**Workshop on Geometric Flows and Relativity**
Punta del Este, March 2024

# A Morse-theoretic glance at phase transitions approximations of mean curvature flows

**Pedro Gaspar (UC Chile)**, joint with J. Chen (UPenn)

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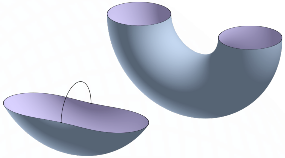
# Motivations: Morse theory

**Main theorem of Morse Theory**

For a critical level $c \in \mathbb{R}$ of a *Morse* $f : X \to \mathbb{R}$, the flow generated by $-Df$ retracts $\{f < c + \delta\}$ onto:

$$\{f < c - \delta\} \cup \bigcup_{p \in f^{-1}(c) \cap \mathrm{crit}(f)} \mathbb{D}^{\mathrm{ind}\, p}$$



Deep link between topology of $X$ and dynamics of gradient flow (equilibrium points and connections between them).

**Question**

Can we implement these ideas for $f = \mathrm{area}$ (= codimension 1 volume)? (**Almgren** '62, **Pitts** '81, **Marques-Neves** ∼ '13)

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# Motivation: area and mean curvature flows

**1st variation formula:** for a hypersurface $\Sigma^{n-1} \subset (M^n, g)$ and a (compactly supported) vector field $X$ along $\Sigma$,

$$D\,\mathrm{area}[\Sigma](X) = \int_\Sigma g(X, -\mathbf{H}_\Sigma)\, dvol_\Sigma$$

**Mean curvature flow (MCF):** gradient flow of area, for $F \colon I \times \Sigma \subset \mathbb{R} \times \Sigma \to M$

$$\partial_t F = \mathbf{H}_{\Sigma_t}$$

Lichnewsky, Temam (PDEs), **Brakke** (GMT), Ecker, Gerhardt, Huisken, Gage, Grayson...

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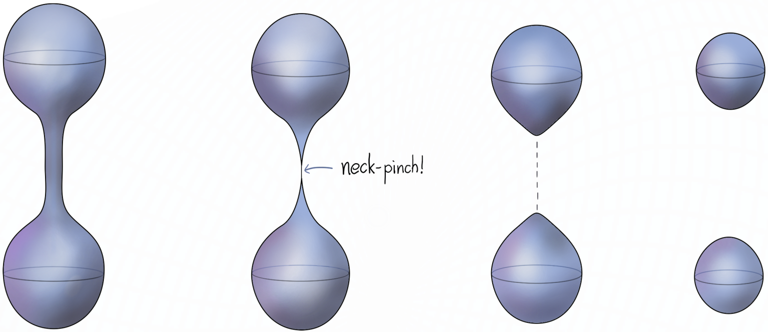

# Phase transitions

**Allen-Cahn equation** (with $\varepsilon > 0$):

$$\partial_t u = \Delta_g u - \frac{1}{\varepsilon^2}(u^3 - u) \qquad \text{(AC)}$$

in a compact $(M^n, g)$.
(Phase separation models, **S. Allen-J. Cahn**, '77)

$L^2$-gradient flow of the **AC energy:**

$$E_\varepsilon(u) = \int_M \left( \varepsilon \cdot \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon} \cdot W(u) \right)$$

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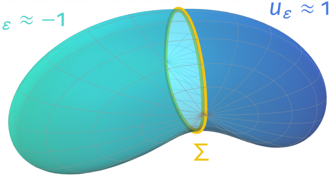
# Phase transitions and geometry

**$\varepsilon$-regularization of area:** If $\{E_\varepsilon(u_\varepsilon)\}_\varepsilon$ is bounded, then $u_\varepsilon \to \pm 1$ a.e. in $M$, while

**diffuse interfaces** $\{u_\varepsilon \approx 0\}$ for stationary solutions $\xrightarrow{\varepsilon \downarrow 0}$ **limit interface** $\Sigma$: critical point of area ($(n-1)$-dimensional)

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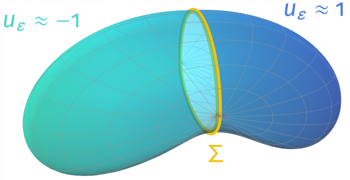

# Convergence

**Ilmanen '93, Tonegawa '03:** Energy densities of solutions $u_\varepsilon$ converge (in measure) to measure theoretic MCFs (Brakke flows):




$$V_{\varepsilon, u_\varepsilon(\cdot,t)} = \frac{1}{2\sigma}\left(\frac{\varepsilon|\nabla u_\varepsilon(\cdot,t)|^2}{2} + \frac{W(u_\varepsilon(\cdot,t))}{\varepsilon}\right) d\,\mathrm{vol}_g$$

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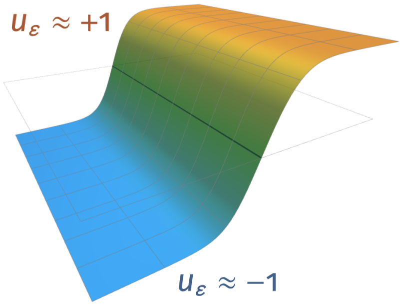


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# Constructing **heteroclinic** solutions and MCFs

From an unstable stationary solution $u_\varepsilon^{-\infty}$,

I. find eternal solutions that converge to $u_\varepsilon^{-\infty}$ as $t \longrightarrow -\infty$, and subconverge to some **nonconstant** $u_\varepsilon^{+\infty}$ as $t \longrightarrow +\infty$.

   Existence of **ancient solutions & unstable manifold** (e.g. Choi-Mantoulidis '22)
   + topological (shooting) arguments, maximum principles.

II. Study $u_\varepsilon^{+\infty}$ and its limit interface: full convergence in time?

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**Theorem (Pacard-Ritoré '03; + Caju-G. '19; + Chodosh-Mantoulidis '20)**

$\Sigma^{n-1}$: nondegenerate, separating, minimal ($\mathbf{H}_\Sigma \equiv 0$) hypersurface in a compact $(M^n, g)$.

For small $\varepsilon > 0$, there exist *stationary* solutions $u_\varepsilon : M \to \mathbb{R}$ of (AC) s.t.:

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Here $\mathrm{ind}(\Sigma) = \mathrm{ind}(D^2\,\mathrm{area}[\Sigma]) = \mathrm{ind}(-\Delta_\Sigma - (|A_\Sigma|^2 + \mathrm{Ric}(\nu_\Sigma, \nu_\Sigma)))$.

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# Unstable manifold and heteroclinic solutions of AC

If $\{\phi_j\}_{j=1}^k$ are eigenfunctions of $-\Delta_g + \frac{3(u_\varepsilon^{-\infty})^2-1}{\varepsilon^2}$ with eigenvalues $\lambda_j < 0$, then

$$\mathcal{S}(a_1, \ldots, a_k; x, t) := u_\varepsilon^{-\infty}(x) + \sum_{j=1}^{k} a_j e^{-\lambda_j t}\phi_j(x)$$

are approximate solutions near $u_\varepsilon^{-\infty}$.

**Choi-Mantoulidis '22:** ancient solutions of (AC) converging back to $u_\varepsilon^{-\infty}$ by fixed point methods on parabolic Hölder spaces.

**Chen-G. '23** (JGA): existence of **heteroclinic solutions** of (AC) from $u_\varepsilon^{-\infty}$ in any compact $(M^n, g)$, for generic $g$ and $\mathrm{ind}(u_\varepsilon^{-\infty}) \geq 2$.

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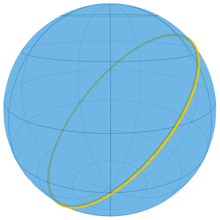
# A model case: solutions in $S^3$

Minimal surfaces of lowest area in the round sphere $S^3$:

### Equatorial spheres

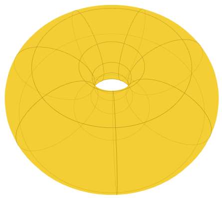
- $\{x \in S^3 \mid \langle x, a\rangle = 0\} \quad (a \in S^3)$
- area $= 4\pi$; ind $= 1$.



### Clifford tori

(Urbano ’90, Marques-Neves ’14)

- $\mathcal{C} = \{x \in S^3 \mid x_1^2 + x_2^2 = \frac{1}{2} = x_3^2 + x_4^2\}$ (and its rotations)
- area $= 2\pi^2$; ind $= 5$.



# Main result

Let $u_\varepsilon^{-\infty}$ be an AC approximation of a Clifford torus, for small $\varepsilon > 0$.

**Theorem (Chen-G., arXiv '23)**

Any ancient solution of (AC) in $S^3$ that converges back in time to $u_\varepsilon^{-\infty}$ is defined for all $t > 0$, and converges smoothly, as $t \to +\infty$, to either an AC approximation of an equator, or $\pm 1$.

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# Key ideas: min-max for $E_\varepsilon$ in compact $(M^n, g)$

**Caju-G.-Guaraco-Matthiesen ’22, Hiesmayr ’20** (arXiv): in $S^3$, the first min-max critical energy levels

$$\omega_1(E_\varepsilon) = \omega_2(E_\varepsilon) = \omega_3(E_\varepsilon) = \omega_4(E_\varepsilon) \qquad \text{and} \qquad \omega_5(E_\varepsilon)$$

correspond to AC approximations of equators and Clifford tori, respectively.
(Allen-Cahn counterpart of classification of low area minimal surfaces)

Here $\omega_p(E_\varepsilon) = \inf_{A\in\mathcal{F}_p} \sup_A E_\varepsilon$ are the min-max critical levels for $E_\varepsilon$ (**G.-Guaraco ’18**), where

$$\mathcal{F}_p \text{ is a family of symmetric } A \subset W^{1,2} \text{ with } H^p(A/(u\sim -u), \mathbb{Z}_2) \neq 0$$

Phase transition analgogues of volume spectrum (**Gromov, Marques-Neves**).

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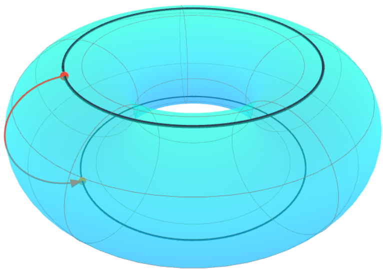
Here $\omega_p(E_\varepsilon) = \inf_{A \in \mathcal{F}_p} \sup_A E_\varepsilon$ are the min-max critical levels for $E_\varepsilon$ (**G.-Guaraco '18**), where

$$\mathcal{F}_p \text{ is a family of symmetric } A \subset W^{1,2} \text{ with } H^p(A/(u \sim -u), \mathbb{Z}_2) \neq 0$$

Phase transition analgogues of volume spectrum (**Gromov, Marques-Neves**).

# Key ideas: Morse perturbation

Choose a *suitable* Morse perturbation $\{E_{\varepsilon,\delta}\}_{\delta\in(0,1)}$ of $E_\varepsilon$.



Perturbed Morse-Bott functions: Banyaga-Hurtubise '13

Morse Theory says, for $k = 1, 2, 3, 4$

$$H^*(\{E_{\varepsilon,\delta} < \omega_k(E_{\varepsilon,\delta}) + (0.001)\}/\sim, \mathbb{Z}_2) \simeq \frac{\mathbb{Z}_2[\lambda]}{\langle\lambda^{k+1}\rangle}.$$

If the unstable manifold of $(-DE_{\varepsilon,\delta})$ at $(u_\varepsilon^{-\infty})$ misses a $u \in \mathrm{crit}(E_{\varepsilon,\delta}) \cap \{E_{\varepsilon,\delta} = \omega_4(E_{\varepsilon,\delta})\}$, then we can construct

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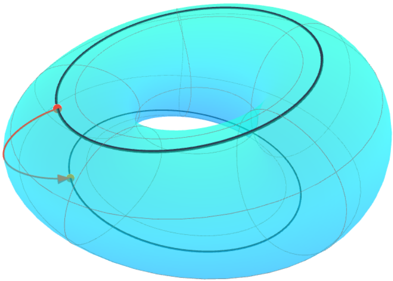

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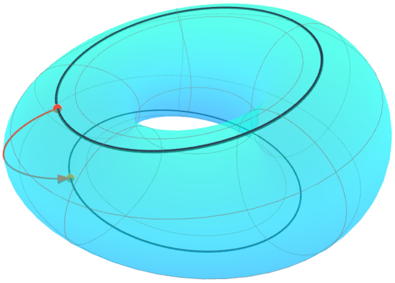

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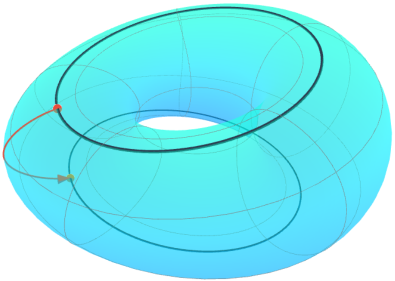

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# Key ideas: analytic technicalities

**Quasilinear perturbation:** $\partial_t u = -DE_{\varepsilon,\delta}(u) = -(1+\delta\theta(u))\Delta u + Q_\delta(u)$

Existence theory for abstract ODEs in $W^{2\alpha,2}(M)$ (e.g. **Amann** 88', **Lunardi** '95)

Dynamics of $(-DE_{\varepsilon,\delta})$ near $u_\varepsilon^{+\infty}$:

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$\implies$ Orbits don't break!

## Question

What kind of MCF (and singularities) can arise from these constructions?

**Chen-G. '23** (JGA): many eternal, symmetric, nonvanishing Brakke flows in $S^3$;

**J. Chen '24**: similar construction in $S^n$, $n \geq 4$.

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¡Muchas gracias!

Thank you!