Pricing perpetual American options for processes with jumps

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PART I: Classical Results

1 Brownian Motion

Brownian motion (or Wiener process) is a stochastic process $W = \{W_t\}_{t\geq 0}$ such that it has

- continuous paths
- independent increments: for $0 \leq t_1 \leq \ldots \leq t_n$, we have that

$$W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$$

  are indepent random variables

- homogeneous and gaussian increments

$$W_t - W_s \sim \mathcal{N}(0, t-s).$$

X is Gaussian or normal ($X \sim \mathcal{N}(\mu, \sigma^2)$) if its probability distribution is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

2 Black-Scholes Financial Market Model

Two assets,

- Bonds $B = \{B_t\}_{t\geq 0}$, such that

$$B_t = B_0 e^{rt},$$

  r is the interest rate.

- Stock $S = \{S_t\}_{t\geq 0}$ defined by

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}$$

  with W a Wiener process. σ is the *volatility*. For short $S_0 = x$, $B_0 = 1$.

Remark: $\{B_t\}$ is the solution of the ODE

$$\frac{dB_t}{B_t} = r dt, \quad B_0 = 1.$$

Similarly, $\{S_t\}$ is the solution of the SDE

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_0 = x.$$

3 Option Pricing

A *call option* is a compromise to sell one unit of S at time τ at price K.

- If $\tau = T$ is fixed we have an *european* option.
- If τ is chosen by the holder we have an *american* option, generally $\tau \leq T$.
- If $0 \leq \tau \leq +\infty$ we have a *perpetual* american option.

The problem is to find the price $V(x, T)$ of the contract, and, in the american case the optimal exercise time τ^*. In words

- How much to pay for such a contract?
- In the american case: When to excercise?

4 Black Scholes Formula

Gives price for european options.

Perfect hedging: It is possible to construct a dinamic portfolio with β_t bonds and γ_t stocks, capital $X_t = \beta_t B_t + \gamma_t S_t$ such that:

1. It is self-financing, i.e.

$$B_t d\beta_t + S_t d\gamma_t = 0$$

2. It gives the exact value of the option at time T,

$$X_T = \beta_T B_t + \gamma_T S_T = (S_T - K)^+$$

The value of the european option was shown to be

Black-Scholes Formula

$$V(x, T) = x\Phi(x_b) - Ke^{-rT}\Phi(x_s)$$

with

$$x_b = \frac{\ln(x/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$x_s = \frac{\ln(x/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$$

where $\Phi(x)$ is the normal distribution.

Remarks:

1. Two main tools of stochastic calculus are employed in the proof of this result:
   - Itô formula: that gives a formula for stochastic diferentials
   - Girsanov Theorem: Changes the measures P into P^*, gives Radom-Nykodym densities in this context.
2. There exist *only one* measure P^*, called the *risk-neutral* probability measure such that

$$V(x, T) = E^*[e^{-rT}(S_T - K)^+]$$

characterized by

$$\frac{S_t}{B_t} \text{ is a } P^* \text{ martingale}$$

This means that the market is *complete*.

5 American options and optimal stopping

Denote by τ the excercise time (choosen now by the holder)

$$0 \leq \tau \leq T$$

If $V_A(x,T)$ is the price of the American option we have to find

- The *price* of the option $V_A(x,T)$
- The *optimal excercise time* τ^* of the option.

Based on *perfect hedging* we obtain that

$$V_A(x,T) = \sup_{\tau} E^*[e^{-r\tau}(S_\tau - K)^+] = E^*[e^{-r\tau^*}(S_{\tau^*} - K)^+]$$

i.e. the pricing of an american option is solved through an optimal stopping problem.

Example: An american call (withouth dividends) is equivalent in prices to the european call, and that the optimal excercise is $\tau_0 = T$ (Merton, 1973).

General case: Non-trivial solutions, solved by:

1. Numerics on a partial differential equation with moving boundary

$$\frac{1}{2}\sigma^2 x^2 V_{xx}(x,t) + rxV_x(x,t) + V_t(x,t) = rV(x,t)$$

   under condition

$$V(x,T) \geq (K-x)^+$$

2. Binomial trees (discrete time-space model) and backward induction.

6 Perpetual Options: $T = +\infty$.

A perpetual option is obtained when $T = +\infty$. Its price gives an upper bound of finite maturity options, and approximate them if T is big. Taking $T \to \infty$ the PDE is an ODE. The price of a *Perpetual put option* is (Mc. Kean, 1965)

$$V_A(x, +\infty) = \begin{cases} Cx^{-\alpha} & \text{if} \quad x > x^* \\ K - x & \text{if} \ 0 \le x \le x^* \end{cases}$$

where

$$\alpha = \frac{2r}{\sigma^2}$$

and

$$C = \alpha^{\alpha}(\frac{K}{1+\alpha})^{1+\alpha} \qquad x^* = \frac{\alpha K}{1+\alpha}$$

Optimal stopping time is

$$\tau^* = \inf\{t \ge 0 \colon S_t \le x^*\}$$

PART II: New directions

7 Criticism and proposals

Two observations:

- Observed stock data does not exhibit statistical properties of BS model (gaussian, independence, ...)
- Theoretical prices of options does not coincide with observed prices (smile phenomena)

We distinguish innovations in at least two direcctions (despite the popularity of BS),

- Stochastic volatility models: forget independence of increments in time, (having continuity)

$$dS_t = S_t(\mu dt + \sigma(S_t, S'_t)dW_t)$$

  with possibility of dependence of other sources of uncertainty S'

- Models with jumps: forget continuity, conserve independence and homogeneous increments

$$S_t = S_0 e^{X_t}$$

  where X is a Lévy process (i.e. a stochastic process with homogeneous independent increments)

8 Lévy processes

$X = \{X_t\}_{t\geq 0}$ is a Lévy process if

- X has paths continuous from the right with limits from the left
- $X_0 = 0$, and has independent increments
- The distribution of $X_t - X_s$ depends on the difference $t - s$.

Lévy-Kinchine formula states

$$E(e^{zX_t}) = e^{t\psi(z)},$$

where ψ is given by

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy)$$

where b and $\sigma \geq 0$ are real constants, and Π is a positive measure on $\mathbf{R} - \{0\}$ such that $\int (1 \wedge y^2)\Pi(dy) < +\infty$, called the Lévy measure. We have a triplet (b, σ, Π).

9 Models with jumps: Lévy markets

Two assets, $B_t = B_0 e^{rt}$ as before, and stock given by

$$S_t = S_0 e^{X_t}$$

with $X = \{X_t\}_{t\geq 0}$ a LP.

10 Examples

Black Scholes

$$X_t = \sigma W_t + (\mu - \frac{1}{2}\sigma^2)t,$$

is a LP with triplet $(\mu - \frac{1}{2}\sigma^2, \sigma, 0)$. No jumps means $\Pi = 0$.

Poisson processes

Let $T_1, T_2, \ldots$ be a sequence of independent exponential random variables with parameter λ. Let

$$N_t = \inf\{k\colon T_1 + T_2 + \ldots T_k \leq t\}.$$

$N = \{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ. $X_t = cN_t - bt$ is a LP with triplet $(-b, 0, \Pi)$, where

$$\Pi(dt, dx) = \lambda dt \delta_c(dx).$$

Compound Poisson process

Actuarial models:

$$X_t = bt + \sum_{k=1}^{N_t} Y_k,$$

$Y = \{Y_k\}_{k\in\mathbf{N}}$ is a sequence of independent random variables with distrubution $F = F(y)$. The triplet is $(b, 0, \lambda F(dx))$.

Mixed exponential jumps

$X = \{X_t\}_{t\geq 0}$ given by

$$X_t = at + \sigma W_t + \sum_{i=1}^{N_t} Y_k,$$

$N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda + \mu$ and $Y = \{Y_k\}_{k\in\mathbf{N}}$ is a sequence of i.i.d. random variables with density

$$h(y) = \frac{\lambda}{\lambda+\mu}\sum_{k=1}^{n} a_k\alpha_k e^{-\alpha_k y}\mathbf{1}_{\{y>0\}} + \frac{\mu}{\lambda+\mu}\sum_{k=1}^{m} b_k\beta_k e^{\beta_k y}\mathbf{1}_{\{y<0\}}$$

for $\lambda \geq 0$, $\mu \geq 0$. The process X jumps upwards at rate λ, downwards at rate μ, and the magnitude of the jumps has mixed-exponential distribution. X has triplet

$$(b, \sigma, (\lambda+\mu)h(x)dx$$

Advantages: Enough rich, allows closed formulae.

11 Non-complete markets and pricing alternatives

We have the equivalences:

Complete market
Perfect hedging
One and only one martingale measure P^*
Existence of rational price.

But: *Non-completeness:* exists more than one martingale measure. Any price is an *admissible price*. How to choose the measure:

- Continous processes: Föllmer and Schweizer introduced the *minimal measure*. Minimizes quadratic loss of a hedge.
- Lévy processes, Gerber and Shiu proposed Esscher transform, from actuarial practice. It minimizes relative entropy (Chan, 1999).
- LP with mixed-exponential jumps are stable under Esscher transforms.

How many martingale measures? If,

$$\psi(1) = b + \frac{1}{2}\sigma^2 + \int_{\mathbf{R}} (e^y - 1 - y\mathbf{1}_{\{|y|<1\}})\Pi(dy) = r,$$

we obtain a martingale measure. In short, we are computing admissible prices of perpetual American options under Lévy information.

12 General results on perpetual options

Assumptions:

(a) The contract has no expiration date, that is, we are considering perpetual American options.

(b) The martingale measure P^* chosen, is such that X:

- X is a Lévy process under P^*,
- $(S_t/B_t)_{t\geq 0}$ is a martingale under P^*

Black-Scholes type formula

Compute

$$V(S_0) = \sup_{\tau\in\mathcal{M}} E^*(e^{-(\gamma+r)\tau}(S_\tau - K)^+),$$

where $\gamma > 0$ is a discount factor. (Darling et al. (1972) r.w.)

Proposition 1 (B-S type formula.) *Consider any Lévy market and $\gamma > 0$. Perpetual call option (put similar) is*

$$V(S_0) = S_0\tilde{P}(M \geq \log(S^*/S_0)) - KP^*(M \geq \log(S^*/S_0))$$

where $M = \sup_{0\leq t<\tau(\gamma+r)} X_t$ $\tau(\gamma + r)$ is an exponential, $S^* = KE^*(e^M)$, and $\tilde{P}$ is the measure defined by

$$d\tilde{P} = \frac{e^M}{E^*(e^M)} dP^*.$$

The optimal strategy is given by

$$\tau^* = \inf\{t \geq 0 \colon S_t \geq S^*\}.$$

13 Closed solutions with mixed-exponential jumps

Proposition 2 *Let X be a LP with mixed-exponential jums. Then*
(a) Denote, for $\gamma > 0$,

$$S^* = K \sum_{k=1}^{n+1} A_k \frac{p_k}{p_k - 1}.$$

with $p = (p_1, \ldots, p_{n+1})$, $0 < p_1 < \ldots < p_{n_a+1}$ the positive roots of $\psi(p) = r + \gamma$, and the coefficients $A = (A_1, \ldots, A_{n+1})$

$$A_j = \frac{\Pi_{k=1}^{n}(\frac{p_j}{\alpha_k} - 1)}{\Pi_{k=1,k\neq j}^{n+1}(\frac{p_j}{p_k} - 1)} \qquad j = 1, \ldots, n+1,$$

The price of a perpetual call is

$$V_c(S_0) = \begin{cases} \Sigma_{k=1}^{n+1} \frac{A_k}{p_k - 1} (\frac{S_0}{S_c^*})^{p_k} & S_0 \leq S_c^*, \\ \\ S_0 - K & S_0 > S_c^*, \end{cases}$$

The optimal exercise is given by the stopping time

$$\tau^* = \inf\{t \geq 0 \colon S_t \geq S^*\}.$$

(b) Put: Analogous.

Remarks: When $h = 0$ (no jumps), we obtain Mc. Kean's result. With $h(x) = \alpha e^{-\alpha x}$ for $x > 0$ we obtain previous results in F&S.

14 Open questions and remarks

- How to choose the martingale measure P^* in an imcomplete market. Continous trajectories: Föllmer and Schweizer introduced the *minimal measure*. For Lévy processes, Gerber and Shiu proposed Escher transforms. General semimartingale models: completely open question.
- Find larger classes of jump distributions with closed solutions: Erlang type distributions or, phase-type.
- Is it possible to obtain closed solutions for finite time T in american options?

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