Dynamical decomposability & topology

$\boxed{\text{GOAL}}$ Take a deterministic law of change with finite precision and give checkable conditions to be able to express meaningful information about asymptotic behavior.

More precisely:

$f^n = \underbrace{f\circ\dots\circ f}_{n \text{ times}}$.

You are given a transformation law $f: M \to M$ (say f is diffeomorphism of closed manifold)

$\mathcal{O}_f(x) = \{f^n(x)\}_{n\in\mathbb{Z}}$

Can you say something about orbit structure using information that can be extracted in "finite time"?

Example:

information about orbit structure complexity

Shub's entropy conjecture: The topological entropy of a smooth map on a closed manifold is bounded below by the spectral radius of its action in homology. (Yomdim)

"static information"

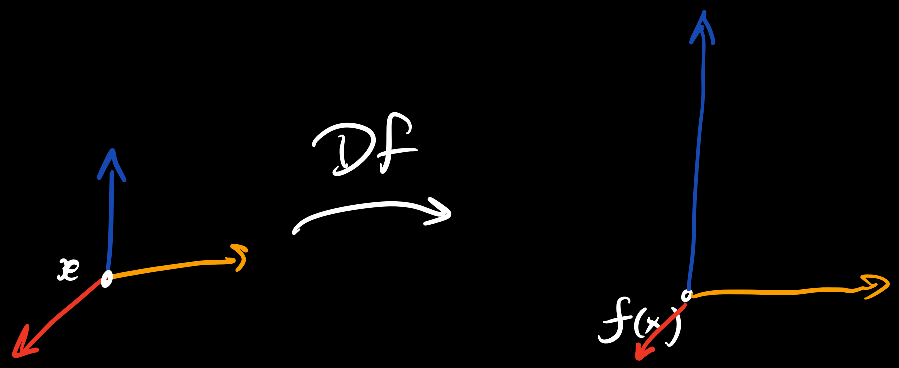
§ GEOMETRIC (STATIC) STRUCTURES

$f : M \to M$ is **partially hyperbolic**

if $TM = E^s \oplus E^c \oplus E^u$ is Df-invariant splitting and $\exists \ell > 0$ such that

$\forall x \in M$, $v^s \in E^s_x$ $v^c \in E^c_x$ $v^u \in E^u_x$ unit vectors

one has
$$\|Df^\ell v^s\| \leq \frac{1}{2} \min\{1, \|Df^\ell v^c\|\}$$
$$\|Df^\ell v^u\| \geq 2 \max\{2, \|Df^\ell v^c\|\}$$



[Cone fields: Allow to detect partial hyperbolicity of a given strength $\ell > 0$ with "static information".

(open condition, many relevant examples, ...)
↳ e.g. algebraic, geometric...)]

(**Problem**: Can one get meaningful dynamical information from such a structure?)

1) **Perturbative approach**: Robust transitivity & Stable ergodicity. Very succesful (still in progress).

Problem: if you are given a system, you can expect to enjoy the dynamical properties, but cannot be sure.
(e.g. like: is π a "normal" number?)

2) **Non perturbative approach**: Can you precisely describe the exceptions? Can you add static hypothesis to promote "almost all" to "ALL".

Many works related to this problem in low dimensions ($\dim M = 3$) and the conservative case (a volume form is preserved).

THIS TALK:
- Dimension $M = 3$
- $f : M \to M$ partially hyp.
$\dim E^s = \dim E^c = \dim E^u = 1$.
- No volume form is preserved.

Question: Is there some manifold M and isotopy class $\rho \in MCG(M)$ such that $\forall f$ as above in the class ρ we have that f is "indecomposable".

"chain transitive" $\Longleftrightarrow$ ~~f is decomposable if $\exists\, U \subseteq M$ open set $\phi \neq U \neq M$ / $f(\bar{U}) \subseteq U$~~

$\nearrow$ f is decomposable if $\exists\, U \subseteq M$ open set $\emptyset \neq U \neq M$ / $f(\bar{U}) \subseteq U$

Many negative evidence:

1) time one maps of Anosov flows $\longrightarrow$ Bonatti-Guelman

2) nilmanifold automorphisms $\longrightarrow$ Shi

$\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$

Question: What about $f: \mathbb{T}^3 \to \mathbb{T}^3$ homotopic to linear Anosov automorphism?

$\Big(f_*: \underbrace{H^1(\mathbb{T}^3)}_{= \mathbb{Z}^3} \to H^1(\mathbb{T}^3) \in GL_3(\mathbb{Z})$ has all its eigenvalues of modulus $\neq 1 \Big)$

Remark: There are many other isotopy classes where this question makes sense and are open for investigation.

Insight: Maybe understanding the topological structure of f helps.

E.g. If $\dim M = 2$

$$
\begin{tikzcd}
\text{Robust transitive} \arrow[rr, Rightarrow, "\text{Mañe}"] & & \text{Anosov in } \mathbb{T}^2 \arrow[dl, Rightarrow, "\text{Franks}"] \arrow[ll, bend left, phantom] \\
& \text{Conjugate to linear} \arrow[ul, "\text{easy}"] &
\end{tikzcd}
$$

In dim 3, in particular for $M = \mathbb{T}^3$ then it makes sense

Thm (Brin, Burago, Ivanov, Hammerlindl, P.–)
Let $f: \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeo homotopic to a linear Anosov automorphism, then E^c integrates uniquely into an f-inv. foliation and f is "leaf conjugate" to its linearization.

to be defined but for instance, $f \sim \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Typically, the object which is difficult to analyse is the strong unstable foliation when f is homotopic to an Anosov automorphism with 2-dimensional unstable direction. (Such as $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$)

Reason: The strong unstable foliation is a one dimensional object inside a 2 dimensional expanding object, so the behaviour is expected to be very unstable (and fractal rather than smooth).

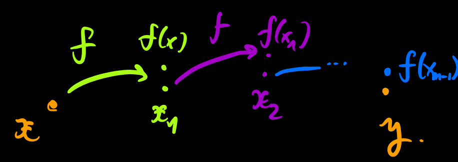
Recently some important progress has been done in order to understand this.

In joint work with Avila, Crovisier, Eskin, Wilkinson, Zhang

Some new techniques have been introduced to study this problem. One consequence of these techniques:

Thm: Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic difeomorphism homotopic to a linear Anosov automorphism. Then, f is chain-recurrent ($\nexists$ proper attractors)

§ Atractors/Repellers and Chain recurrence



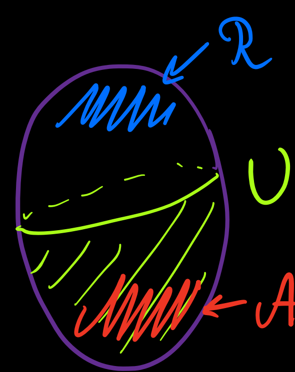
$f: M \to M$ is chain recurrent if $\forall\, x, y \in M$ and $\varepsilon > 0$, $\exists\ x = x_0, x_1, \dots, x_m = y$ such that

$$d(f(x_i), x_{i+1}) < \varepsilon.$$

[(Conley) If and only if $\forall$ open set $U \neq \emptyset$ such that $f(\bar{U}) \subseteq U$ one has $U = M$.]

So, if f is NOT chain recurrent, $\exists\, U$ open set (non empty, $U \neq M$) such that

$$f(\bar{U}) = U.$$



Define: "atractor" $\mathcal{A} = \bigcap_{n \geq 0} f^n(\bar{U})$ compact $\neq \emptyset$ invariant

"repeller" $\mathcal{R} = \bigcap_{n \geq 0} f^{-n}(M \setminus U)$ compact $\neq \emptyset$ invariant.

Remark: $d(\mathcal{A}, \mathcal{R}) > \delta > 0$. (compactness)

- If f is partially hyperbolic then:

  $\mathcal{A}$ is saturated by $\mathcal{W}^u$

  $\mathcal{R}$ is saturated by $\mathcal{W}^s$.

This motivates the study of (minimal) laminations saturated by $\mathcal{W}^u$ or $\mathcal{W}^s$.

When $f \sim$ Anosov, much can be said.

Franks semiconjugacy:

A acts on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ as $A(x+\mathbb{Z}^3) = Ax + \mathbb{Z}^3$ well defined since $A \in SL_3\mathbb{Z}$.

$f \sim A \in SL_3(\mathbb{Z})$ hyperbolic matrix (no eigenvalues of modulus $=1$) → This means $f \sim$ Anosov

$\exists\, h : \mathbb{T}^3 \to \mathbb{T}^3 \sim id$ continuous such that

$$h \circ f = A \circ h$$

So, dynamics of f is an "extension" of the dynamics of A.

Thm above implies that $\forall x \in \mathbb{T}^3$, $h^{-1}(\{x\})$ is a (possibly trivial) segment tangent to E^c.

How is dynamics of A?

$\lambda_A^s < \lambda_A^c < \lambda_A^u$ ← eigenvalues

A must decompose $E_A^s \oplus E_A^c \oplus E_A^u$ ← eigenspaces

up to taking f^{-1}, we can assume that eigenvalue

$\lambda_A^c > 1$ (all positive...)

Its structure allows to show:

→ h maps leaves of $\mathcal{W}^s$ to lines parallel to E^s_A

→ h maps leaves of $\mathcal{W}^u$ <u>inside</u> planes parallel to $E^c_A \oplus E^u_A$.

<u>Consequence</u>: Repeller is $\mathcal{W}^s$-saturated and compact $\Rightarrow h(\mathcal{R}) = \mathbb{T}^3$.

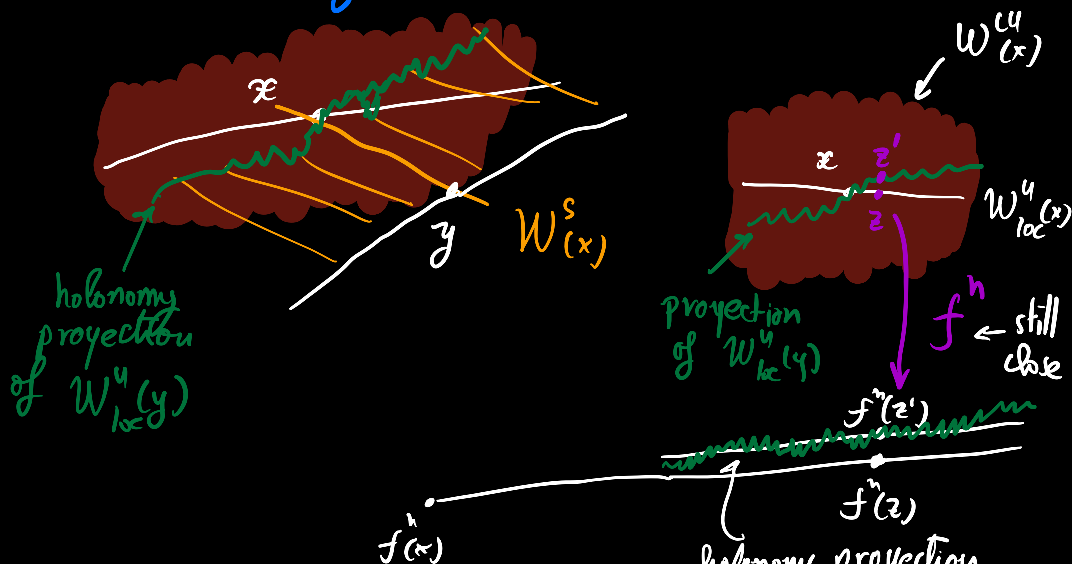
<u>Issue</u>: $h(\mathcal{A})$ cannot make same argument because $h(\mathcal{W}^u)$ may not be a(n irrational) line inside $E^c_A \oplus E^u_A$.

However, if $h(\mathcal{A}) \subsetneq \mathbb{T}^3$ there are some geometric properties which can be exploited.

<u>Transversality</u>:
If $h(\mathcal{A}) \subsetneq \mathbb{T}^3$ then $\forall x \in \mathcal{A}$ there is $y \in \mathcal{W}^u(x) \setminus \{x\}$ such that $y \in \mathcal{W}^s(x)$ and such that stable holonomy from $h(\mathcal{W}^u_{loc}(y))$ to $E^c_A \oplus E^u_A(h(x))$ topologically crosses $h(\mathcal{W}^u_{loc}(x))$.

(One consequence of this: unique minimal u-saturated set, other: recently used by Hammerlindl–Shi to deduce accesibility ...)

Key idea (ACW) This configuration allows to produce "new" strong unstable manifolds of the attractor inside the same W^{cu}-leaf.



Using the fact that

$$d(A, R) > \delta > 0$$

one can perform this argument finitely many times to get a contradiction. $\blacksquare$