

24. Pricing Fixed Income Derivatives

through Black's Formula

MA6622, Ernesto Mordecki, CityU, HK, 2010.

References for this Lecture:

John C. Hull, *Options, Futures & other Derivatives (Fourth Edition)*, Prentice Hall (2000)

Plan of Lecture 24

(24a) Bond Options

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24a. Bond Options

A **bond option** is a contract in which the underlying asset is a bond, in consequence, a derivative or secondary financial instrument.

An example can be the option to buy (or sell) a 30 US Treasury Bond at a determined strike and date¹.

Bond options are also included in **callable bonds**. A callable bond is a coupon bearing bond that includes a provision allowing the issuer of the bond to buy back the bond at a predetermined price and date (or dates) in the future.

When buying a callable bond we are:

- Buying a coupon bearing bond
- Selling an European bond option (to the issuer of the

¹Options on US bonds of the American Type, i.e. they give the right to buy/sell the bond at any date up to maturity.

bond)

Similar situations arises in **putable bonds**, that include the provision for the holder to demand an early redemption of the bond at certain predetermined price, and at a predetermined date(s).

When buying a putable bond, we are

- Buying a coupon bearing bond
- Buying a put option on the same bond.

This are called **embedded** bond options, as they form part of the bond buying contract.

24b. Black's Model for European Options

A standard procedure to price bond options is Black's Formula (1976)² that was initially proposed to price commodities options.

Assume that we want to price an option written on a financial instrument with value V , in a certain currency. Define

- T : Maturity of the option.
- F : Forward price of V .
- F_0 : Value of F at time $t = 0$.
- K : strike price of the option.
- $P(t, T)$: Price at time t of a zero coupon bond paying 1 unity of the currency at time T
- $V(T)$: value of V at time T .

²Black, F. "The Pricing of Commodity Contracts" Journal of Financial Economics, 1976.

- σ : volatility of F .

The assumptions of Black's model are

- $V(T)$ has a lognormal distribution with standard deviation of $\log V(T)$ equal to $\sigma\sqrt{T}$.
- The expected value³ of $V(T)$ is F_0 .

Under this conditions, Black showed that the option price is

$$Call = P(0, T) [F_0 \Phi(d_1) - K \Phi(d_2)],$$

where

$$d_1 = \frac{\log(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}},$$

$$d_2 = \frac{\log(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}.$$

³under the log-normal risk neutral probability measure

Similarly, the value of a put option is given by

$$Put = P(0, T) [K\Phi(-d_2) - F_0\Phi(-d_1)].$$

The similarity with the BS formula is clear, being the main differences:

- There is no assumption on the **time dynamics** of the price of the financial instrument, the assumption is on time T
- The risk-free interest rate does not appear, it is taken into account in the zero coupon bond.
- The price of the financial instrument is substituted by the its forward price, that includes the (risk neutral) expectation about future behaviour of prices.

In this respects Black's formula is a generalization of Merton's time dependent Black-Scholes formula⁴.

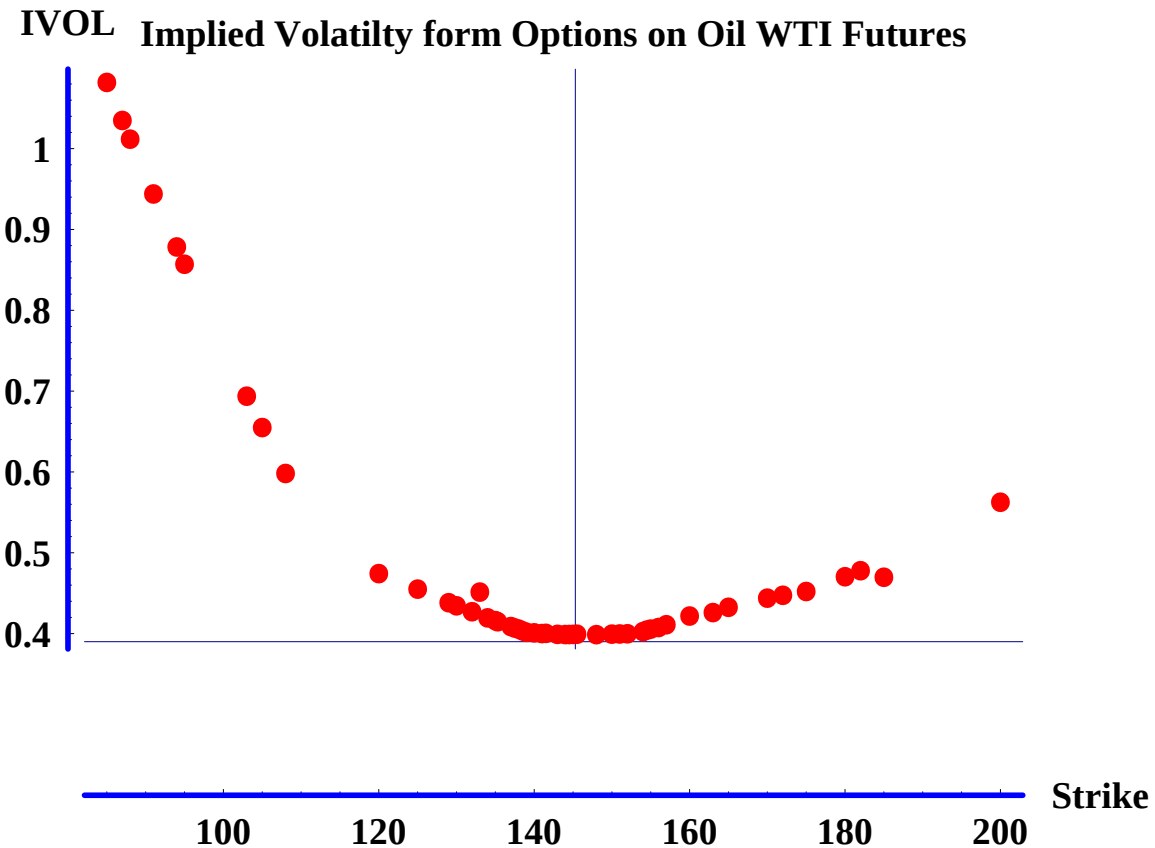
⁴Remember Lecture 16 "Time dependence in Black Scholes".

Example: Options on WTI Futures⁵

Black's 1976 formula was devised to price options on futures on commodities. Let us compute the implied volatility of Oil prices from quoted option prices.

- T : Today: 4/7/2008. Expire on 17/7/2008. $T=14/365$
- Quoted futures price is $F_0 = 145.49$ per barrel (NYMEX web page)
- Strike: all strikes with non zero volume (ranging from 85 to 200 US per barrel)
- $P(0, T) = e^{-rT}$. 4-week US T-Bill: $r = 0.0184$ (Federal reserve web page)

⁵In fact the options are american, and assumed to be european in this example



- Minimum implied vol is 40%, for at the money options.
- Spot futures price is 145.29.

24c. Pricing Bond Options

The pricing computations under the Black Model are similar to the BS pricing, with some minor differences.

One main difference is that here the quoted price, or clean price, should be corrected in order to obtain the cash (or dirty) price. This correction applies both for the spot and the strike price.

Example Compute the Bond Call Option price under the following characteristics⁶. The underlying is a 9.75 year Bond with a face value of \$1,000. Suppose that

- The option expires in 10 months.
- Current quoted (clean) bond price is \$935
- The (clean) strike price is \$1,000

⁶Taken from John C. Hull, *Options, Futures & other Derivatives (Fourth Edition)*, Prentice Hall (2000)

- The ten month risk free interest rate is 10% p.a.
- The volatility of the forward bond price in 10 months is 9% p.a.
- The bond pays a semiannual coupon of 10% p.a. interest rate
- Coupon payments (\$50 each) are expected in three and nine months.
- Risk-free interest rates for three and nine months are 9.0% and 9.5% p.a. respectively.

The procedure to determine the necessary input in Black's formula follows.

Step 1. The maturity is $T = 10/12$ is in the contract.

Step 2. We compute the cash bond price. The accrued interest is \$25, as the coupon is \$50 to be paid in three months (i.e. $f = 1/2$).

In consequence the bond cash price is

$$935 + 25 = 960.$$

Step 3. Now we compute the forward price of the bond, beginning by discounting the coupons the holder will receive.

The present value of the two coupons to be paid are

$$50 \exp(-0.09 \times 0.25) + 50 \exp(-0.095 \times 0.75) = 95.45.$$

Then, the futures price is

$$F_0 = (960 - 95.45) \exp(0.1 \times 10/12) = 939.68$$

Step 4. We compute the **cash** strike price to use in the formula, taking into account that at month 10, we have 1 month since the last coupon payment date. The cash strike price is

$$1,000 + 50 \times (1/6) = 1008.33$$

Step 5. We have all the input for Black's formula:

$$P(0, T) = \exp(-0.1 \times 10/12) = 0.92$$

$$F_0 = 939.68$$

$$K = 1008.33$$

$$\sigma = 0.09$$

$$T = 10/12 = 0.833$$

We obtain

$$d_1 = \frac{\log(939.68/1008.33) + (0.09)^2(0.833)/2}{0.09\sqrt{0.833}} = -0.681,$$

$$d_2 = d_1 - 0.09\sqrt{0.833} = -0.749.$$

So the call price is

$$0.92[939.68\Phi(-0.681) - 1008.33\Phi(-0.749)] = 7.968.$$

24d. Yield volatilities

Undoubtedly the most difficult parameter to determine in the previous computation is the volatility. The usual procedure is to infer some implied volatility to be used in the formula.

If instead of the bond volatility, we get quotes of **yield** volatilities, we obtain the approximate bond volatilities by the following procedure.

We know the form of dependence of the value of a bond and its yield $V = V(y)$, and we have obtained that

$$\frac{V'(y)}{V(y)} = -D,$$

where D is the duration of the underlying bond.

If we approximate $V'(y) = \Delta V / \Delta y$, multiplying and divid-

ing by y , we have

$$\frac{\Delta V}{V} = -(Dy) \left(\frac{\Delta y}{y} \right).$$

Taking the variance in both sides

$$\sigma_{Bond}^2 = \mathbf{var} \left(\frac{\Delta V}{V} \right) = (Dy)^2 \mathbf{var} \left(\frac{\Delta y}{y} \right) = (Dy)^2 \sigma_y^2,$$

that gives

$$\sigma_{Bond} = (Dy) \sigma_y.$$

- We are assuming that

$$\frac{dV}{V} = \mu dt + \sigma_{Bond} dW,$$

- We see that the larger the duration, for the same yield, the higher the Bond's volatility.

24e. Interest rate options

Instead of writing an option on a bond, it is possible and usual to write an option on a **floating interest rate**, typically the Libor.

These options are produced in order to protect the buyer against large up or down movements of interest rates, and are called respectively ⁷ **caps**, to protect against high floating interest rates, and **floors** to protect against low floating interest rates.

Consider a series of dates $t_1 < \dots < t_n$, with $\Delta = t_{k+1} - t_k$ fixed (the **tenor**) and denote by $r_1 < \dots < r_n$ the floating interest rate, applied for the period between dates t_k and t_{k+1} .

⁷You can think that you have a short position on a floating interest rate coupon Bond, and want to protect yourself against interest rates over a certain level.

Consider a principal P and a fixed interest rate r_c , the **cap rate**.

The holder of a **cap** will receive the difference of the interest generated in the period t_k, t_{k+1} computed with the floating rate r_k and the fixed cap rate r_c , if this difference is positive, i.e. when $r_k > r_c$.

The first accrued interest is

$$P \times \Delta \times r_k,$$

while the second is

$$P \times \Delta \times r_c.$$

So the holder of a cap will receive

$$\max(P\Delta r_k - P\Delta r_c, 0) = P\Delta \max(r_k - r_c, 0) = P\Delta (r_k - r_c)^+$$

In consequence, a cap will provide the holder with the following cash-flow:

Dates	t_2	t_3	$\dots$	t_n
Amount	$P\Delta(r_1 - r_c)^+$	$P\Delta(r_2 - r_c)^+$	$\dots$	$P\Delta(r_{n-1} - r_c)^+$

Similarly, a **floor** with principal P and interest rate r_f , on dates $t_1, \dots, t_n$ will provide to the holder the following pay-off:

Dates	t_2	t_3	$\dots$	t_n
Amount	$P\Delta(r_f - r_1)^+$	$P\Delta(r_f - r_2)^+$	$\dots$	$P\Delta(r_f - r_{n-1})^+$

Remarks

- Each individual payment of a cap is called a **caplet**, and is in fact a call option written on the floating interest rate observed at t_k to be paid at t_{k+1} .
- There is a put-call parity relating prices of caps and floors. If both contracts have the same scheduled dates, on a same principal, and if $r_c = r_f$, then

$$\text{cap price} = \text{floor price} + \text{value of swap}$$

as is equivalent to hold a floor and a swap than a cap.

- A **collar** is a contract that pays when the interest rate is above a prefixed rate r_c or below another prefixed rate r_f , and is a combination of a long position in a cap and a short position in a floor.

24f. Pricing Interest rate options

The pricing of interest rate options can also be priced with Black's formula.

Consider an individual caplet, that provides, at time t_{k+1} , a payoff

$$P\Delta(r_k - r_c)^+,$$

where r_k is the (floating) interest rate observed at time t_k .

In order to price the caplet, assuming that the rate r_k is lognormal with volatility σ_k , as we have to different time dates involved:

- Date t_k when the rate is observed,
- Date t_{k+1} when the payment is effective,

we use the following **time modified Black's formula**, dis-

counting up to the moment of payment:

$$Call = P \Delta P(0, t_{k+1}) [F_k \Phi(d_1) - r_c \Phi(d_2)]$$

where

$$d_1 = \frac{\log(F_k/r_c) + \sigma_k^2 t_k/2}{\sigma_k \sqrt{t_k}}, \quad d_2 = d_1 - \sigma_k \sqrt{t_k}.$$

Example Consider a contract that caps the interest rate on a $P = \$10,000$ loan at $r_c = 8\%$ p.a., with quarterly compounding (all interest rates will be with quarterly compounded), for three months, starting in one year. Assume that we know:

- The forward interest rate for 3 months, starting in one year is 7%,
- The current 15 months interest rate is 6.5%,
- The volatility of the forward three month rate underlying

the caplet is 20%,

The dates of the contract are:

$t = 0$	$t_k = 1$ (year)	$t_{k+1} = 1 + \Delta, \Delta = 1/4$ (3 mths)
Pay the caplet	Observe r_k	Receive $P\Delta(r_k - r_c)^+$

With this information:

- The forward interest rate is $F_k = 0.07$.
- The principal is $P = 10,000$.
- The cap interest is 0.08
- $\Delta = 3/12 = 0.25$
- The discount is $P(0, t_{k+1}) = \exp(-1.25 \times 0.065) = 0.922$,
- $\sigma_k = 0.20$, and $t_k = 1$.

We compute

$$d_1 = -0.5677, \quad d_2 = -0.7677$$

to obtain the price of the caplet as

$$\begin{aligned} &0.25 \times 10,000 \times 0.922 \\ &\times [0.07\Phi(-0.5677) - 0.08\Phi(-0.7677)] = 5.19 \end{aligned}$$

This should be compared to

$$P \times \Delta \times (0.09 - r_c) = 10,000 \times 0.25 \times 0.01 = 25$$

the difference of interest to be paid if $r_k = 0.09$.

25. Interest rates models

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Plan of Lecture 25

- (25a) Modelling the interest rates
- (25b) Equilibrium spot rates models (Vasicek)
- (25c) Other equilibrium models
- (25d) Pricing Bond options under Vasicek model
- (25e) Arbitrage Free pricing Models
- (25f) Comments on more advanced models

25a. Modelling the interest rates

Black's Formula to price interest rate derivatives makes the assumption of log-normality at the maturity time of the underlying.

In order to price time dependent derivatives (as American options) or simply, more general distributed financial instruments it is necessary to develop a model for the evolution of the whole term structure of interest rates.

Let us now define the quantities we observe in the market:

- The price of zero coupon bonds at t , paying 1 at maturity T , denoted by $P(t, T)$.
- Equivalently, we observe the continuously compounded yield of the zero coupon bonds, defined as

$$Y(t, T) = -\frac{\log P(t, T)}{T - t}.$$

based on the relation

$$P(t, T) = 1 \times \exp \left[- (T - t)Y(t, T) \right].$$

25b. Equilibrium spot rates models (Vasicek)

A first approach to model the zero coupon bond prices, based on economics equilibrium arguments, assumes the existence of a

- a stochastic **instantaneous interest rate**, denoted by $r(t)$, also called the **spot rate** or **short rate**, that (theoretically) is the interest corresponding to a short time interval of the form $t, t + h$.

that satisfies the diffusion equation:

$$dr(t) = a(b - r(t)) dt + \sigma dW(t),$$

where a, b and σ are positive constants, $\{W(t)\}$ is a Wiener Process, and the process departs from some $r(0) = r_0 > 0$.

This is the Vasicek model⁸.

⁸Vasicek, O. "An equilibrium characterization of the term structure", Journal of Financial Economics, 1977, pp. 177-188.

It can be proved (with the help of Itô's formula) that the interest rates can be computed through:

$$r(t) = b + e^{-at}(r_0 - b) + \sigma e^{-at} \int_0^t e^{as} dW(s)$$

what gives that $r(t)$ is a gaussian random variable, with

$$\begin{aligned}\mathbf{E} r(t) &= b + e^{-at}(r_0 - b), \\ \mathbf{var} r(t) &= \frac{\sigma^2}{2a} [1 - e^{-2at}].\end{aligned}$$

In particular, for big values of t we obtain

$$\mathbf{E} r(t) \sim b, \quad \mathbf{var} r(t) \sim \frac{\sigma^2}{2a}.$$

Vasicek shows that the price of a zero coupon bond is

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

where r is the value of r at time t , and

$$B(t, T) = (1 - \exp(-a(T - t))) / a,$$

$$A(t, T) = \exp \left[\frac{(B(t, T) - T + t)(a^2 b - \sigma^2 / 2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a} \right]$$

From this we obtain the equation for the yield curve, as

$$Y(t, T) = \frac{-1}{T - t} \log A(t, T) + \frac{1}{T - t} B(t, T) r(t).$$

In order to **calibrate** the model, we introduce $\tau = T - t$, and $r(t) = r_0$. Then (with a slightly abuse of notation), we have

$$Y(\tau) = \frac{1}{\tau} \left[r_0 B(\tau) - \log A(\tau) \right],$$

where

$$B(\tau) = \frac{1 - e^{-a\tau}}{a},$$

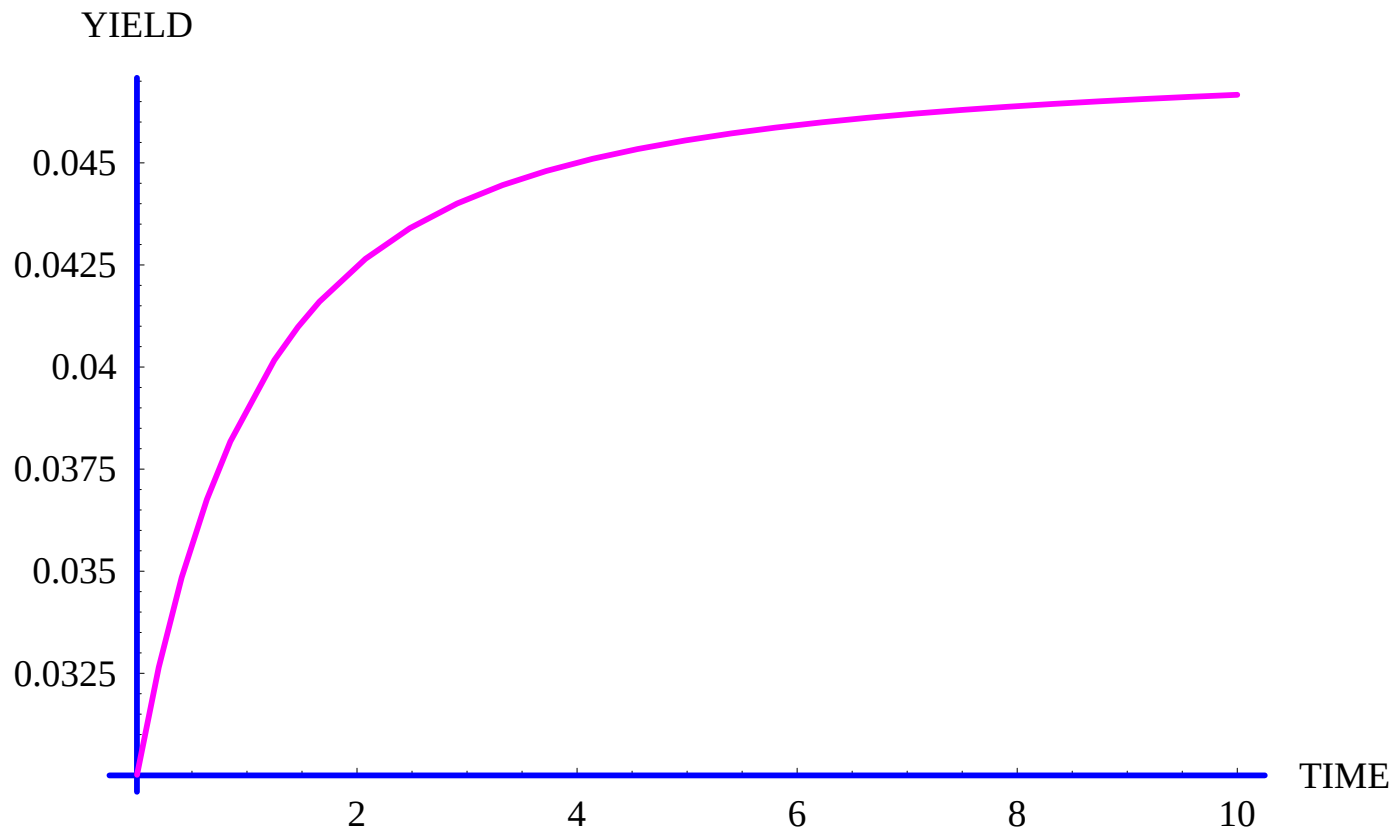
and

$$-\log A(\tau) = \left(b - \frac{\sigma^2}{2a^2}\right)(\tau - B(\tau)) + \frac{\sigma^2}{4a}B(\tau)^2.$$

We then observe:

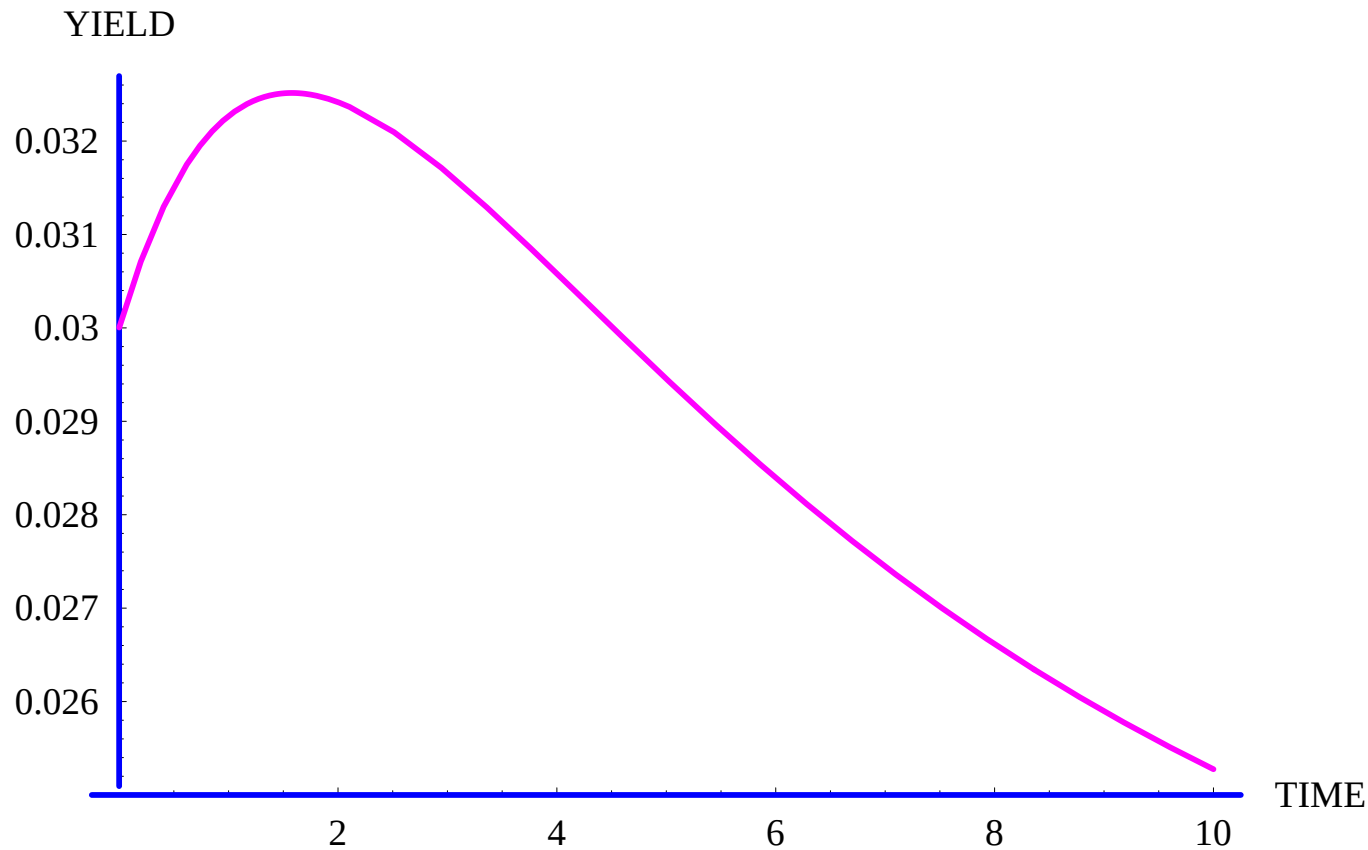
- $Y(0) = r_0$,
- $Y'(0) = a(b - r_0)/2$, so in order to have positive slope at $\tau = 0$ we require $b > r_0$.
- $\lim_{\tau \rightarrow \infty} Y(\tau) = b - \frac{\sigma^2}{2a^2}$ and this defines the form of the yield curve.

Let us see the three typical forms the yield curve has in Vasicek model:



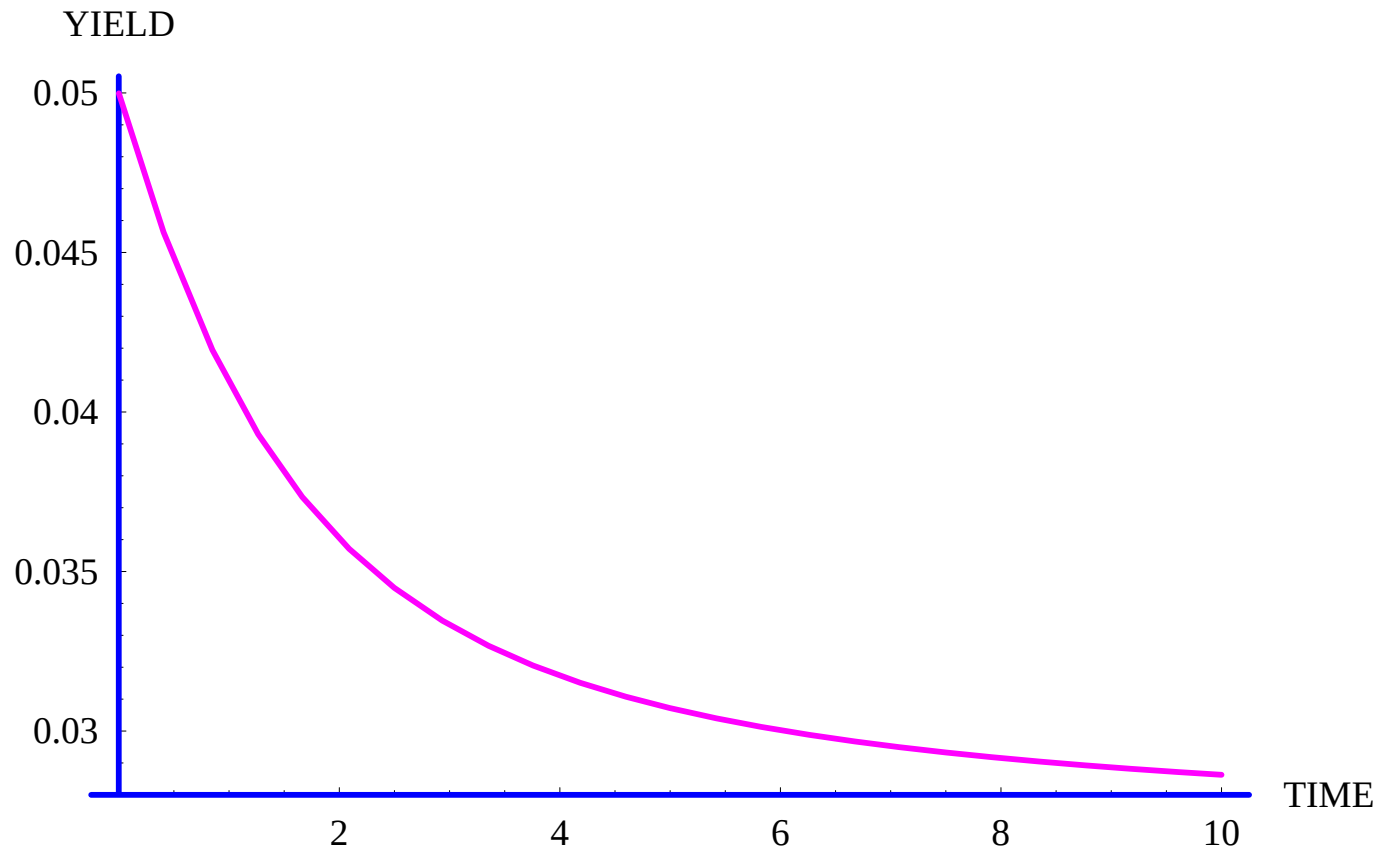
The value of the parameters are $a = 1.5$, $r_0 = 0.03$, $b = 0.05$, $\sigma = 0.10$. The characteristic values are $Y(0) = r_0 = 0.03$, $Y'(0) = 0.015$, $Y(\infty) = 0.0477$,

This is a **normal** yield curve.



The value of the parameters are $a = 0.3$, $r_0 = 0.03$, $b = 0.05$, $\sigma = 0.10$, The characteristic values are $Y(0) = r_0 = 0.03$, $Y'(0) = 0.03$, $Y(\infty) = 0$,

This is a **humped** yield curve.



The value of the parameters are $a = 1.2$, $r_0 = 0.05$, $b = 0.03$, $\sigma = 0.10$, The characteristic values are $Y(0) = r_0 = 0.05$, $Y'(\infty) = -0.012$, $Y(\infty) = 0.03$,

This is an **inverted** yield curve.

25c. Other equilibrium models

It must be observed that, as the interest rates in Vasicek model are gaussian random variables, the interest rates can take negative values⁹.

An alternative similar model was proposed by Cox, Ingersoll and Ross¹⁰. It assumes that the interest rate process satisfies the diffusion

$$dr(t) = a(b - r(t)) dt + \sigma \sqrt{r(t)} dW(t),$$

where a, b and σ are positive constants, $\{W(t)\}$ is a Wiener Process, and the process departs from some $r(0) = r_0 > 0$.

In this model, called the CIR model, Bond prices have the

⁹If the model is correctly calibrated this will happen with very small probability

¹⁰J.C. Cox, J.E. Ingersoll Jr, S.A. Ross, A Theory of the Term Structure of Interest Rates
Econometrica, Vol. 53, No. 2 (March, 1985)

same form as in Vasicek model

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

but with different formulas for $A(t, T)$ and $B(t, T)$.

Some other characteristics of the CIR model are:

- The model produces the same type of yield curves as in Vasicek model.
- The authors provide closed formulas for European call and put options on zero coupon bonds.
- In order to calibrate it, it is necessary to fit parameters a, b, σ .
- The interest rates always remain positive.

25d. Pricing Bond options under Vasicek model

First consider zero coupon bonds. Jamshidian¹¹ obtained the formula for a call option on a zero coupon bond with:

- a, σ the parameters in Vasicek model,
- The principal Pr .
- Bond maturity T ,
- Option maturity $t < T$,
- Strike K ,

as

$$Call = Pr. \times P(0, T)\Phi(h) - K \times P(0, t)\Phi(h - \sigma_P),$$

where the constants h and σ_P are:

¹¹Jamshidian, Farshid “An Exact Bond Option Formula” The Journal of Finance (1989)

$$h = \frac{1}{\sigma_P} \log \left[\frac{Pr. \times P(0, T)}{K \times P(0, t)} \right] + \frac{\sigma_P}{2},$$

$$\sigma_P = \frac{\sigma}{a} \left[1 - e^{-a(s-T)} \right] \sqrt{\frac{1 - e^{-2aT}}{2a}}.$$

Decomposing a coupon bearing bond, as a series of zero coupon bonds, corresponding to Jamshidian obtains the price of coupon bearing bonds.

25e. Arbitrage Free pricing Models

The tale of bond pricing only begins with Vasicek (1976) model.

The paradigm of **risk-neutral martingale measures** enters into the fixed-income securities to produce **arbitrage free** pricing models.

In particular, this models fit exactly (theoretically) the term structure of interest rates.

In equilibrium models today's term structures of interest rates is an output of the model, whereas in arbitrage free interest rates it is an input.

This is possible at the cost of assuming that the drift in the equation governing the interest rate is time dependent.

An example of a risk-free interest rate is the Ho and Lee model¹²

$$dr(t) = \theta(t)dt + \sigma dW(t),$$

where σ is constant, and $\theta(t)$ is a function chosen consistent with the initial term structure.

In this simple model the function θ can be calculated analytically, as

$$\theta(t) = \frac{\partial}{\partial t}F(0, t) + \sigma^2 t.$$

where $F(0, t)$ is the instantaneous forward rate for maturity t .

¹²T. Ho; S. Lee “Term Structure Movements and Pricing Interest Rate Contingent Claims”
The Journal of Finance, (1986).

Characteristics of the Ho and Lee model are:

- It gives arbitrage free prices for bond options
- Zero coupon bond prices can be computed also as

$$P(t, T) = A(t, T)e^{-B(t, T)r}$$

but with corresponding analytic formulas for $A(t, T)$ and $B(t, T)$.

25f. Comments on more advanced models

- Two factor models for interest rates have also been proposed in the literature.
- A tree construction, similar to the Derman Kani tree can also be used to price fixed income securities.

An relevant step in the modelling of fixed income securities has been proposed by Heath Jarrow and Morton.

The main innovation in the HJM approach is that they model the zero coupon bonds directly, instead of modelling the interest rate.

In consequence, the HJM models is called a **forward rate model**, in contrast with the previous models, known as **short rate models**.

Consider the formula

$$P(t, T) = \exp \left[- \int_t^T f(t, s) ds \right]$$

where $f(t, s)$ is the forward interest rate observed today of a loan with duration $s, s + h$.

In particular, when $s = t$ we obtain as a particular case

$$f(t, t) = r(t),$$

the spot interest rate.