Arrêt optimal et options perpétuelles pour les processus de Lévy

Optimal stopping and perpetual options for Lévy processes

Ernesto Mordecki

Centro de Matemática, Facultad de Cencias
Universidad de la República, Montevideo, Uruguay

e-mail: mordecki@cmat.edu.uy
http://www.cmat.edu.uy/~mordecki

Plan

1. Financial market models
2. Options and pricing
3. European and American options in B-S market
4. Perpetual options in B-S market
5. Criticism and proposals
6. Perpetual options in Lévy markets
7. Distribution of maxima of Lévy processes
8. Closed solutions: mixed exponential jumps
9. Open questions and remarks

1 Financial Market Models

Two assets,

- Bonds $B = \{B_t\}_{t\geq 0}$, such that

$$B_t = B_0 e^{rt},$$

  r is the interest rate.

- Stock $S = \{S_t\}_{t\geq 0}$ defined by

$$S_t = S_0 e^{X_t}$$

  with $X = \{X_t\}_{t\geq 0}$ is a random process on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P^*)$ and $\{S_t/B_t\}$ is a martingale under P^*. For simplicity $B_0 = 1$.

Consider the following two cases:

Black-Scholes market model. If

$$X_t = \sigma W_t + (r - \frac{\sigma^2}{2})t$$

and $W = \{W_t\}_{t\geq 0}$ is a Wiener process we have the well known Black-Scholes (1973) model (B-S).

Lévy market model. Suppose that X is a Lévy process, i.e. we add a jump process in the B-S model,

$$X_t = \sigma W_t + at + J_t$$

where $J = \{J_t\}_{t\geq 0}$ is certain jump process (Poisson, compund Poisson, ...).

2 Options and pricing

A *call option* is a compromise to sell one unit of S at time τ at price K.

European option: If $\tau = T$ is fixed.

American option: If τ is chosen by the holder, under $0 \leq \tau \leq T$.

Perpetual option: τ chosen by the holder without expiration date, $0 \leq \tau \leq \infty$.

Problem: Find the price $V(S_0, T)$ of the contract, and, in the american case the optimal exercise time τ^*. In words

- How much to pay for such a contract?
- In the american case: When to excercise?

3 European and American options in a B-S

The value of the european option is given by the B-S formula

$$V(S_0, T) = E^*(e^{rT}(S_T - K)^+ =$$
$$S_0\tilde{P}(S_T > K) - Ke^{-rT}P^*(S_T > K)$$

with

$$d\tilde{P} = e^{X_T - rT}dP^*$$

American options and optimal stopping

τ, the excercise time, is chosen by the holder under the restriction $0 \leq \tau \leq T$. If $V_A(S_0, T)$ is the price of the American option we have also to find the *optimal excercise time* τ^* of the option. Based on *perfect hedging* we obtain that

$$V_A(S_0, T) = \sup_\tau E^*[e^{-r\tau}(S_\tau - K)^+] = E^*[e^{-r\tau^*}(S_{\tau^*} - K)^+]$$

i.e. the pricing of an american option leds to an optimal stopping problem. Non-explicit solutions, solved by: (a) Numerics on a partial differential equation with moving boundary

$$\frac{1}{2}\sigma^2 x^2 V_{xx}(x, t) + rxV_x(x, t) + V_t(x, t) = rV(x, t)$$

under condition $V(x, T) \geq (K - x)^+$, or (b) binomial trees (discrete time-space model) and backward induction.

4 Perpetual Options in B-S: $T = +\infty$.

A perpetual option is obtained when $T = +\infty$.

- $V_A(S_0, T) \leq V_A(S_0, \infty) = V_A(S_0)$
- $V_A(S_0, T) \to V_A(S_0), \quad T \to \infty$
- The PDE becames an ODE

The price of a *Perpetual put option* is (Mc. Kean, 1965)

$$V_A(S_0) = \sup_{\tau \geq 0} E^* e^{-r\tau}(K - S_\tau)^+ = \begin{cases} CS_0^{-\alpha} & S_0 > S^* \\ K - S_0 & S_0 \leq S^* \end{cases}$$

where

$$\alpha = \frac{2r}{\sigma^2}, \quad S^* = K\frac{\alpha}{1+\alpha}, \quad C = \alpha^\alpha \left(\frac{K}{1+\alpha}\right)^{1+\alpha}.$$

Optimal stopping time is

$$\tau^* = \inf\{t \geq 0 \colon S_t \leq S^*\}$$

5 Criticism to B-S and proposals

(a) Observed stock data does not exhibit statistical properties of BS model (gaussian, independence, ...)
(b) Theoretical prices of options does not coincide with observed prices (smile phenomena)

We distinguish innovations in at least two direcctions (despite the popularity of B-S),

- Stochastic volatility models: forget independence of increments in time, (having continuity)

  $$dS_t = S_t(\mu dt + \sigma(S_t, S_t')dW_t)$$

  with possibility of dependence of other sources of uncertainty S'

- Models with jumps: forget continuity, conserve independence and homogeneous increments

  $$S_t = S_0 e^{X_t}$$

  where X is a Lévy process.

Lévy-Khinchine formula states $E(e^{zX_t}) = \exp(t\psi(z))$

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy),$$

If $\tau(\gamma)$ an exponential time with parameter γ independent of X

$$M = \sup_{0\le t<\tau(\gamma)} X_t \quad \text{and} \quad I = \inf_{0\le t<\tau(\gamma)} X_t,$$

denote the supremum (infimum) of X killed at rate γ.

6 Perpetual options in Lévy markets

Theorem 1 *Let X be a Lévy process and $\delta > 0$. Put $\gamma = \delta + r$*

(a) We have $E(e^M) < \infty$, and for the optimal stopping problem

$$\sup_{\tau \geq 0} E^*(e^{-(\delta+r)\tau}(S_\tau - K)^+) = \frac{E[S_0 e^M - KE(e^M)]^+}{E(e^M)},$$

and optimal stopping time

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S^*\},$$

with $S^* = KE(e^M)$.

(b) Put option, similar with I.

Corollary 1 (B-S type formula) *(a) For $\gamma > 0$*

$$V(S_0) = S_0 \tilde{P}(S_0 e^M \geq S^*) - KP^*(S_0 e^M \geq S^*),$$

and $\tilde{P}$ is the measure defined by

$$d\tilde{P} = \frac{e^M}{E^*(e^M)} dP^*.$$

(b) Put similar.

7 Distribution of Maxima for LP

Theorem 2 *Let X be a Lévy process with $\sigma > 0$, and*

$$\Pi(y) = \begin{cases} \lambda \sum_{k=1}^{n} a_k \alpha_k e^{-\alpha_k y}, & y > 0, \\ \\ \pi(dy) & y < 0, \end{cases}$$

$\pi(dy)$ arbitrary, i.e. X has mixed exponential positive jumps. Then M, when proper, has density

$$f_M(y) = \sum_{k=1}^{n+1} A_k p_k e^{-p_k y}, \quad y > 0,$$

where $p = (p_1, \ldots, p_{n+1})$, with

$$0 < p_1 < \alpha_1 < p_2 < \ldots < \alpha_n < p_{n+1}$$

are the real and positive roots of $\psi(p) = \gamma$ and $A = (A_1, \ldots, A_{n+1})$ are given by

$$A_j = \frac{\prod_{k=1}^{n}(\frac{p_j}{\alpha_k} - 1)}{\prod_{k=1,k\neq j}^{n+1}(\frac{p_j}{p_k} - 1)} \qquad j = 1, \ldots, n+1,$$

Remarks:

- $\psi(z)$ denotes the analitical continuations of ψ.
- Analogous results for I when *negative* jumps are a mixture of exponentials.

8 Closed solutions: mixed exponential jumps

Theorem 3 *Let X be a LP with positive mixed exponential jums. Then*
(a) Denote, for $\gamma > 0$,

$$S^* = K \sum_{k=1}^{n+1} A_k \frac{p_k}{p_k - 1},$$

with $p = (p_1, \dots, p_{n+1})$, $0 < p_1 < \dots < p_{n_a+1}$ the positive roots of $\psi(p) = r + \gamma$, and the coefficients $A = (A_1, \dots, A_{n+1})$

$$A_j = \frac{\prod_{k=1}^{n}(\frac{p_j}{\alpha_k} - 1)}{\prod_{k=1, k\neq j}^{n+1}(\frac{p_j}{p_k} - 1)} \qquad j = 1, \dots, n+1,$$

The price of a perpetual call is

$$V_c(S_0) = \begin{cases} \sum_{k=1}^{n+1} \frac{A_k}{p_k - 1}(\frac{S_0}{S^*})^{p_k} & S_0 \leq S^*, \\ \\ S_0 - K & S_0 > S^*. \end{cases}$$

The optimal exercise is given by the stopping time

$$\tau^* = \inf\{t \geq 0 \colon S_t \geq S^*\}.$$

(b) Put: Analogous.

Remarks:

- When $\Pi = 0$ (no jumps), we obtain Mc. Kean's result.
- With $h(x) = \alpha e^{-\alpha x}$ for $x > 0$ and $\pi = 0$ we obtain previous results in Finance and Stochastics (1999).

9 Open questions and remarks

- How to choose the martingale measure P^* in an incomplete market. Continous trajectories: Föllmer and Schweizer introduced the *minimal measure*. For Lévy processes, Gerber and Shiu proposed Escher transforms. General semimartingale models: completely open question.
- Find larger classes of jump distributions with closed solutions: Erlang type distributions or, phase-type.
- Details at `http://www.cmat.edu.uy/~mordecki`