18. Diffusion processes for stocks and interest rates

MA6622, Ernesto Mordecki, CityU, HK, 2006.

References for this Lecture:

P. Willmot, *Paul Willmot on Quantitative Finance*. Volume 1, Wiley, (2000)

A. N. Shiryaev *Essentials of Stochastic Finance*, World Scientific (1999)

Plan of Lecture 18

(18a) Diffusion models

(18b) Diffusions and Time Series

(18c) Two factor models

(18d) Extensions of Black Scholes

(18d) Calibration in parametric models

18a. Diffusion models

A useful way to model the random evolution of a financial instrument in continuous time, for instance a stock, an index, or a stochastic interest rates, is provided by diffusion processes.

A diffusion process $X = \{X(t)\}$, where $0 \le t \le T$, departs from a fixed value $X(0) = x_0$, and follows a dynamics of the form

$$dX(t) = \alpha(t, X(t)) \, dt + \beta(t, X(t)) \, dW(t).$$

that, can be also (more formally) written as

$$X(t) = x_0 + \int_0^t \alpha(s, X(s)) ds + \int_0^t \beta(s, X(s)) \, dW(s).$$

Here

W

- $\{W(t)\}$ is a Wiener Process (or Brownian motion),
- $\alpha(t, x)$ the drift, and $\beta(t, x)$, the variance, are regular¹ functions of two variables, time and space,
- the last term is a stochastic integral.

Suppose that today is t, and we observe the value X(t) = x. Compute the numbers $\alpha = \alpha(t, x)$ and $\beta = \beta(t, x)$. A financial instrument can be modeled through a diffusion if it is reasonable to assume that the future value of X at time $t + \Delta$ is

$$X(t + \Delta) = \alpha \, \Delta + \beta \, \Delta W,$$
here $\Delta W \sim \mathcal{N}(0, \Delta).$

¹For details, see A. N. Shiryaev. Essentials of Stochastic Finance, World Scientific (1999)

In other terms, is is reasonable to model through diffusions when we expect movements on a short time intervals with gaussian distribution, and

- expected value $\alpha \Delta$,
- variance $\beta^2 \Delta$.

Example Assume that α and β are constants. The corresponding diffusion process is

$$X(t) = x_0 + \alpha t + \beta W(t)$$

This is the Bachelier model introduced by L. Bachelier in 1900^2 to describe the movements in the "Bourse de Paris".

²L Bachelier (1900), Thorie de la spéculation, Gauthier-Villars, 70 pp

Example Assume that

$$\alpha(t,x) = a - bx, \qquad \beta(t,x) = \beta,$$

i.e. β is constant. In this way we obtain Vasicek model³ for the instantaneous interest rates:

$$dX(t) = (a - bX(t)) dt + \beta dW(t).$$

It has the property of mean reversion:

- If X(t) < b/a then the drift α is positive, and the proces tends to go up,
- If X(t) > b/a then the drift is negative, and the process tends to go down.

It can be seen that for large time values, the process reaches an equilibrium around the mean b/a.

³Vasicek, O. "An equilibrium characterization of the term structure", Journal of Financial Economics, 1977, pp. 177-188.

Example Assume now that the coefficients do not depend on time, and are linear in space:

$$\alpha(t,x)=\mu x,\quad \beta(t,x)=\sigma x.$$

We obtain a diffusion with equation

$$dX(t) = \mu X(t) dt + \sigma X(t) dW(t)$$

= $X(t) [\mu dt + \sigma dW(t)].$

This is Black Scholes model (as we can verify with the use of Ito's Formula)

Note The reference to BS model is so important, that in finance is more usual to write the diffusions as

$$dX(t) = X(t) \left[\mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(t) \right]$$

i.e. to assume that

$$\alpha(t,x) = x\mu(t,x), \quad \beta(t,x) = x\sigma(t,x),$$

to obtain the BS model when μ and σ are constant.

Example The Constant Elasticity of Variance Model⁴, generalizes BS, trying to capture the smile:

$$\mu(t,x) = \mu, \qquad \sigma(t,x) = \sigma x^{-\alpha},$$

where $0 \le \alpha \le 1$. For $\alpha = 0$ we obtain BS.

⁴J.C. Cox an S.A. Ross "The valuation of Options for Alternative Stochastic Processes", Journal of Financial Economics, 3 (1976).

18b. Diffusions and Time Series

Consider a discrete time scheme with N steps:

$$t_0 = 0, t_1 = \frac{T}{N}, t_2 = \frac{2T}{N}, \dots, t_{N-1} = \frac{(N-1)T}{N}, t_N = T.$$

A good approximation of the diffusion is obtained through the time series $\{Y_n\}$, beginning from $Y_0 = x_0$, and iteratively, for $n = 1, \ldots, N - 1$, computing the values

$$Y_{n+1} = Y_n + \alpha(t_n, Y_n) \Delta + \beta(t_n, Y_n) \Delta W_n,$$

where

$$\Delta = \frac{T}{N}, \qquad \Delta W_n = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, \Delta)$$

It can be shown that $X(t_n) \sim Y_n$. This fact can be used to compute option prices through Monte Carlo simulation method. We call $\{Y_n\}$ the discretized diffusion, or also the Euler approximation of the diffusion.

Example Let us consider the discretized Vasicek model of interest rates, with $Y_0 = x_0$, and

$$Y_{n+1} = Y_n + (a - bY_n)\,\Delta + \beta\,\Delta W_n$$

$$= a\Delta + (1 - b\Delta)Y_n + \beta\,\Delta W_n.$$

If we write

$$a \Delta = \omega, \qquad \phi = 1 - b \Delta, \qquad \beta \Delta W_n = \varepsilon_n,$$

we obtain

$$Y_{n+1} = \omega + \phi Y_n + \varepsilon_n,$$

where $\{e_n\}$ is a gaussian white noise with variance $\beta^2 \Delta$. We have obtained that the discretized Vasicek model is a non centered AR(1) time series.

18c. Two factor models

In order to capture the volatility smile of option prices we can model the value and the volatility by a two dimensional diffusion:

$$dX(t) = X(t) \left[\mu(t, X(t)) dt + \sigma(t) dW_1(t) \right]$$

$$d\sigma(t)^{2} = p[t, X(t), \sigma(t)^{2}] dt + q[t, X(t), \sigma(t)^{2}] dW_{2}(t),$$

departing from a value $(X(0), \sigma(0)) = (x_0, \sigma_0)$, where⁵:

- The functions $\alpha(t, x)$, p(t, x, s) and q(t, x, s) are regular,
- The source of randomness $(W_1(t), W_2(t))$ is a two dimensional Wiener process with correlation ρ .

⁵Sometimes σ is modelled instead of σ^2 .

• We have a two-factor source of randomness.

$$d\sigma(t)^2 = a(b - \sigma(t)^2)dt + c\sigma(t)^2 dW_2(t).$$

Observe that the drift term is mean reverting, as in Vasicek Model.

The discretized diffusion is a multivariate time series $\{Y_n, s_n\}$, beginning at $(Y_0, s_0) = (x_0, \beta_0)$, with:

$$Y_{n+1} = Y_n + \alpha(t_n, Y_n) \Delta + Y_n s_n \Delta W_{1,n}$$

$$s_{n+1}^2 = s_n^2 + p(t_n, Y_n, s_n^2) \Delta + q(t_n, Y_n, s_n^2) \Delta W_{2,n}$$

⁶The pricing of options on assets with stochastic volatility J Hull, A White - Journal of Finance, 1987

where

$$(\Delta W_{1,n}, \Delta W_{2,n}) \sim \mathcal{N}\left(\mathbf{0}, \Delta \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

Example GARCH diffusion. Consider the two factor model

$$dX(t) = \sigma(t)X(t)dW(t)$$

$$d\sigma(t)^2 = [a + b\sigma(t)^2]dt.$$

The discretized time series is:

$$Y_{n+1} = Y_n + Y_n s_n \,\Delta W_n$$

$$s_{n+1}^{2} = s_{n}^{2} + (a + bs_{n}^{2})\Delta = a\Delta + (1 + b\Delta)s_{n}^{2}.$$

Write

$$\omega = a\Delta, \quad \beta = 1 + b\Delta, \quad \varepsilon_n = \Delta W_n,$$

and observe that $\{e_n\}$ is a gaussian white noise with variance $\beta^2 \Delta$.

The previous time series, for the returns $R_n = \frac{Y_{n+1}}{Y_n} - 1$ are:

$$R_n = s_n \varepsilon_n, \qquad s_n^2 = \omega + \beta s_{n-1}^2,$$

that is a GARCH time series with $\alpha = 0$.

18d. Extensions of Black Scholes

We are ready to reveiw the three main approaches to capture the volatility smile in option pricing:

• One Factor diffusion modelling. It assumes that prices follow a diffusion

$$dX(t) = X(t) \big[\mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(s) \big].$$

The proposal is to calibrate the function $\sigma(t, x)$ in order to obtain theoretical prices as close as possible as observed prices.

One possibility is to assume some parametric form, as in the constant elasticity of variance model, where

$$\sigma(t,x) = x^{-\alpha},$$

for $0 \leq \alpha \leq 1$. The calibration in this case consist in determining σ and α that better fit the smile.

In general, the idea is to construct the function $\sigma(t, x)$ departing from the implied volatility matrix, obtained from observed prices.

• Stochastic Volatitliy Models. This are two factor models, as the ones described in the previous sections. The calibration in general is numerically complex, and it seems that they have not entered in the practitioners routines.

Diffussion with jumps. Generates volatility smiles by adding jumps to the Black Scholes diffusion dynamics.

Introduced by Merton⁷, it is assumed that intervals between jumps are random variables with exponential distribution, independent from the other source of random-

⁷R.C. Merton, "Option Pricing When Underlying Asset Returns are Discontinuous", Journal of Financial Economics (1976)

ness, and that the magnitudes of the jumps are normally distributed.

The diffusion with jumps models are a particular class of the Lévy models.

18e. Calibration in parametric models

Suppose that we want to fit certain model, depending on a vectorial parameter θ , consistently with a certain set of call option prices $C(T_i, K_j)$.

The proposal is to find the value of θ that fits better to the observed prices, in the following sense:

• Find θ such that

$$\sum_{i,j} w_{ij} \left(Call(\theta, T_i, K_j) - C(T_i, K_j) \right)^2$$

is minimum.

Here

- $Call(\theta, T_i, K_j)$ are the option prices produced by the model we want to calibrate,
- the weights w_{ij} are usually selected as

$$w_{ij}^{-1} = \operatorname{vega}(\mathbf{v}) = S(0)\sqrt{T}\phi(d_1),$$

where \mathbf{v} is the implied volatility (computed through BS) of the corresponding observed price.

The idea is that large vega's, indicating large variation of prices for small variations of volatilities should be less relevant that small vega's.

19. Calibrating one factor Diffusion Models

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References for this Lecture:

P. Willmot, *Paul Willmot on Quantitative Finance*. Volume 1, Wiley, (2000)

B. Dupire. "Pricing with a Smile", Risk Journal 7, 18-20 (2004)

Plan of Lecture 19

(19a) Risk Neutral density from Option prices

(19b) The Local Volatility Surface

(19c) Calibrating the approximated Local Volatility Surface

19a. Risk Neutral density from Option prices

In this lecture we assume that our price process follows a one factor diffusion model:

$$dX(t) = X(t) \big[\mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(s) \big].$$

Suppose that, for a given maturity T we have an enough rich amount of option prices C(K,T) for different strikes. Denote by

$$q(t,s,T,y)=q(t,S(t)=s,T,S(T)=y)$$

the risk neutral transition probability density, given that at time t we are in position S(t) = x.

The price of a call option with strike K and expiry T can

be computed as

$$\begin{split} C(T,K) &= e^{-r(T-t)} \, \mathbf{E}_{\mathbf{Q}}(S(T)-K)^+ \\ &= e^{-r(T-t)} \int_K^\infty (y-K)^+ q(t,s,T,y) dy. \end{split}$$

If we differentiate with respect to K we obtain

$$\frac{\partial}{\partial K}C(T,K) = -e^{-r(T-t)}\int_{K}^{\infty}q(t,s,T,y)dy.$$

A second differentiation gives

$$\frac{\partial^2}{\partial K^2}C(T,K) = e^{-r(T-t)}q(t,s,T,K)$$

This means that the second derivative of the price of a call option with respect to the strike gives (discounted), gives the risk neutral probability. In formulas:

$$q(t, s, T, y) = e^{r(T-t)} \frac{\partial^2}{\partial K^2} C(T, K)$$

As we do not have the call prices for all strikes, we approximate:

$$f''(x) \sim \frac{f(x+2h) + f(x) - 2f(x+h)}{h^2}.$$

Example Let us compute the risk neutral approximate density for the HSI, for June 29, if today is June 15.

We take quoted option call prices from SCMP (see web page), and, for simplicity, assume r = 0 (this does not change the shape of the density).

We have h = 200, and 22 values, so the (approximate)

values of the density are:

	a(k) =		c(k+2) + c(k) - 2c(k+1)		
q(n) -			200^{2}		
Month	Strike	k	Price	$q(k) \times 200^2$	
June	13000	1	c(1) = 2246	1	
June	13200	2	c(2) = 2048	2	
June	13400	3	c(3) = 1851	1	
June	13600	4	c(4) = 1656	4	
June	13800	5	c(5) = 1462	4	
June	14000	6	c(6) = 1272	5	
June	14200	7	c(7) = 1086	9	
June	14400	8	c(8) = 905	10	
June	14600	9	c(9) = 733	15	
June	14800	10	c(10) = 571	19	
June	15000	11	c(11) = 424	31	

Month	Strike	k	Price	$q(k) \times 200^2$
June	15200	12	c(12) = 296	22
June	15400	13	c(13) = 199	23
June	15600	14	c(14) = 124	18
June	15800	15	c(15) = 72	14
June	16000	16	c(16) = 38	9
June	16200	17	c(17) = 18	7
June	16400	18	c(18) = 7	2
June	16600	19	c(19) = 3	2
June	16800	20	c(20) = 1	0
June	17000	21	c(21) = 1	-
June	17200	22	c(22) = 1	-



The second graph is the risk-neutral BS density. BS assumes that

$$S(T) = S(0) \exp\left[(r - \sigma^2/2)T + \sigma W(T)\right]$$

We assume that r = 0, and estimate⁸ $\sigma = 0.20$. As we have 10 trading days, T = 10/247. Then

$$S(T) = 15248 \exp\left[-0.0008 + 0.04\mathcal{N}\right]$$

⁸As we know that there is no unique σ we use an intermediate value of implied volatilities

where \mathcal{N} is a standard normal random variable.

Application Let us use the risk neutral density to compute the price of an european call digital option, also called cash-or-nothing binary option. It pays a fixed amount of money if it expires in the money and nothing otherwise.

Let us assume that the strike is K = 15000. Then we will recive 1 if $S(T) \ge 15000$, and nothing otherwise.

The price of such an instrument is the discounted expected value of the payoff under the risk probability measure. As we assume that r = 0, the price is

$$D = \mathbf{E}_{\mathbf{Q}} \mathbf{1}_{\{S(T) \ge 15000\}}$$

= $\mathbf{Q}(S(T) \ge 15000) = \int_{15000}^{\infty} q(0, 15816.5, T, y) dy,$

i.e the price is the risk-neutral probability of the asset value

resulting larger that the strike.

As 15000 corresponds to k = 11, We compute this probability from our estimated q(k):

$$D = \sum_{k=11}^{20} 200 \times q(k) = 0.64.$$

where 200 is the distance between consecutives strikes (in fact we are computing an area).

The BS price is

$$D_{BS} = 0.651.$$

In case we take a strike K = 14700, the results are

$$D = \sum_{k=10}^{20} 200 \times q(k) = 0.735, \text{ while } D_{BS} = 0.813.$$

19b. The Local Volatility Surface

Assume that our price process follows a one factor model:

 $dX(t) = X(t) \big[\mu(t, X(t)) \, dt + \sigma(t, X(t)) \, dW(s) \big].$

The function $\sigma = \sigma(t, x)$ is called the local volatility surface, and calibrating this models means finding an adecuate function σ that reproduces correctly the observed option prices.

In 1994 Dupire⁹ found that there is a way to compute the function σ knowing prices C(T, K) for all excercise times and strikes.

The approach is similar to Derman and Kani proposal of implied trees, and is based in the analysis of the Kolmogorov Backward or Fokker Planck equation.

⁹B. Dupire. "Pricing with a Smile", Risk 7, 18-20 (2004)

The obtained formula is

$$\sigma(t,x)^{2} = \frac{C_{t}(t,x) + rxC_{x}(t,x)}{\frac{1}{2}x^{2}C_{xx}(t,x)}$$

where

$$C_t(t,x) = \frac{\partial}{\partial t}C(t,x)$$

is the first derivative of the function C(t, x) with respect to the time variable, and similarly, $C_x(t, x)$ is the first derivative with respect x (space coordinate), and $C_{xx}(t, x)$ the second derivative with respect to space.

In other words, Dupire found that if we know all call option prices, for all maturities and strikes, we can find a one factor model that produces the smile corresponding to these prices. In practice, one only has a finite set of prices, so the proposal is to find an approximate local volatility function.

A further development of this formula gives the function $\sigma(t, x)$ in terms of the implied volatility $\mathbf{v}(t, x)$ obtained by applying BS formula to the observed prices.

An approximation of this formula, used in practice, is obtained assuming that

$$\mathbf{v}(T, K) = a(T)(K - S(0)) + b(T),$$

We are assuming then that the implied volatility, once the expiry is fixed, is a linear function of the strike.

If we remember the volatility smile of the period June 16 (today) June 29,



with a spot price of 15248, we see that (in this case) this is a reasonable assumption in the interval 13000 - 16800. We need to know the local volatility for all intermediate time values $t \in [t^*, T]$, and all price values S. Today is t^* and the spot price today is S^* . Denote by $\tau = T - t$.

The approximated formula obtained, is

$$\sigma(t,S)^{2} = \frac{\mathbf{v}^{2}(t,S) + 2\tau \mathbf{v}(t,S)\mathbf{v}_{t}(t,S) + 2rS\tau \mathbf{v}(t,S)a(t)}{(1 + Sd_{1}\sqrt{\tau}a(t))^{2} - S^{2}\tau^{3/2}\mathbf{v}(t,S)d_{1}a(t)^{2}}$$

where

$$d_1 = \frac{\log[S^*/S] + (r + \mathbf{v}(t, S)^2/2)\tau}{\mathbf{v}(t, S)\sqrt{\tau}}$$
$$\mathbf{v}(t, S) = a(t)(S - S^*) + b(t)$$
$$\mathbf{v}_t(t, S) = a'(t)(S - S^*) + b'(t)$$

In order to calibrate the model we must determine the functions a(t) and b(t)

19c. Calibrating the approximated Volatility Surface

Calibration of b(t).

We begin by b(t) based on the fact that, if $S = S^*$ we have

$$\mathbf{v}(t, S^*) = b(t).$$

This means that we need to know the implied volatility of options at the money.

Then b(t) is the term forward volatility, that we have seen how to calibrate with options at the money.

An important difference with our previous calculations, is that here is that the derivative b'(t) is also necessary to compute $\sigma(t, x)$, so we need more frequent traded options, in order to obtain a reasonable approximation of the derivative, as

$$b'(t) \sim \frac{b(t+h) - b(t)}{h}$$

In this context, practitioners calibrate the a(t) with prices of an at the money straddle.

A straddle is made up of a (long) call and a (long) put, with the same strike and expires. The value a(t) is the implied volatility. Based on put-call parity, we can also use prices of (at the money) call (or put) options.

Calibration of a(t).

The calibration of a(t) is not so direct. For this we use prices of a risk-reversal conformed with a long call struck slightly above the current spot, i.e. $K = S^* + \varepsilon$; plus a short put, slightly below the current spot, i.e. with strike $K' = S^* - \varepsilon$.

Knowing the price V_{RR} of the risk reversal, assuming that ε is small, after some approximations, one obtains¹⁰:

$$a(T) = \frac{1}{2\varepsilon S^* \sqrt{\tau} \phi(d_1)} [V_{RR} - S^*(1 - e^{r\tau})] + \frac{e^{-r\tau} \Phi(d_2)}{S^* \sqrt{\tau} \phi(d_1)}$$
$$= \frac{1}{S^* \sqrt{\tau} \phi(d_1)} \Big[\frac{V_{RR} - S^*(1 - e^{r\tau})}{2\varepsilon} + e^{-r\tau} \Phi(d_2) \Big].$$

¹⁰The details can be found in page 360 of P. Willmot's Vol.1

Here Φ is the standard normal distribution function, ϕ its derivative, and

$$d_1 = \left[\frac{r + b(T)^2/2}{b(T)}\right]\sqrt{\tau}, \quad d_2 = d_1 - b(T)\sqrt{\tau}$$