\partial^{\text{out}} M_0$

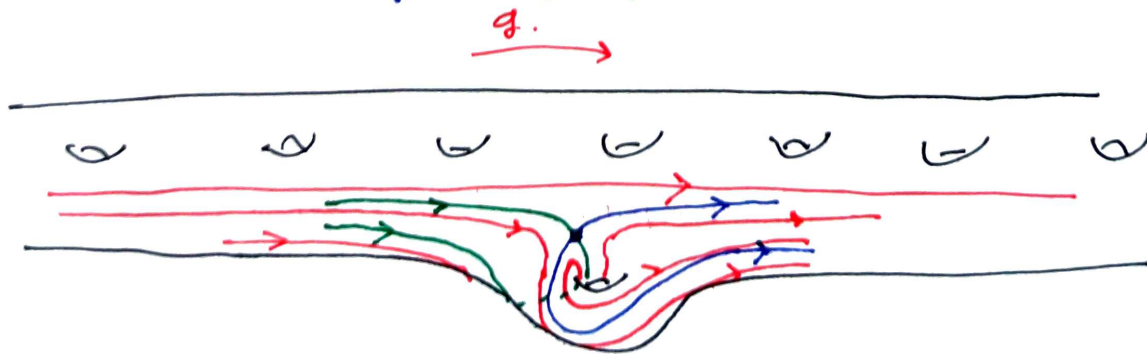
s.t. $\partial^{\text{in}} M_0 \simeq \partial^{\text{out}} M_0$ but M_0 not equal
to $\partial^{\text{in}} M_0 \times [0, 1]$



Then glue $\partial^{\text{out}} M_0$ to $\partial^{\text{in}} M_0$ so that resulting M is atoroidal

One expects that in $\tilde{M} \simeq \mathbb{H}^3$ lift of $S = (\text{quotient of } \partial^{\text{in}} M_0 \sim \partial^{\text{out}} M_0)$
are some bounded distance apart and orbits either go from
one boundary to the other in some bounded type, or spiral a lot
around the periodic orbit in the "middle". This spiraling
forces to cross some lift very far, so, produces actual QG behavior.

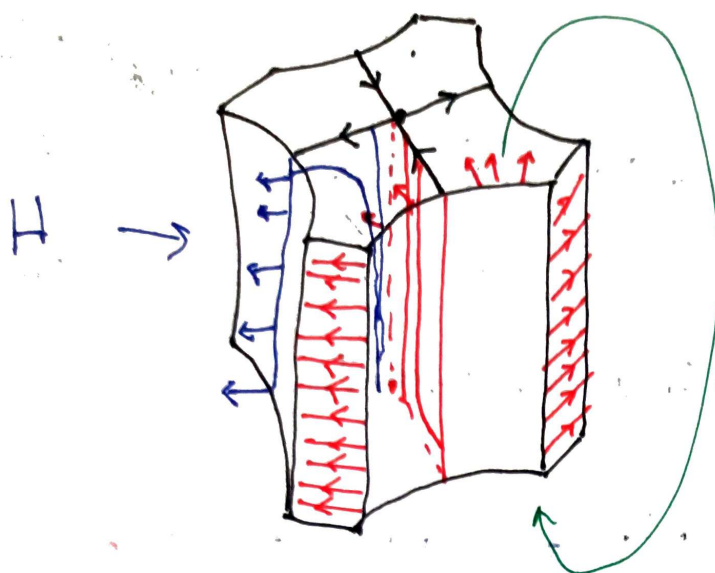
How to see this as $\mathbb{H}^3$ to finite depth foliation (if enough time)



Take suspension of this and "compactify" at infinity
get some examples.

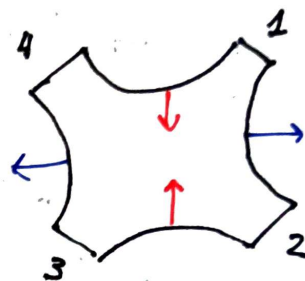
Concrete examples (with & without perfect fits) (similar to BL)

Q65



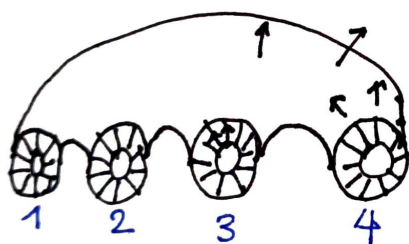
maximal invariant of H
set, single periodic orbit

$$\varphi = (\text{id})$$



$\partial H = 8 \text{ annuli} \rightarrow 4 \text{ trivially foliated } (1,2,3,4)$
 $\rightarrow 2 \text{ inward } (14, 23)$
 $\rightarrow 2 \text{ outward } (12, 34)$

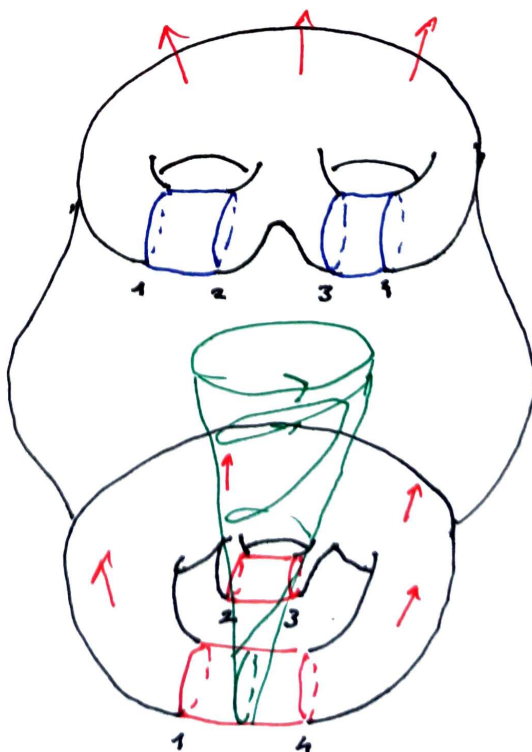
Example 1:



$$\approx S \times [0,1] \text{ where } S = \begin{pmatrix} x & t \\ \downarrow & \downarrow \end{pmatrix} \text{ vector field is } \frac{\partial}{\partial t}$$

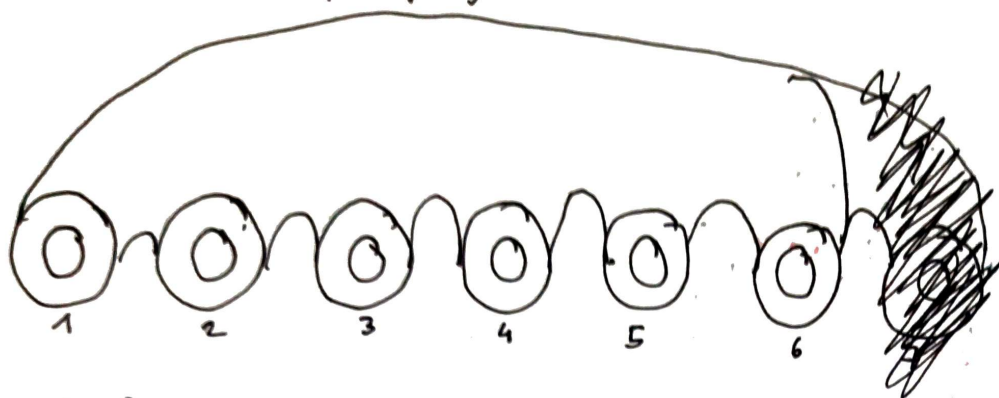
Glue with H where

Then glue boundaries in order
to kill all possible tori
(can do so remains pA,
else, show QG and
apply Frankel-Landry)



Example 2 (with perfect fits)

Q66

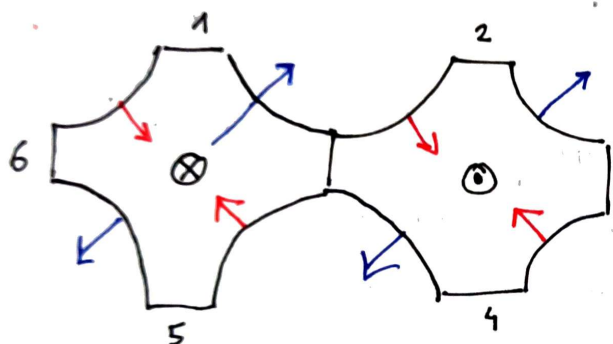


$$S_{0,1} \times [0,1]$$

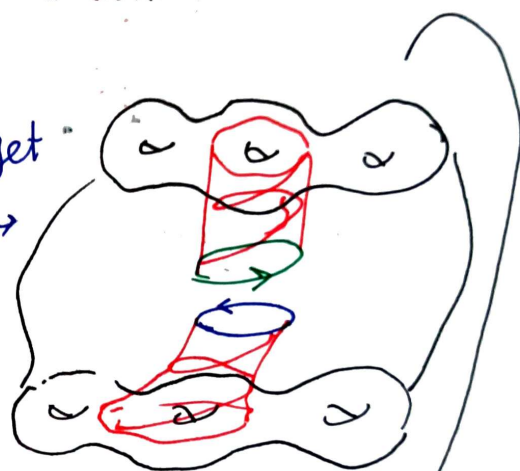
same vector field

$$\frac{\partial}{\partial t}$$

glue with two copies of H in different orientation



get



glue