

11. Value at Risk through Monte Carlo

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References for this Lecture:

The portable financial analyst (Second Edition). Mark P. Kritzman, Wiley (2003).

An Overview of Value at Risk. D Duffie, J Pan, Journal of Derivatives (1997)

[Available at: <http://www.mit.edu/~junpan/>]

Main Purposes of Lectures 10 and 11:

- Introduce the notion of Value at Risk (VaR)
- Notice the relevance of this risk measurement notion, in particular in relation to the Basel Second Accord (Basel II)
- Review the analytical and historical computation of VaR with emphasis on its tail behaviour dependence
- Discuss the difference between long and short positions in VaR (Lecture 11).
- Present the Monte Carlo approach to VaR (Lecture 11).
- Comment on VaR and derivatives (nonlinearity) (Lecture 11)

Plan of Lecture 11

(11a) VaR for portfolios with short positions

(11b) Monte Carlo Approach to VaR

(11c) Monte Carlo VaR for AR(1) log-returns

(11d) Multivariate Monte Carlo Approach to VaR

(11e) Comments on Derivatives and VaR

11a. VaR for portfolios with short positions.

Up to now we have assumed that the portfolio value can be obtained from price quotations, i.e. we assumed that the portfolio in question is itself a **liquid asset** (i.e. an asset that can easily and cheaply turned into cash).

In general we can not assume this, as financial institutions held portfolios constituted by different type of assets.

In consequence it is necessary to compute the VaR when only prices (or returns) of the individual components of the portfolio are available.

Furthermore, when computing the VaR for long positions we focus on the **left tail** of the return distribution. If instead we held a **short** position in this asset, we should focus on the **right tail** as we suffer a loss for a rise in the asset value.

In general, for a portfolio π with d assets containing

$$\pi = (\pi_1, \dots, \pi_d)$$

units of different assets A_1, \dots, A_d , with prices

$$(S_1(t), \dots, S_d(t)),$$

then the **value** of your portfolio at time t will be

$$V_\pi(t) = \pi_1 S_1(t) + \dots + \pi_d S_d(t).$$

Assets held in long positions have $\pi_i > 0$, short positions $\pi_i < 0$.

Assuming that the distribution of each individual asset is log-normal, although not exact, it is reasonable (to simplify computations) that the value of the portfolio V_π is also lognormal (we drop the t for notational ease), with expectation

$$\mathbf{E} V_\pi = \pi_1 \mathbf{E} S_1 + \dots + \pi_d \mathbf{E} S_d,$$

and variance

$$\mathbf{var} V_\pi = \sum_{k=1}^d \pi_k^2 \mathbf{var} S_k + 2 \sum_{1 \leq j < k \leq d} \pi_j \pi_k \mathbf{cov}(S_j, S_k).$$

If we compute these values from historical data, (as we assume that they are liquid assets), we can perform the lognormal computation of the VaR.

But, as reported in the literature¹ this approximation does not hold when the portfolio includes **short** positions.

The situation is that we **dont know** the distribution of the PL random variable, we do not have records of its evolution. Then it is necessary to use the Monte Carlo (or Simulation) approach.

¹Chow, G. and Kritzman, M.P., Value at Risk with Portfolios with Short Positions, Journal of Portfolio Management, summer (2001)

11b. Monte Carlo Approach to VaR

An alternative approach to the historical **var**, is the simulation or Monte Carlo approach. The main problem that we face when trying to compute **var** is that we do not know the exact PL distribution, and, furthermore, the VaR is a property that depends on the tail behavior.

The Monte Carlo Approach consists in

STEP 1. Simulate the returns with the prescribed mean and variance covariance structure,

STEP 2. Compute the corresponding PL value for the simulated returns,

STEP 3. Order the obtained PL values,

STEP 4. Compute $-VaR$ from the simulated sample, as the empirical 0.05 quantile.

How to carry out the simulations of **STEP 1** depends on assumptions that we make about our returns (i.e. the **model** that we choose).

In what follows we review two different models:

- We assume that the log-returns of our portfolio follow an AR(1) model with high-kurtosis strict white noise,
- We assume that the vector of log-returns of our portfolio is normally distributed with a given variance-covariance matrix.

11c. Monte Carlo VaR for AR(1) log-returns

Assume the log-returns $X(t)$ of portfolio $V_\pi(t)$ satisfies an AR(1) model of the form

$$X(t) = \frac{1}{4}X(t-1) + \varepsilon(t),$$

where, trying to obtain the kurtosis property of financial data², we assume that the sequence ε has a [mixed normal distribution](#), with density

$$f_\varepsilon(x) = \frac{1}{4}\varphi\left(\frac{x}{\sqrt{2}}\right) + \frac{3}{4}\varphi\left(\frac{x}{\sqrt{2/3}}\right)$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

is the density of a standard normal random variable.

²Taken from: An Overview of Value at Risk. D Duffie, J Pan, Journal of Derivatives (1997)

The noise $\varepsilon(t)$ is centered, has unit variance, i.e $\mathbf{var} \varepsilon(t) = 1$, but

$$\kappa_{\varepsilon} = \mathbf{E} \varepsilon(t)^4 - 3 = 1.$$

As we don't know the distribution of $X(t)$, we simulate the observations.

Simulation of $\varepsilon(t)$

In order to simulate a mixed normal r.v. ε with the given density we perform:

STEP 1. Simulate a uniform random variable U in $[0, 1]$.

STEP 2. Simulate a standard normal random variable $Z \sim \mathcal{N}(0, 1)$

STEP 3. If $U < 1/4$ then $\varepsilon = \sqrt{2}Z$.

STEP 4. If $U > 3/4$ then $\varepsilon = \sqrt{2/3}Z$

In order to obtain one value of our portfolio V_π we run our time series from $t = 0$ to $t = 100$, to obtain $X(100)$:

STEP 1. Set $x(0) = 0$, set $k = 1$.

STEP 2. Simulate $\varepsilon(k)$, and set $x(k) = (1/4)x(k - 1) + \varepsilon(k)$,
 $k = k + 1$.

STEP 3. If $k = 101$ then $X(100) = x(k)$ and we are done. If not, we repeat STEP 2.

When we finish, we set $V = V(0) \exp[X(100)]$.

Repeat this last algorithm 1000 times, we obtain a sample of 1000 values of the portfolio. We then choose the VaR such that the 50 least values of the 1000 obtained are smaller than $-\text{VaR}$.

11d. Multivariate Monte Carlo Approach to VaR

The objective is to estimate the *risk* of a portfolio, with 1 year horizon, that comprises two assets:

- A bond A with $\mu_A = 0.08$ expected annual log-return, and $\sigma_A = 0.10$ standard deviation,
- A stock B with $\mu_B = 0.10$ expected annual log-return, and $\sigma_B = 0.20$ standard deviation,
- A positive correlation of $\rho = \rho_{AB} = 0.30$ between this two log-returns.

We assume that we hold π_1 units of A and π_2 units of B , including the possibility of $\pi_1 < 0$ or $\pi_2 < 0$, i.e. being short in one of the two assets.

The value of the portfolio at time $t = 1$ will be

$$V_\pi(1) = \pi_1 S_A(0)e^{X_A} + \pi_2 S_B(0)e^{X_B}.$$

As we do not know the probability distribution of $V_\pi(1)$ we simulate it, simulating $\mathbf{X} = (X_A, X_B)'$ i.e. a gaussian vector with

$$X_A \sim \mathcal{N}(0.08, (0.10)^2),$$

$$X_B \sim \mathcal{N}(0.10, (0.20)^2),$$

$$\rho(X_A, X_B) = 0.30.$$

Departing from uniform independent r.v. on $[0, 1]$, to simulate (X_A, X_B) we perform:

STEP 1. Simulate U_1, U_2 , a pair of independent uniformly distributed r.v. on $[0, 1]$.

STEP 2. Compute

$$\begin{aligned} Y_1 &= \sqrt{-2 \log U_1} \cos(2\pi U_2) \\ Y_2 &= \sqrt{-2 \log U_1} \sin(2\pi U_2) \end{aligned}$$

It can be proved that $(Y_1, Y_2) \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_2)$.

STEP 3. Compute

$$\begin{aligned} X_A &= \mu_A + \sigma_A \sqrt{1 - \rho^2} Y_1 + \sigma_A \rho Y_2 \\ X_B &= \mu_B + \sigma_B Y_2 \end{aligned}$$

It can be checked that the random vector (X_A, X_B) has the desired properties.

For each simulation we compute the value of our portfolio as

$$V_{\pi}(1) = \pi_1 S_A(0)e^{X_A} + \pi_2 S_B(0)e^{X_B}.$$

Repeat the simulation, for instance, 10,000 times, the VaR will be such that the 500 least values of $V_{\pi}(1)$ are smaller than $-\text{VaR}$, and the others 9,500 are bigger than $-\text{VaR}$.

11e. Comments on Derivatives and var.

Up to now we have computed the VaR assuming that the portfolio was conformed by primary financial instruments (as stocks, indices, or bonds).

A characteristic of this financial instruments is that their prices depends **linearly** on the observed market prices.

A further step in VaR is to include in the analysis the situation when the portfolio includes also **derivative** assets, for instance, call options.

The variation of the prices of call options depends in a **nonlinear** way on the stock.

Assuming Black-Scholes formula to compute option prices, if the underlying stock has return

$$S = S_0 \exp(Z),$$

with $Z \sim \mathcal{N}(\mu, \sigma^2)$, then the price of the option $C(S)$ depends in a nonlinear way of the underlying S .

An first order approximation for the price in terms of the underlying is

$$C(S) \sim C(S_0) + C'(S_0)(S - S_0).$$

The derivative $C'(S_0)$ is called the [delta](#) of the option.

A more reliable approach is provided by the [second order](#) approximation, given by

$$C(S) \sim C(S_0) + C'(S_0)(S - S_0) + \frac{1}{2}C''(S_0)(S - S_0)^2.$$

The second derivative $C''(S_0)$ is the [gamma](#) and the substitution of the option for this approximation in a portfolio in order to compute the VaR is known as the [delta-gamma](#) approach.