14-15. Calibration in Black Scholes Model and Binomial Trees

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References for this Lecture:

John C. Hull, *Options, Futures & other Derivatives (Fourth Edition)*, Prentice Hall (2000)

Marco Avellaneda and Peter Laurence, *Quantitative Modelling of Derivative Securities*, Chapman&Hall (2000).

Paul Willmot *Paul Willmot on Quantitative Finance*, Volume 1, Wiley (2000).

Main Purposes of Lectures 14 and 15:

- Introduce the notion of Calibration
- Examine how to calibrate the different parameters in BS
- Define and compute implied volatility
- Term structure and matrices of implied Volatilities
- Review Binomial Trees
- Calibrate them, and compute prices of European and American Options.

Plan of Lecture 14

(14a) Calibration

(14b) Black Scholes Formula (Revision)

(14c) Implied Volatility

(14d) Time-dependent volatility

14a. Calibration

The calibration of a mathematical model in finance is the determination of the risk neutral parameters that govern the evolution of a certain price process $\{S(t)\}$.

As we have seen, the martingale hypothesis assumes that there exists a probability measure $\mathbf{Q}$, equivalent to $\mathbf{P}$, such that our discounted price process $\{S(t)/B(t)\}$ is a martingale (here $\{B(t)\}$ is the evolution of a riskless savings account, usually $B(t) = B(0)\exp(rt)$).

- $\mathbf{P}$ is the historical or physical probability measure. We use statistical procedures to fit it to the data. It reflects the past evolution of prices of the underlying.
- $\mathbf{Q}$ is the risk neutral probability measure. It is calibrated through prices of derivatives written on the underlying.

The calibration of a model is performed observing the prices of certain derivatives written on the underlying $\{S(t)\}$, and fitting the parameters of the model, in such a way that it reproduces the observed derivative prices.

The purpose of calibration is to compute prices of not so liquid derivatives instruments, or more complex instruments.

The calibration procedure should be contrasted to the statistical fitting procedure:

- When statistically fitting a model, we take information from the quoted prices of the underlying, to determine $\mathbf{P}$.
- When calibrating a model, we need to know prices of derivatives written on the underlying, to determine $\mathbf{Q}$.

14b. Revision of Black Scholes Formula

Assume that we have a market model with two assets:

- A savings account $\{B(t)\}$ evolving deterministically according to

$$B(t) = B(0)e^{rt},$$

where r is the riskless interest rate in the market, and

- A stock $\{S(t)\}$ with random evolution of the form

$$S(t) = S(0)\exp\left((\mu - \sigma^2/2)t + \sigma W(t)\right),$$

where $\{W(t)\}$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Here σ is the volatility, and μ the rate of return of the considered stock.

Black-Scholes[1] formula gives the price C of an European Call Option written on the stock, as

$$C = C(S(0); K; T; r; \sigma) = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

- $S(0)$ is the spot price of the stock, measured in local currency.
- K is the strike price of the option, in the same currency.
- T is the maturity of the option, measured in years,
- r is the annual percent of risk free interest rate,
- σ is the volatility (also annualized).
- $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$ is the distribution function of a normal standard random variable,

[1] Robert Merton and Myron Scholes received the Nobel Prize in Economics in 1997 "for a new method to determine the value of derivatives". Fischer Black died in August 1995, the Nobel prize was never given posthumously.

- $d_1 = \frac{\log[S(0)/K]+[r+\sigma^2/2]T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$.

The value of a Put Option has a similar formula:

$$P = Ke^{-rT}\Phi(-d_2) - S(0)\Phi(-d_1),$$

Remark The option price does not depend on μ. This is due to the fact that the price is computed under the risk neutral probability $\mathbf{Q}$.

More precisely, the evolution of the stock under $\mathbf{Q}$ is

$$S(t) = S(0) \exp\left((r - \sigma^2/2)t + \sigma W(t)\right), \qquad (1)$$

where now $\{W(t)\}$ is a Wiener process under the risk-neutral measure $\mathbf{Q}$. The value can be computed as

$$C(S(0); K; T; r; \sigma) = \mathbf{E}_{\mathbf{Q}}\, e^{-rT}\big(S(T) - K\big)^+$$

where the expectation is taken with respect to $\mathbf{Q}$ (i.e. for a stock modelled by (1))

Example Let us compute the price of a call option written on the Hang Seng Index[2], with (that day's) price $S(0) = 15247.92$, struck at $K = 15000$ expiring in July, with a volatility of 22%.

In order to compute the other parameters we take into account:

- the underlying of the option is $50\times$ HSI, but this is not relevant for the option price (why?)
- The option was written on June 14, and it expires in the business day immediately preceding the last business of the contract month (July): the expiration date is July 29.
- We then have 32 trading days. As the year 2006 has 247 trading days, we obtain $T = 32/247$.

We compute the risk-free interest rate from the Futures prices, written on the same stock over the same period. We get a futures quotation of $F(T) = 15298$ for July 2006. Then, as $F(T) =$

[2]From the "South China Morning Post", June 15, 2006

$S(0)\exp(rT)$, we obtain

$$r = \frac{\text{year}}{\text{days}} \log\left(\frac{F(T)}{S(0)}\right) = \frac{247}{32} \log\left(\frac{15298}{15248}\right) = 0.025.$$

With this information, we compute

$$C(15248; 15000; 32/247; 0.025; 0.22) = 639.72.$$

(Newspaper quotation was 640.)

Just we are here, we compute by put-call parity the price of the put option with the same characteristics. Put-Call parity states that

$$C + Ke^{-rT} = P + S(0),,$$

that, in numbers, is

$$P = 640 + 15000e^{0.025\times(32/247)} - 15248 = 343.5.$$

(Quoted price was 342.)

14c. Implied Volatility

In the previous example, everything is clear with one relevant exception: Why did we used $\sigma = 0.22$?

In fact, the real computation process, in what respects the volatility is the contrary: we know from the market that the option price is 640, and from this quotation we compute the volatility. The number obtained is what is called implied volatility, and should be distinguished from the volatility in (1).

It must be noticed that there is no direct formula to obtain σ from the Black Scholes formula, knowing the price C.

In other words, the equation

$$C(15248; 15000; 32/247; 0.025; \sigma) = 639.72.$$

can not be inverted to yield σ. We then use the Newton-Raphson method to find the root σ.

Newton-Raphson method

Suppose that you want to find the root x of an equation $f(x) = y$, where f is an increasing (or decreasing) differentiable function, and we have an initial guess x_0. By Taylor expansion

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0).$$

If we want x to satisfy $f(x) = y$, then it is natural to assume that

$$f(x_0) + f'(x_0)(x - x_0) = y,$$

and from this we

$$x = x_0 + \frac{y - f(x_0)}{f'(x_0)}.$$

The obtained value of x is nearer to the root than x_0. The Newton Raphson method consists in computing a sequence

$$x_{k+1} = x_k + \frac{y - f(x_k)}{f'(x_k)}$$

that converges to the root.

Let us then find the implied volatility of given the quoted option price QP. The derivative of the function C with respect to σ is called vega, and is computed as

$$\text{vega}(\sigma) = S(0)\sqrt{T}\varphi(d_1), \text{ with } d_1 \text{ as above.}$$

So, given an initial value for σ_0, we compute $C(\sigma_0)$, and obtain our first approximation:

$$\sigma_1 = \sigma_0 + \frac{C(\sigma) - C(\sigma_0)}{S(0)\sqrt{T}\varphi(d_1)}.$$

where $C(\sigma) = QP$. If this is close to σ_0 we stop. Otherwise, we compute σ_2 and stop when the sequence stabilizes.

Example Let us compute the implied volatility in the previous example. Suppose we take an initial volatility of $\sigma_0 = 0.15$.

Step 1. We compute $C(0.15) = 495$.

Step 2. We compute $\text{vega}(0.15) = 2029$

Step 3. We correct

$$\sigma_1 = 0.15 + \frac{640 - 495}{2029} = 0.2216$$

Step 4. We compute $C(0.2216) = 643.116$. We are really near to the implied volatility.

Step 5. We compute again $\text{vega}(0.2216) = 2101.75$,

Step 6. Finally we obtain

$$\sigma_2 = 0.2216 + \frac{640 - 643.16}{2101.75} = 0.220134,$$

and we are done.

14d. Time-dependent volatility

Black Scholes theory assumes that volatility is constant over time. We have seen that, from a statistical point of view, that volatility varies over time.

What happens with respect to the risk-neutral point of view? In other terms, is implied volatility constant over time?

Month	Strike	Price	Volit %	Futures	r
June	14400	905	27	15241	-0.010
July	14400	1079	24	15298	0.025
August	14400	1117	23	-	-
September	-	-	-	15296	0.010

In this table[3] we see that the implied volatility (Volit) also varies over time. This series of implied volatilities for of at-the-money op-

[3] Taken from "South China Morning Post", June 15, 2006. The spot price is $S(0) = 15247.92$

tions with different maturities is called the term structure of volatilities. We included the Futures prices and the corresponding risk free interest rates[4] computed from this price, through the formula $F(T) = S(0)\exp(rT)$

This has a remedy in the frame-work of BS theory. Assume that

- There exists a deterministic instantaneous (non observable) risk free interest rate $r(s)$,
- There exists a deterministic instantaneous (non observable) volatility $\sigma(s)$.

We define the forward interest rate, and the forward volatility, as

$$r(t,T) = \frac{1}{T-t}\int_t^T r(s)ds, \qquad \sigma(t,T)^2 = \frac{1}{T-t}\int_t^T \sigma(s)^2 ds. \tag{2}$$

here t is today, and T is the maturity of the option.

[4]The interest rate is negative due to the fact that futures price is smaller than the spot price

In this model we have a Black-Scholes pricing formula for a Call option:

$$C(S(t);K;T;r;\sigma) = S(t)\Phi(d_1) - Ke^{-r(t,T)\theta_t}\Phi(d_2)$$

where $\theta_t = T - t$, and

$$d_1 = \frac{\log[S(t)/K] + [r(t,T) + \sigma(t,T)^2/2]\theta_t}{\sigma(t,T)\sqrt{\theta_t}},$$

$$d_2 = d_1 - \sigma(t,T)\sqrt{\theta_t}.$$

In practice, we do not know the complete curves $r(s)$ and $\sigma(s)$. In order to use the time-dependent BS formula we assume that $r(s)$ are $\sigma(s)$ are constant between the different expirations.

Plan of Lecture 15

(15a) Volatility Smile

(15b) Volatility Matrices

(15c) Review of Binomial Trees

(15d) Several Steps Binomial Trees

(15e) Pricing Options in the Binomial Model

(15f) Pricing American Options in the Binomial Model

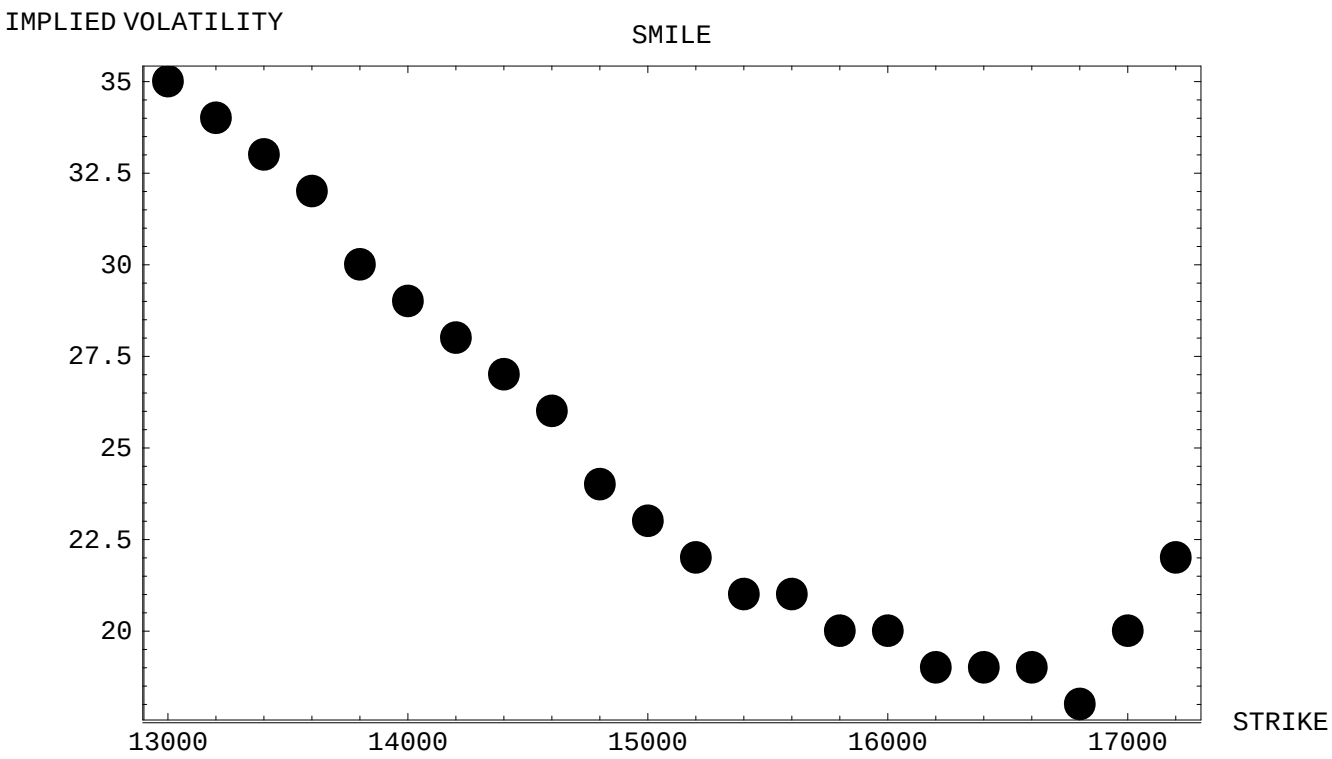
15a. Volatility Smile

Let us see more in details the quotations of option prices[5],

Month	Strike	Price	Volit %
June	13000	2246	35
June	13200	2048	34
June	13400	1851	33
June	13600	1656	32
June	13800	1462	30
June	14000	1272	29
June	14200	1086	28
June	14400	905	27
June	14600	733	26
June	14800	571	24
June	15000	424	23

Month	Strike	Price	Volit %
June	15200	296	22
June	15400	199	21
June	15600	124	21
June	15800	72	20
June	16000	38	20
June	16200	18	19
June	16400	7	19
June	16600	3	19
June	16800	1	18
June	17000	1	20
June	17200	1	22

[5] "South China Morning Post", June 15, 2006



We see that the volatility, far from constant, varies on the strike prices, forming a smile, or, more precisely, a smirk.

This clear fact shows that real markets do not follow Black-Scholes theory.

15b. Volatility Matrices

Volatility matrices combine volatility smiles with volatility term structures, and are used to compute options prices.

	14400	14600	14800	15000	15200	15400	15600
Jun	27	26	24	23	22	21	21
Jul	24	24	23	22	21	21	21
Aug	23	-	-	-	21	20	20

	15800	16000	16200	16400	16600	16800	17000
Jun	20	20	19	19	19	18	20
Jul	21	21	20	20	20	20	19
Aug	20	20	19	19	19	18	18

With this matrix in view, we can compute implied volatilities with a reported strike and arbitrary expiration, and with a reported expiration and arbitrary strike.

In order to compute the implied volatility for a non reported expiration and a non reported strike (for instance, expiration on July 21 and strike 16500) we can compute

- First, by linear interpolation in time, obtain both values of implied volatility at strike 16600 and 16400 for the given expiry.
- Second, use this values, interpolating in strike, to obtain the desired implied volatility.

The problem here is that the process computing first the volatilities interpolating in strike, and second in time, can produce a different value.

In fact, more complex models are needed, as the procedure of strike interpolation has only an empirical basis.

15c. Review of Binomial Trees

Our interest is then to consider more flexible models, with more parameters. Let us first consider the one step binomial tree.

Consider then a risky asset with value $S(0)$ at time $t = 0$, and, at time $t = 1$, value

$$S(1) = \begin{cases} S(0)u & \text{with probability } p \\ S(0)d & \text{with probability } 1-p \end{cases}$$

Here u and d stand for up and down. We are then assuming that the returns X defined by

$$\frac{S(1)}{S(0)} - 1 = X$$

satisfy

$$1 + X = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases}$$

Let us calibrate this model, i.e. determine the values of the parameters u, d, p under the risk neutral measure.

Denoting by r the continuous risk free interest rate, the first condition is that $e^{-rt}S(t)$ is a martingale.

In this simple case, this amounts to $\mathbf{E}\, S(1) = S(0)e^r$, that gives:

$$up + d(1-p) = e^r.$$

Given a value σ of the implied volatility (computed from some traded derivative), we impose $\mathbf{var}\, X = \sigma^2$. Let us compute

$$\mathbf{var}\, X = \mathbf{var}(1+X) = \mathbf{E}\left[(1+X)^2\right] - \left[E(1+X)\right]^2.$$

We have

$$\mathbf{E}(1+X) = up + (1-p)d$$
$$\mathbf{E}(1+X)^2 = u^2p + (1-p)d^2$$

giving the condition

$$\mathbf{var}\, X = u^2p + d^2(1-p) - (up + d(1-p))^2 = (u+d)^2p(1-p) = \sigma^2.$$

We have two equations for three parameters u, d, p. In order to determine the parameters, it is usual to impose $u = 1/d$[6]

These three conditions imply

$$p = \frac{e^r - d}{u - d}, \qquad u = e^{\sigma}, \qquad d = e^{-\sigma}.$$

[6]Cox, J., Ross,S. and Rubinstein, M. *Option Pricing: A simplified approach.* Journal of Financial Economics, 7 (1979).

Remark In practice we take a time increment Δ instead of one, and use annualized values of r and σ. The corresponding formulas are

$$p = \frac{e^{r\Delta} - d}{u - d}, \qquad u = e^{\sigma\sqrt{\Delta}}, \qquad d = e^{-\sigma\sqrt{\Delta}}.$$

Example Let us calibrate the Binomial Tree using the values of our first example on option pricing. We have $\Delta = 32/247$, $r = 0.025$, $\sigma = 0.22$. This gives

$$u = 1.082, \qquad d = 0.924, \qquad p = 0.500677.$$

15d. Several Steps Binomial Trees

In practice one assumes that $t = 0, \Delta, 2\Delta, \ldots, T$, with $T = N\Delta$, and construct the several step binomial tree under the assumption of time-space homogeneity.

This assumption is equivalent to the Black-Scholes assumption that the risk-free rate and the volatility are constant over time and space.

The result of this assumption is that at each of the two nodes, resulting from the first step, the future evolution of the asset price reproduces as in the first step.

In order to model the stock prices we label each node by (n, i), where n is the step, and i the number of upwards movements. At step n we have $i = 0, \ldots, n$, and we denote by $j = n - i$ the number of downwards movements. We obtain, given i, that the

stock price takes the value

$$S(n) = S(0)u^i d^{n-i} = s_n(i),$$

where $s_n(i)$ is a notation. Let us now compute the probability of reaching the value $s_n(i)$. We need exactly i ups (and $j = n - i$ downs), but they can come in different orders. There are

$$C_i^n = \frac{n!}{i!(n-1)!},$$

different ways of obtaining i ups, each has a probability p, they are independent, so

$$\mathbf{P}[S(n) = S(0)u^i d^j] = C_i^n p^i (1-p)^{n-i} = P_n(i).$$

(where $P_n(i)$ is a notation). The conclusion is that the stock price evolves according to the formula

$$S(n) = S(0)u^i d^{n-i}, \quad \text{with probability } P_n(i), \text{ for } i = 0, \ldots, n.$$

15e. Pricing Options in the Binomial Model

The principle applied to price derivatives is:

In a risk-neutral world individuals are indifferent to risk. In consequence, the expected return of any derivative is the risk-free interest rate.

As we have calibrated our probability $\mathbf{Q}$, a put option paying

$$\max(K - S(T), 0),$$

has a price

$$P = e^{-rT}\, \mathbf{E}_{\mathbf{Q}} \max(K - S(T), 0).$$

Which prices give a positive payoff? Denote by

$$i_0 = \max\{i \colon S(0)u^i d^j \leq K.\}$$

Then all values $i \leq i_0$ give positive payoff, while the others give null payoff.

The formula for the price is

$$P = e^{-rT} \sum_{i=0}^{i_0} \big(K - s_N(i)\big) P_N(i)$$

$$= Ke^{-rT} \sum_{i=0}^{i_0} P_N(i) - \sum_{i=0}^{i_0} e^{-rT} s_N(i) P_N(i)$$

$$= Ke^{-rT}\, \mathbf{P}(S(T) \le i_0) - S(0) \sum_{i=0}^{i_0} C_i^N e^{-rT} [up]^i [d(1-p)]^{N-i}$$

In can be shown, for big value of N, using the Central Limit Theorem (that approximates a Binomial random variable by a normal random variable), that

$$\mathbf{P}(S(T) \leq i_0) \sim \Phi(-d_2),$$

$$\sum_{i=0}^{i_0} C_i^N e^{-rT} [up]^i [d(1-p)]^{N-i} \sim \Phi(-d_1)$$

(where d_1 and d_2 are the values in BS formula) obtaining that, for N big

$$P \sim Ke^{-rT}\Phi(-d_2) - S(0)\Phi(-d_1),$$

the Black-Scholes price of a put option.

15f. Pricing American Options in the Binomial Model

Binomial trees are popular due to their simplicity, mainly when implementing numerical schemes.

Example Let us compute the price of an American Put Option written on the HSI[7] with the calibrated Binomial Tree. Assume

$$S(0) = 15248,\ K = 14400,\ T = 32/247,\ r = 0.025,\ \sigma = 0.24.$$

We first calibrate our Binomial Tree:

$$u = 1.01539, d = 0.984845, p = 0.499496, q = 0.500504$$

[7] If the stocks pays no dividends, as in the HSI, the price of the American Call and European Call options coincide

First we compute the Put Option price with the Binomial Tree formula:

$$P = e^{-0.025(32/247)} \sum_{i=0}^{32} \left[14400 - 15248 u^i d^{32-i}\right] C_i^{32} p^i (1-p)^{32-i}$$

$$= 181.934$$

The quoted price is 181, and Black-Scholes price is 182.537.

To compute the price of the American put option we use the method of backwards induction, as follows.

Step 1. Compute the prices $AP(32, i)$ of the option at node $(32, i)$ trough the formula

$$AP(32, i) = \max(14400 - s_{32}(i), 0).$$

Step 2. Time $t = 31$. Compute at each node the expected payoff cor-

responding to holding (not exercising) the option. As from the node $(31, i)$ we can go to up to the node $(31, i+1)$, and down to the node $(31, i)$ this values are

$$H(31,i) = e^{-r\Delta}\big(p\,AP(32,i+1) + (1-p)AP(32,i)\big)$$

Step 3. Time $t = 31$. Compute at each node the expected payoff corresponding to exercising the option for each node $(31, i)$ through

$$E(31,i) = \max(14400 - s_{31}(i), 0).$$

Step 4. Compare the results $H(31, i)$ of holding, against the ones of executing $E(31, i)$, to obtain the price AP of the option at nodes $(31, i)$:

$$AP(31,i) = \max\big(H(31,i), E(31,i)\big).$$

Step 5. With the obtained prices repeat the procedure for time=30,29 and so on, up to time 1.

Step 6. We finally obtain the price for the American Put

$$AP = 183.178$$

Remark The difference $AP - P$ is called the early exercise premium. The algorithm can also provide the optimal stopping rule for the American Option.