5. Portfolio diversification (II) and CAPM

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References for Lecture 5:

- Essentials of Stochastic Finance, Albert N. Shiryaev, World Scientific (1999)

- http://en.wikipedia.org/wiki/Capital_asset_pricing_model

5a. Computing efficient portfolios

Purpose: construct an efficient portfolio in practice, departing from historial prices of the assets.

Suppose: we have three assets A, B, C (in the same currency) to construct our portfolio.

Let $S_i(0), \ldots, S_i(n)$ be the historical prices of asset $i = A, B, C$. (in practice $n = 90$ for daily data, of $n = 12$ for monthly data, etc).

STEP 1: Calculate the return of A, by the equations

$$x_A(1) = \frac{S_A(1)}{S_A(0)} - 1, \ldots, x_A(n) = \frac{S_A(n)}{S_A(n-1)} - 1,$$

and similar for B and C.

STEP 2: Estimate the mean returns of A by

$$\bar{x}_A = \frac{1}{n} \sum_{k=1}^{n} x_A(k).$$

and the same for B, C.

STEP 3: Estimate the variance-covariance matrix:

$$\begin{bmatrix} \sigma_A^2 & \mathbf{cov}_{AB} & \mathbf{cov}_{AC} \\ \mathbf{cov}_{AB} & \sigma_B^2 & \mathbf{cov}_{BC} \\ \mathbf{cov}_{AC} & \mathbf{cov}_{BC} & \sigma_C^2 \end{bmatrix}$$

As this matrix is symmetric we must estimate only 6 values.

For A, the variance is

$$\bar{\sigma}_A^2 = \frac{1}{n-1} \sum_{k=1}^{n} \left(x_A(k) - \bar{x}_A \right)^2.$$

and similarly for B and C.

The covariance between returns of A and B is

$$\bar{c}_{AB} = \frac{1}{n} \sum_{k=1}^{n} x_A(k) x_B(k) - \bar{x}_A \bar{x}_B,$$

and similarly for A, C and B, C.

Now, we can estimate the expected return of a portfolio $\pi = (\alpha, \beta, \gamma)$, with proportions (a, b, c), $a + b + c = 1$. The expected return is

$$\mathbf{E}\, X_\pi = a\bar{x}_A + b\bar{x}_B + c\bar{x}_C,$$

while the variance of the return is

$$\begin{aligned}\mathbf{var}\, X_\pi = a^2\bar{\sigma}_A^2 + b^2\bar{\sigma}_B^2 + c^2\bar{\sigma}_C^2 \\ + 2\Big(ab\bar{c}_{AB} + ac\bar{c}_{AC} + bc\bar{c}_{BC}\Big).\end{aligned}$$

Consider for example the values (in %)

$$\bar{x}_A = 10 \quad \bar{x}_B = 5 \quad \bar{x}_C = 3$$
$$\bar{\sigma}_A = 8 \quad \bar{\sigma}_B = 10 \quad \bar{\sigma}_C = 12$$
$$\bar{c}_{AB} = 0 \quad \bar{c}_{AC} = 0 \quad \bar{\sigma}_{BC} = 0$$

Consider the following two portfolios:

	a	b	c	$\bar{x}$	$\bar{\sigma}$
π	0.333	0.333	0.333	6	5.85
π_{eff}	0.322	0.372	0.305	6	5.81

The second portfolio is efficient, both have the same expected return.

5b. Introduction to CAPM

Consider

- The risk-free interest rate r

- A global market index with (stochastic) return ρ

- An asset A in the market with (stochastic) return ρ_A

Problem: Quantify the risk and return of A in the market

Example: r is the interest rate of a zero coupon US-bond, ρ is the return of the S&P 500, and A is a share of Google.

W. Sharpe (1964) established that there exists a quantity denoted β such that

$$\mathbf{E}\,\rho_A - r = \beta\Big(\mathbf{E}\,\rho - r\Big), \qquad (1)$$

where

$$\beta = \frac{\mathbf{cov}(\rho_A, \rho)}{\mathbf{var}\,\rho} = \frac{\mathbf{E}(\rho\rho_A) - \mathbf{E}\,\rho\,\mathbf{E}\,\rho_A}{\mathbf{E}\left(\rho^2\right) - (\mathbf{E}\,\rho)^2}.$$

The amount $\mathbf{E}\,\rho_A - r$ is the risk premium of asset A.

5c. β and the expected return

Based on (1) we see that

- If $\beta = 0$ then $\mathbf{E}\,\rho_A = r$.

- If $\beta = 0$ then $\mathbf{E}\,\rho_A = \rho$.

$\mathbf{E}\,\rho_A$ is a linear function of β under the equation

$$\mathbf{E}\,\rho_A = r + \beta(\mathbf{E}\,\rho - r).$$

5d. β and the risk

As we are interested in risk, we compute the variance of ρ_A. Define the random variable ε by the relation

$$\varepsilon = \rho_A - \mathbf{E}\,\rho_A - \beta(\rho - \mathbf{E}\,\rho).$$

Taking expectations we verify $\mathbf{E}\,\varepsilon = 0$.

$$\mathbf{cov}(\rho, \varepsilon) = \mathbf{E}\,\rho\varepsilon = \mathbf{E}\,\rho\Big(\rho_A - \mathbf{E}\,\rho_A - \beta(\rho - \mathbf{E}\,\rho)\Big) =$$
$$\mathbf{E}(\rho\rho_A) - \mathbf{E}\,\rho\,\mathbf{E}\,\rho_A - \beta[\mathbf{E}\left(\rho^2) - (\mathbf{E}\,\rho)^2\right] = 0$$

We obtained a decomposition of ρ_A of the form

$$\rho_A - \mathbf{E}\,\rho_A = \beta(\rho - \mathbf{E}\,\rho) + \varepsilon$$

where the two terms in the sum are uncorrelated.

This gives

$$\mathbf{var}\,\rho_A = \beta^2\,\mathbf{var}\,\rho + \mathbf{var}\,\varepsilon.$$

Here

- $\beta^2\,\mathbf{var}\,\rho$ is the systematic (unavoidable) risk of A

- $\mathbf{var}\,\varepsilon$ is the unsystematic (diversifiable) risk of A.

Then, β measures the systematic or market risk of A

It is important to distinguish between this two risks:

- the first is the systematic, intrinsic of the market, can not be reduced.
- The second, called unsystematic, can be diversified, for instance, buying other assets.

In practice β ranges, approximately from $1/2$ to 2, and, for instance

- If $\beta = 1/2$, the asset A has half of the expected return of the market, but $1/4$ of the systematic risk, we have a defensive asset.

- If $\beta = 2$, the asset doubles the expected return of the market, but the systematic risk is raised four times. We have an aggressive asset.

5d. Computation of β

The parameter β results as the slope in a simple linear regression model of the form

$$\rho_A - r = \beta(\rho - r) + \varepsilon,$$

where ε is a statistical error.

In order to estimate β, you need:

- Prices $(S_A(0), S_A(1), \ldots, S_A(n))$ of the asset A,

- corresponding prices $(S(0), S(1), \ldots, S(n))$ of the market index.

STEP 1: Compute the corresponding returns by the formula

$$x(k) = \frac{S(k)}{S(k-1)} - 1, \qquad y(k) = \frac{S_A(k)}{S_A(k-1)} - 1.$$

STEP 2: Find the mean returns

$$\bar{x} = \frac{1}{n} \sum_{k=1}^{n} x(k), \qquad \bar{y} = \frac{1}{n} \sum_{k=1}^{n} y(k),$$

STEP 3: Estimate the variance of ρ by

$$\bar{\sigma}_x^2 = \frac{1}{n} \sum_{k=1}^{n} \Big(x(k) - \bar{x} \Big)^2,$$

STEP 4: Estimate the covariance

$$\bar{c}_{xy} = \frac{1}{n} \sum_{k=1}^{n} (x(k)y(k)) - \bar{x}\bar{y}.$$

FINAL STEP: The estimation of β is

$$\beta = \frac{\bar{c}_{xy}}{\bar{\sigma}_x^2}.$$