Asymptotic counting of surface subgroups and foliated Plateau problems

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joint with

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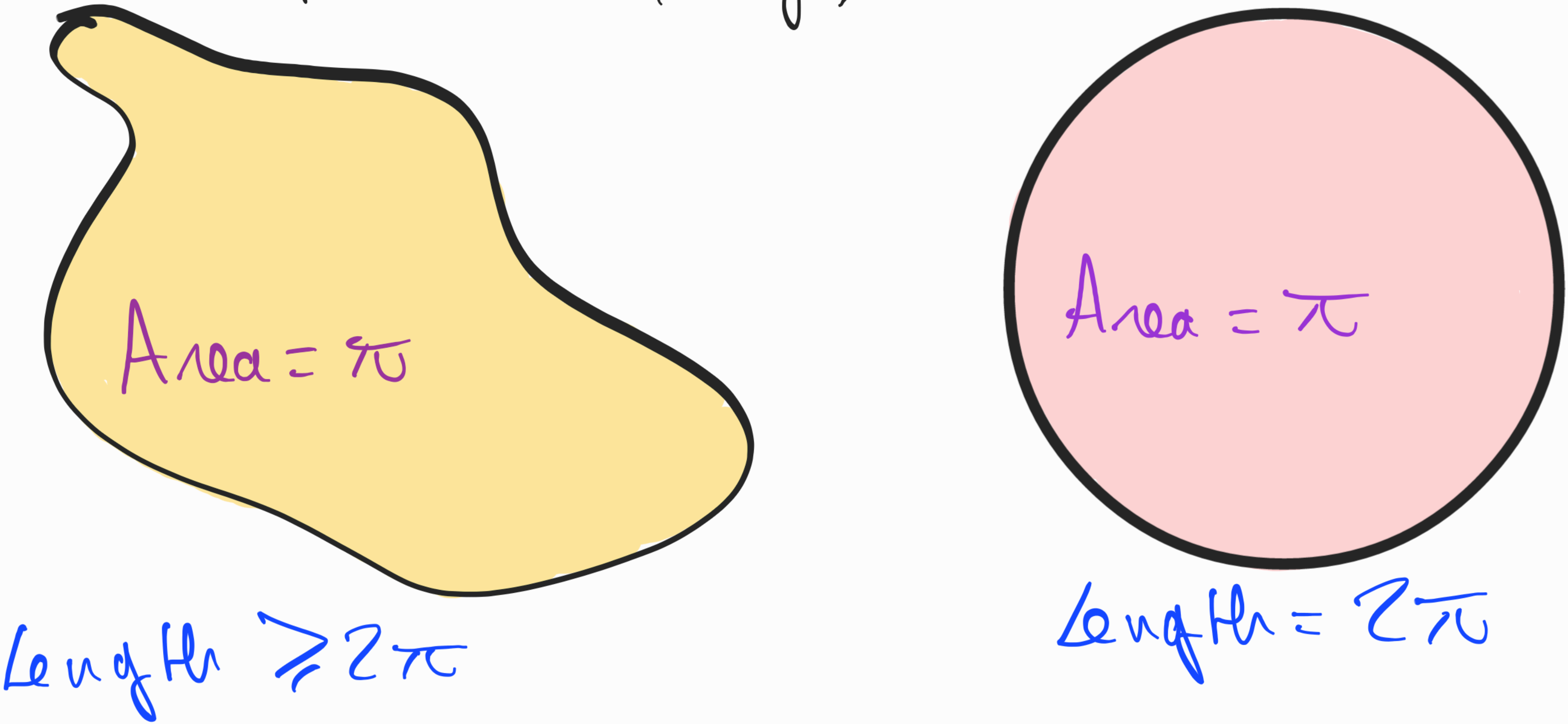
Geometric rigidity.

(isoperimetric inequality)



Setting

(M, h_0): closed 3-dim hyperbolic mfd.

$M = \mathbb{H}^3 / \pi$, $\pi \leq PSL_2(\mathbb{C})$ $(= \mathrm{Isom}^+(\mathbb{H}^3))$

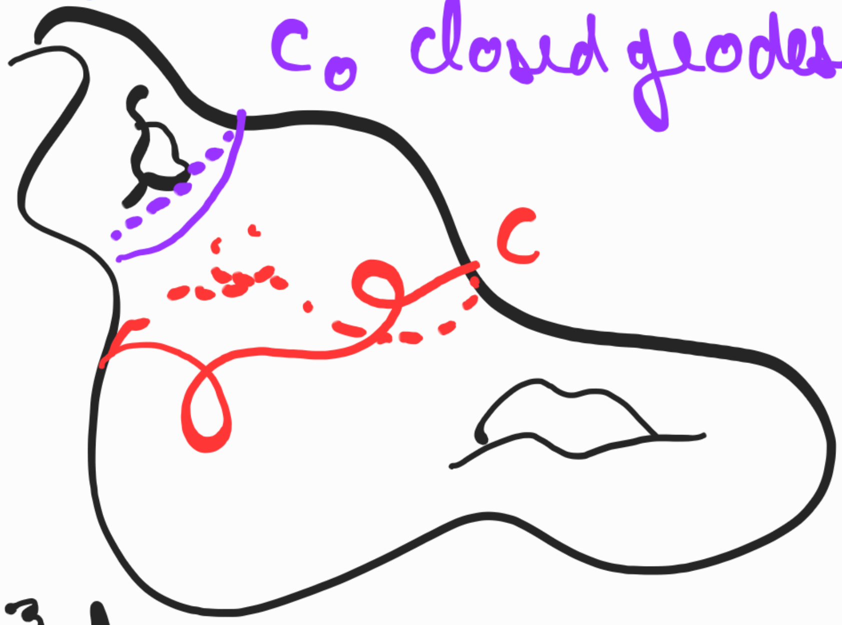
Poincaré plane

cocompact lattice.

Problem

$\boxed{\text{Recognize } h_0 \text{ (symmetric case) among } \{h \text{ Riemannian metric on } M \text{ s.t. } \mathrm{sec}_h < 0\}}$

Asymptotic counting of closed curves.



Def [Entropy]:

$$\boxed{H(M,h) = \lim_{R\to\infty} \frac{1}{R} \log \# \left\{ \begin{array}{l} c_0 \text{ closed geodesic s.t.} \\ \text{length}(c_0) \le R \end{array} \right\}}$$

Well defined: Bowen-Margulis ($\simeq$ 70)

THM [Besson-Courtois-Gallot]

Let h with $\sec_h < 0$,

$\mathrm{Vol}(M,h) = \mathrm{Vol}(M,h_0) \Longrightarrow H(M,h) \geq 2 = H(M,h_0)$

w. equality iff $h \simeq h_0$.

Asymptotic counting of surfaces

(M^3, h_0)

Immersion $\iota : S \looparrowright M^3$ essential that is

$\iota_\# : \pi_1(S) \hookrightarrow \pi_1(M^3)$ injective

Existence: Kahn–Markovic (2012).

Much better:

Topological counting of surfaces [Kahn–Markovic, 2012]:

$$\lim_{g\to\infty} \frac{1}{2g\log g} \log \# \left\{ [\Gamma] : \begin{array}{l} \Gamma \simeq \pi_1(S) \\ \text{genus}(S) \le g \end{array} \right\} = 1$$

$[\Gamma]$: conjugacy class in π

$\#\{\ldots\} < \infty$: Thurston

Problem: Counting geometric representative.

Quasi_Fuchsian surfaces

$[\cdot,\cdot,\cdot,\cdot]$: cross ratio (conformal invariant)

$$[x,y,z,t] = \frac{(t-x)\,(y-z)}{(t-z)\,(y-x)}$$

1/ C-Quasi_circles

$(C \geq 1)$

s.t. $\forall x,y,z,t \in \mathbb{RP}^1$, $|[h(x),h(y),h(z),h(t)]| \leq C\,|[x,y,z,t]|$

$C=1 \Longrightarrow$ Round circle.

2/ C_quasi_Fuchsian $\Gamma \leq PSL_2(\mathbb{C})$ discrete, $\Gamma \simeq \pi_1(\,\cdots\,)$ s.t.

Limit set Λ is a C_quasi_circle.

$(\Gamma\Lambda = \Lambda)$.

3/ KM surfaces are quasi_Fuchsian.

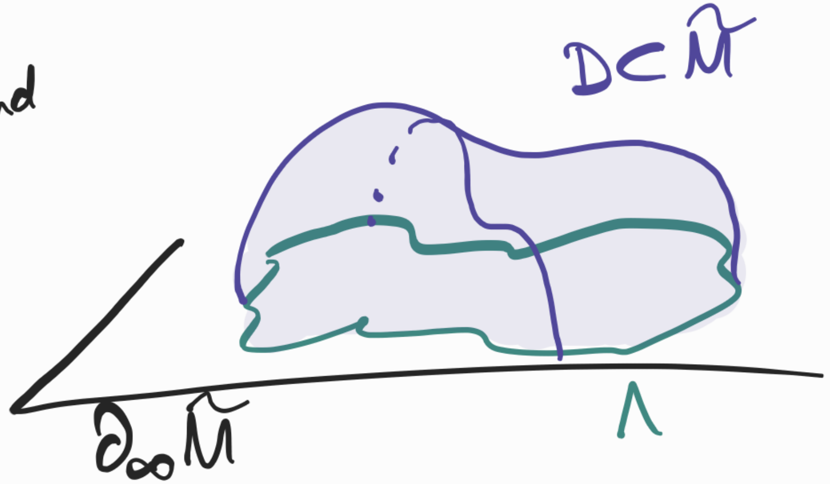
$$\lim_{g\to\infty} \frac{1}{2g\log g} \log \# \left\{ [\Gamma] : \begin{array}{c} \Gamma \simeq \pi_1(S) \\ \text{genus}(S) \leq g \end{array} \text{Quasi\_Fuchsian} \right\} = 1$$

Asymptotic Plateau problem.

From now on: $\sec_h \leq -1$

Given $\Lambda \subset \partial_\infty \tilde{M} \simeq \partial_\infty \mathbb{H}^3 = \mathbb{CP}^1$ C-quasi circle

Look at $D \subset \tilde{M}$ embedded disc s.t. $\partial_\infty D = \Lambda$ and



$0 < k < 1$ fixed.

D k-surface : $K_{ext} = \lambda_1 \lambda_2 = k$ (pcpl curv.)

Existence + Uniqueness } Yes (Labourie, Smith)

D minimal surface: $H = \frac{\lambda_1 + \lambda_2}{2} = 0$.

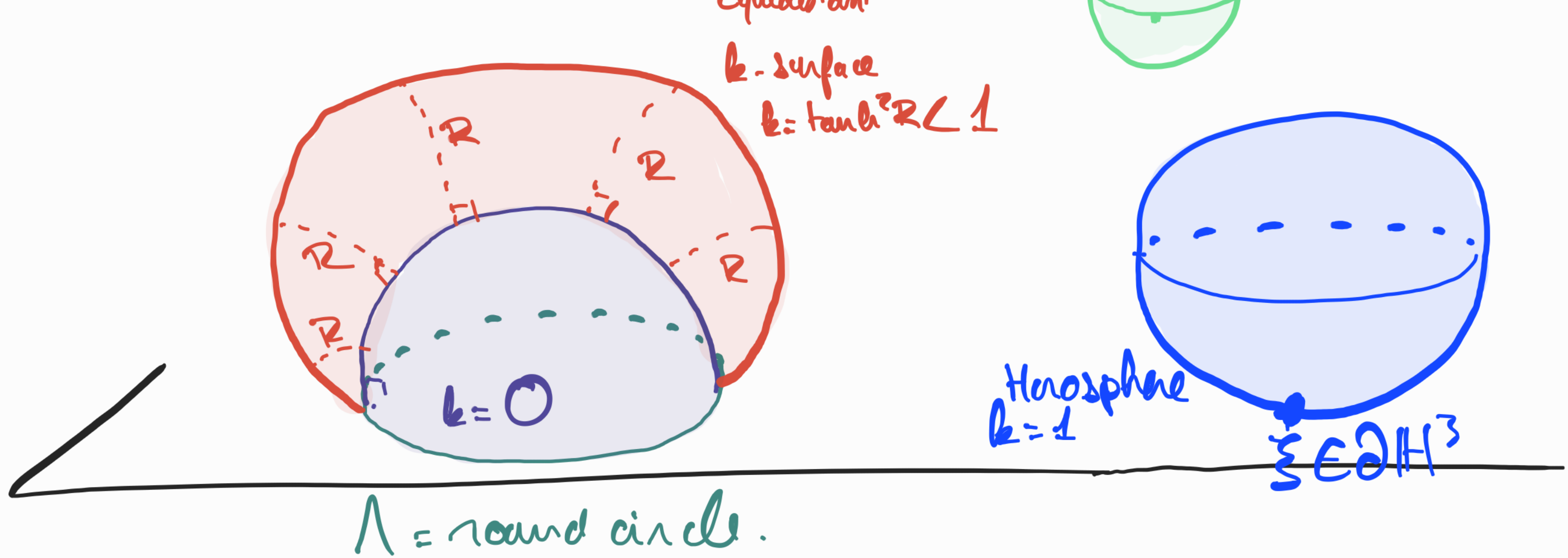
Existence: Yes (Uhlenbeck, Anderson)

Uniqueness: Only when $C \simeq 1$.

We will be interested in k-surfaces.

Examples of k-surfaces in $\mathbb{H}^3$.

(totally umbilical)

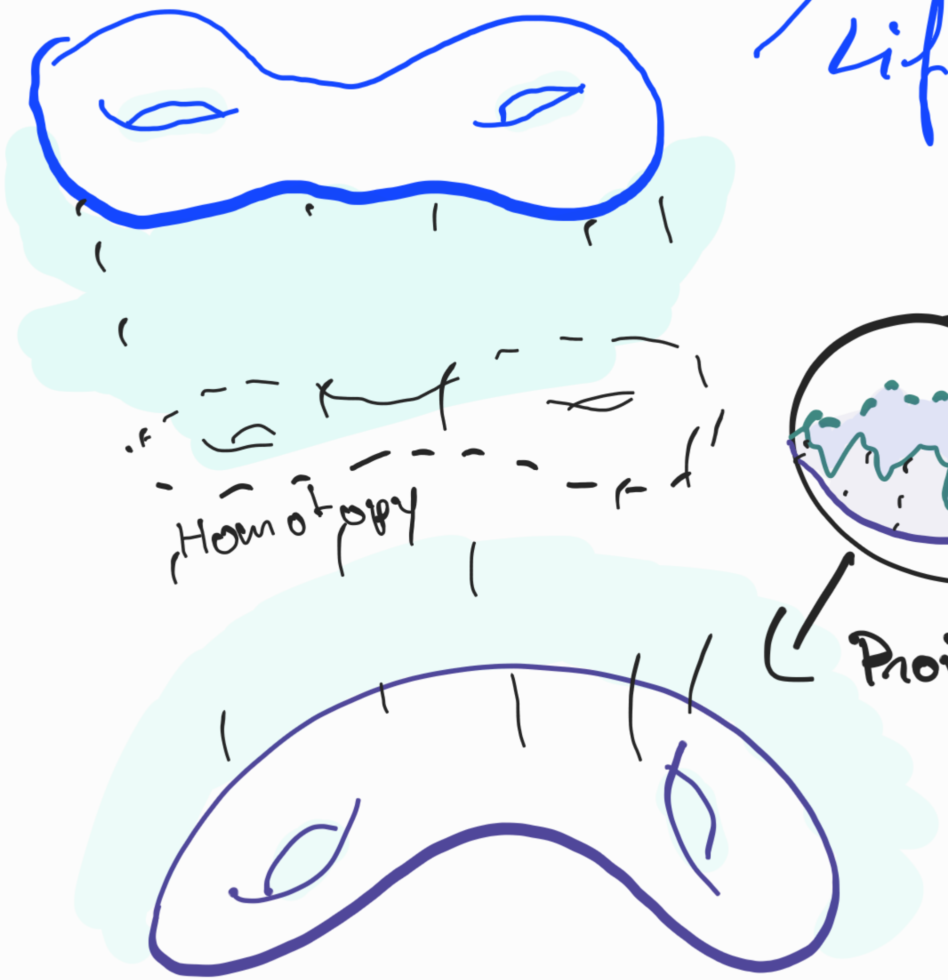


Rk: Gauss equation: if S k-surface,

$$K_{int} = \underbrace{K_{ext}}_{= k} + \underbrace{\text{sect}|TS}_{\leq -1} \leq k - 1 < 0 \quad (\text{if } 0 < k < 1).$$

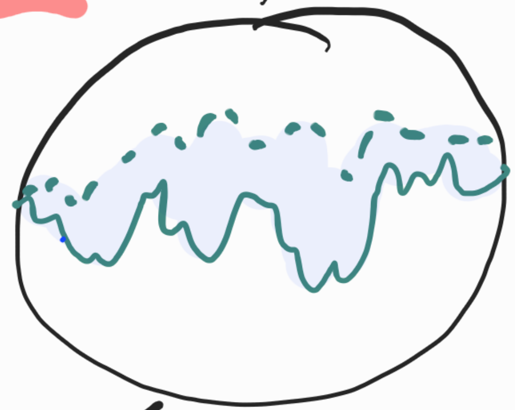
k-surface representative

$S \subset M$
quasi fuchsian
$\Gamma = \pi_1(S)$



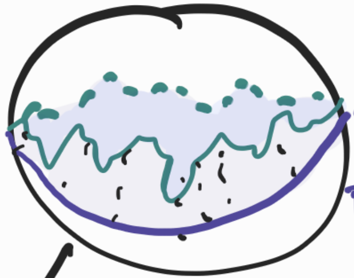
lift

$\tilde{M}^3$



Λ. Γ-invariant.

Solve asymptotic Plateau problem.



Project down

$D_{k,\Gamma}$: k-disc.
Γ-invariant: $\rightarrow \Gamma . D_{k,\Gamma}$ k-disc $(\Gamma \leq \mathrm{Isom})$
$\rightarrow \Gamma \Lambda = \Lambda$
$\rightarrow$ Uniqueness of the sol of asymptotic Plateau problem

$S_{k,\Gamma}$: k-surface representing $[\Gamma]$.

Geometric counting.

Thm [ALS]: Assume $\sec_h \leq -1$. Let $0 < k < 1$. Then:

$\frac{H(M,h)^2}{2\pi}$

$$\mathrm{Ent}(M,h) = \lim_{A \to \infty} \frac{1}{A \log A} \log \#\{[\Gamma]\ \mathrm{QF} : \mathrm{Area}_h(S_{k,\Gamma}) \leq A\}$$

$$\geq \frac{1-k}{2\pi} = \mathrm{Ent}(M, h_0)$$

w. equality iff $h \simeq h_0$.

Calegari - Marques - Neves: Similar result w. minimal surfaces.

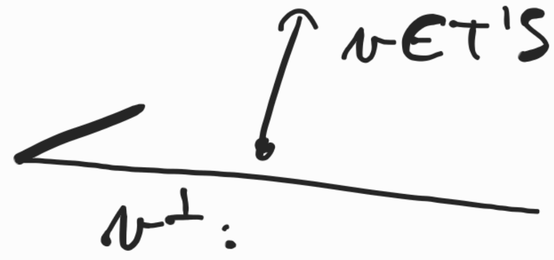
Marked area spectrum rigidity.

$0 < k < 1.$

Th [ALS]: Assume $\forall [\Gamma]$ QF.,

$\mathrm{Area}_h(S_{k,\Gamma}) = \mathrm{Area}_{h_0}(S_{k,\Gamma})$.

Then $h \simeq h_0$.



Idea of proof

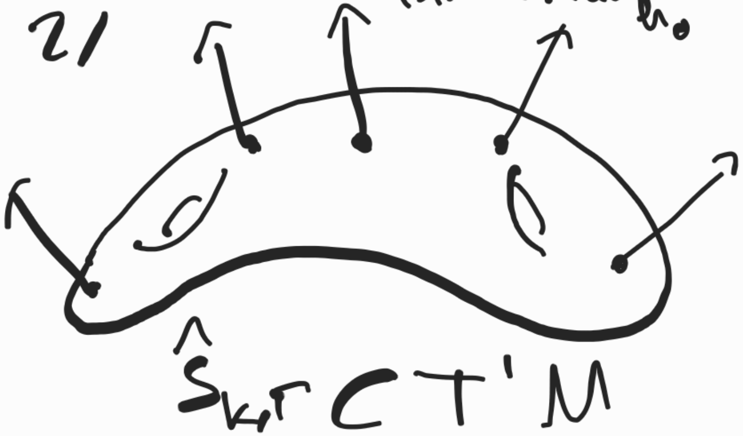
1/ $\mathrm{Area}_h(S_{k,\Gamma}) = \mathrm{Area}_{h_0}(S_{k,\Gamma}) \Longrightarrow \mathrm{sect}_h|_{TS_{k,\Gamma}} = -1$

Gauss eqn: $K_{int} = -\underbrace{K_{ext}}_{k} + \mathrm{sect}|_{TS} < 0$

Gauss Bonnet: $4\pi(g-1) = \int_{S_{k,\Gamma}} |K_{int}| \, d\mathrm{Area}_h = \int_{S_{k,\Gamma}} (\underbrace{|\mathrm{sect}|}_{\geq 1} - k) \, d\mathrm{Area}_h \geq (1-k)\mathrm{Area}_h(S_{k,\Gamma})$

$\quad = (1-k)\mathrm{Area}_{h_0}(S_{k,\Gamma})$

2/



These surfaces equidistribute in T^1M

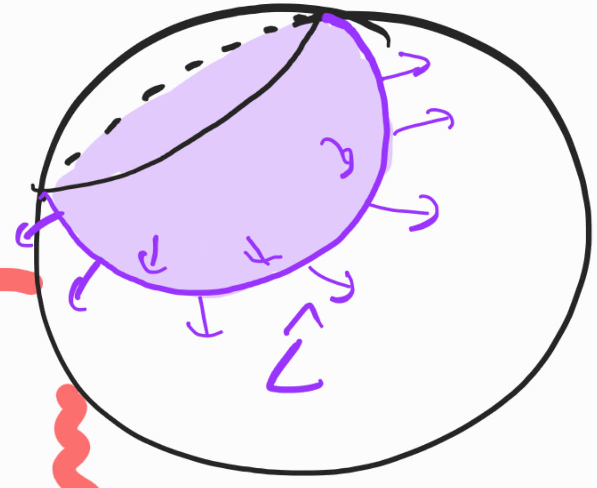
$\longrightarrow \mathrm{sect}_h = -1$ everywhere

$\longrightarrow h \simeq h_0$ (Mostow's rigidity)

The limit object: foliated Plateau problem.

$\mathrm{sect}_h \leq -1, \quad 0<k<1$

$$\tilde{\mathcal{F}} = \left\{ \hat{\Sigma} \subset T^1\hat{M} : \begin{matrix} \Sigma \ k\text{-disc} \\ \partial_\infty \Sigma \text{ round circle} \end{matrix} \right\}$$



Th [ALS]: 1/ $\tilde{\mathcal{F}}$ forms a foliation of $T^1\tilde{M}$.

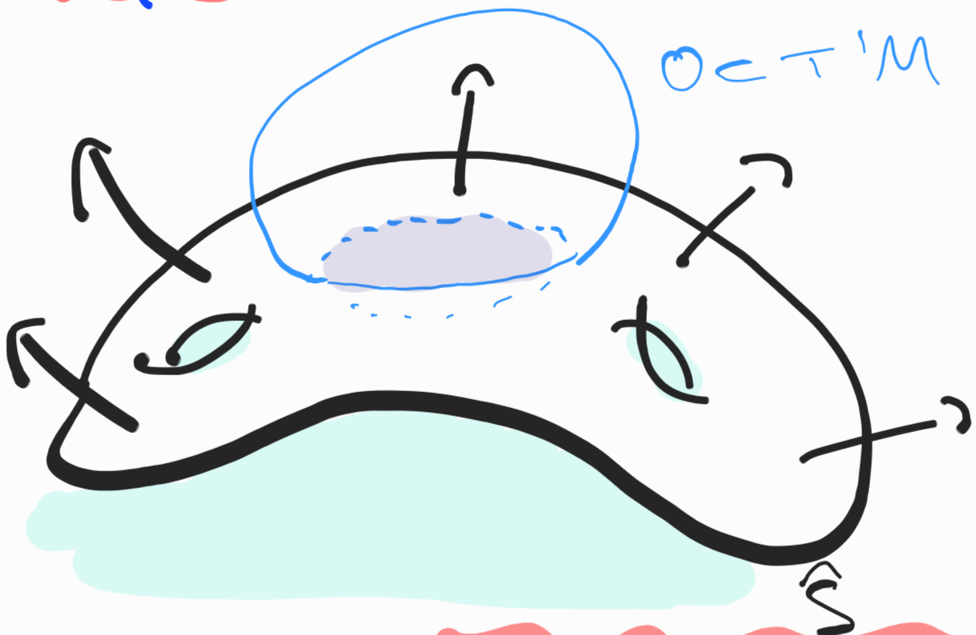
2/ $\tilde{\mathcal{F}}$ is π-invariant.

3/ $\tilde{\mathcal{F}}$ projects down to a foliation $\mathcal{F}$ of T^1M.

⚠ The analogous results for minimal surface only holds for h close to h_0 (Gromov, Lowe)

Lowe proves that there exist h with $\mathrm{sect}_h \leq -1$ st T^1M has NO foliation by minimal surfaces.

Equidistribution result.



$S \subset M$, $\hat{S} \subset T^1M$.

$\mathrm{proj}: T^1M \longrightarrow M$
$(x,v) \longmapsto x$

$$\delta_{\hat{S}}(O) = \frac{\mathrm{Area}(\mathrm{proj}(\hat{S} \cap O))}{\mathrm{Area}(S)}.$$

(Proba on T^1M: analogous to a Birkhoff average for the geodesic flow) $\subset M$

Th [ALS]: There exist a sequence of compact surfaces $(S_i)_{i \in \mathbb{N}}$ s.t.

1/ S_i is C_i-quasiFuchsian $\lim_{i \to \infty} C_i = 1$

2/ $(\delta_{\hat{S}_i})$ converges to a proba. measure ν on T^1M s.t.

a/ ν totally invariant by $\mathcal{F}$ (locally: $\mathrm{area}_{\mathrm{Leaf}} \times \underbrace{\mu_{\mathrm{Transv}}}_{\text{Holonomy invariant}}$)

b/ $\mathrm{Supp}\,\nu = T^1M$.

Cor: If $\mathrm{sect}_h = -1$ in all $\hat{S}_i$ we have $\mathrm{sect} = -1$ ν-a.e $\Rightarrow$ $\mathrm{sect} = -1$ everywhere.

Rigidity of the energy spectrum

$S \subset M$. The energy of S is $W_h(S) = \int_S |H| \, dArea_h$

Th [ALS '24]. $\begin{matrix} -1 \leq sect_h \leq -\varepsilon \\ 0 < k < \varepsilon \end{matrix}$ $\forall \, \Gamma$ quasi fuchsian,

$W_h(S_{k,\Gamma}) = W_{h_0}(S_{k,\Gamma})$ ← --- (area of $\hat{S}_{k,\Gamma}$ in T^1M ...)

$\Longrightarrow h \simeq h_0$

Th [ALS '24]: $-b \leq sect_h \leq -a < 0$, $0 < k < a$.

Assume that the (normalized) energy and area spectra of (M,h) are asymptotic (this is the case in constant sect curvature):

$\forall (S_i)$ C_i-QF k-surface w. $C_i \longrightarrow 1$

$$\frac{W_h(S_i)}{\sqrt{k}\, Area_h(S_i)} \xrightarrow[i \to \infty]{} 1$$

Then $sect_h$ is constant.

For the last thm, we prove that if the hypothesis holds, then all leaves of $\mathcal{F}$ are totally umbilical w. constant mean curvature.

Criterion of Leung-Nomizu:

If through any plane $P \subset TM$ passes a totally umbilical surface w. constant mean curvature, then M is locally symmetric

How to use it?

Prove that if S_i is a sequence of q.F surface that equidistribute for $\mathcal{F}$, then $\left(\frac{H}{\sqrt{k}}\right)_{S_i} \longrightarrow 1$ so $H = \sqrt{k}$ for ν-a.e. leaves.