16. Time dependence in Black Scholes

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References for this Lecture:

Marco Avellaneda and Peter Laurence,
Quantitative Modelling of Derivative Securities,
Chapman&Hall (2000).

Plan of Lecture 16

(16a) Term Structure of Interest rates

(16b) Term Structure of Volatilities

(16c) A detailed Example

16a. Term Structure of Interest rates[1]

A first simple extension in Black and Scholes model is to incorporate time dependence.

For example, the following table gives the interest rates of Treasury Bills (zero coupon bonds issued by the US Treasury Department), for June 8, 2006:

	4-week	3-month	6-month
Yield	4.70	4.74	4.87

Treasury bills can be considered to be the most risk free investment, and this is why they are widely used in Black-Scholes formula[2].

The given yields are in percent per annum on a 360-days basis.

[1] This section expands section 14d.

[2] It must be taked into account that these rates apply for USD.

360 calendar day

The 360 day calendar is a method of measuring durations used in financial markets. It is based on the assumption of a 360 day year, consisting of 12 months of 30 days each. To arrive at such a calendar from the standard 365/366 day Gregorian calendar, certain days are skipped, in our case, we do not count July 31, August 31, and October 31 (as our computations will refer to this time period).

Instantaneous forward rates

In order to model the interest rates with different maturities, we assume:

- Exists and instantaneous deterministic forward rate $r(t)$,
- $r(t)$ is constant between consecutive maturities,
- the observed term rates $r(t,T)$, satisfy:

$$r(t,T) = \frac{1}{T-t}\int_t^T r(s)ds. \tag{1}$$

To calibrate the term structure of interest rates:

- Given a set of term rates $r(t,T)$, for different maturities T, find the instantaneous forward interest rates $r(t)$.

In our example

- Maturities are $\quad T_1 = \frac{28}{360}, \quad T_2 = \frac{90}{360}, \quad T_3 = \frac{180}{360}.$
- Find r_1, r_2 and r_3:

$$r(t) = \begin{cases} r_1 & \text{for } 0 \le t \le T_1 \\ r_2 & \text{for } T_1 \le t \le T_2 \\ r_3 & \text{for } T_2 \le t \le T_3 \end{cases}$$

such that (1) gives:

$$r(0,T_1) = 0.0470, \; r(0,T_2) = 0.0474, \; r(0,T_3) = 0.0487.$$

Step 1. We find r_1. As the integrand takes a constant value, we have

$$
\begin{aligned}
r(0, T_1) &= \frac{1}{T_1} \int_0^{T_1} r(s) ds \\
&= \frac{1}{28/360} (28/360) r_1 = r_1 = 0.0470
\end{aligned}
$$

So $r_1 = r(0, T_1)$[3]

Step 2. For the second period the integrand takes two values:

$$
\frac{1}{(90/360)} \left[\frac{28}{360} r_1 + \frac{62}{360} r_2 \right] = 0.0474
$$

Observe that the 360 simplifies, to give

$$
\frac{1}{90} \left(28 r_1 + 62 r_2 \right) = 0.0474
$$

Here $62 = 90 - 28$. This gives $r_2 = 0.0476$.

[3] Always the first forward rate coincides with the first term rate.

Step 3. For the third interest rate r_3 we arrive to a similar computation:

$$\frac{1}{180}\Big(28r_1 + 62r_2 + 90r_3\Big) = 0.0487$$

This gives $r_3 = 0.0500$

In conclussion, from the term rates

Duration	4-weeks	3 months	6 months
In days	28	90	180
$r(t,T)$ %	4.70	4.74	4.87

we find the forward rates:

Intervals	Jn 9 - Jl 6	Jl 7 - Sp 8	Sp 9 - Dc 8
In days	28	62	90
$r(t)$ %	4.70	4.76	5.00

Example Compute the forward interest rate for 60 days.

Time interval is June 9 - August 8:

- The first 28 days (June 9 - July 6) apply r_1,
- The second days (July 7 - August 8) apply r_2:

$$r(\text{June 9, August 8}) = \frac{1}{T-t}\int_t^T r(s)ds$$

$$= \frac{28 \times r_1 + 32 \times r_2}{60} = 0.0473$$

16b. Term Structure of Volatilities

The same procedure applies to the implied volatilities.

Here the situation is more complex: for a maturity T we have different implied volatilies, depending on the strike price (volatility smile).

We choose at the money options $(S(0) = K)$ to compute the term implied volatilies $\sigma(t, T)$.

Our model assumes

- Exists an instantaneous deterministic forward volatility $\sigma(t)$,
- $\sigma(t)$ is constant between consecutive maturities,
- The term volatilities satisfy

$$\sigma^2(t, T) = \frac{1}{T - t} \int_t^T \sigma^2(s) ds.$$

Merton (1973)[4] proved that Black-Scholes formula holds for time dependent interest rates and volatilities.

For an option written today t, expiring at time T we must use BS formula with:

- $\sigma(t, T)$ instead of σ
- $r(t, T)$ instead of r.

The calibration of the Volatility Term Structure consists in:

- Given a set of at the money option prices with different maturities, compute the implied term volatilities $\sigma(t, T)$.
- Compute the instantaneous forward volatility $\sigma(t)$,

[4] R.C. Merton, *Theory of Rational Option Pricing*, The Bell Journal of Economics and Management Science, 4, (1973) PP 141-183.

16c. A detailed Example

We want to price[5] an at the money call option on the Hang Seng Index, written over the counter[6] on June 8 with a duration of 60 calendar days. More precisely:

- Time interval of the option is June 9 - August 8.
- Spot price is closing price on June 8, $S(0) = 15816.5$
- Strike price is 15800,

This time we use risk-free interest rates from zero coupon US Bond corresponding to the maturity (see previous example).

Here we do not take into account

- currency bid-ask spread when converting from USD to HKD

[5] This is an improvement over Example in 14d.
[6] I.e. A direct agreement between the buyer and the holder

- Exposure to currency risk (i.e. HKD/USD exchange rate is fixed and risk-free).

In other terms we assume that we have a risk-free instrument in HKD with the same interest rate structure of the US Bond (Treasury Bills in this case)

This is the information that we have:

Month	Strike	Price	$r(t,T)$	$\sigma(t,T)$
June, 29	15800	335		
July, 28	15800	507		
August, 8	15800	**Price?**		
August, 30	15800	625		

Step 1. Compute the term interest rates, from the table of last section, using calendar days:

$$r(\text{June } 9, \text{June } 29) = 0.0470,$$

$$r(\text{Jn } 9, \text{Jl } 28) = \frac{0.0470 \times 28 + 0.0476 \times 22}{28 + 22} = 0.04726,$$

$$r(\text{Jn } 9, \text{Ag } 30) = \frac{0.0470 \times 28 + 0.0476 \times 54}{28 + 54} = 0.04740.$$

Step 2. Compute the term volatilities implied by the quoted prices (trading days), with this interest rates (calendar days).

$$C(S(0); K; r; \sigma) = QP,$$

where QP is the quoted price (We use Newton Raphson to solve this equation in σ).

– For June 29 we have 15 trading days and $r = 0.047$:

$$C(15816.5; 15088, 15/247, 0.047, \sigma) = 335.$$

gives $\sigma = 0.195$.

– For July 28 we have 36 trading days and $r = 0.04726$:

$$C(15816.5; 15088, 36/247, 0.04726, \sigma) = 507.$$

gives $\sigma = 0.184$.

– For August 30 we have 59 trading days and $r = 0.0474$:

$$C(15816.5; 15088, 59/247, 0.0474, \sigma) = 625.$$

gives $\sigma = 0.170$.

So our table is now:

Month	Strike	Price	$r(t,T)$ %	$\sigma(t,T)$ %
June, 29	15800	335	4.70	19.5
July, 28	15800	507	4.724	18.4
August, 8	15800	**Price?**	4.73	
August, 30	15800	625	4.740	17.0

Step 3. Compute the instantaneous volatility for each time interval, to fit the implied term volatilities just computed.

– The first volatility, between June 9 and June 29, $\sigma_1 = 0.195$.

– The second value σ_2, between June 30 and July 28 satisfies:

$$\frac{15 \times \sigma_1^2 + 21 \times \sigma_2^2}{15 + 21} = 0.184^2,$$

gives $\sigma_2 = 0.175722$.

– For σ_3, we solve

$$\frac{15 \times \sigma_1^2 + 21 \times \sigma_2^2 + 23\sigma_3^2}{15 + 21 + 23} = 0.17^2,$$

that gives $\sigma_3 = 0.145406$

Step 4. Use the instantaneous forward volatilies to compute our term volatility:

$$\sigma(\text{Jn } 9, \text{Ag } 8) = \sqrt{\frac{15\sigma_1^2 + 21\sigma_2^2 + 7\sigma_3^2}{15 + 21 + 7}} = 0.178287$$

Step 5. Use this volatility to compute the price of the option, having 43 trading days:

$$C(15816.5; 15088, 43/247, 0.0473, 0.1783) = 543.768.$$

The final table is

Month	Strike	Price	$r(t,T)$ %	$\sigma(t,T)$ %
June, 29	15800	335	4.70	19.5
July, 28	15800	507	4.724	18.4
August, 8	15800	**543.8**	4.73	0.1783
August, 30	15800	625	4.740	17.0

Remember

- $r(t,T)$ are the term interest rate, observed.

- $r(t)$ is the forward interest rate, computed.

- We use a 360 calendar day.

- $\sigma(t,T)$ are the term volatilities, implied by QP and computed through Newton Raphson.

- $\sigma(t)$ is the forward volatility, computed.

- We use trading days.