Optimal stopping and perpetual options for Lévy processes

Arrêt optimal et options perpétuelles pour les processus de Lévy

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Plan

1 Financial Market Models

Two assets,

- Bonds $B = \{B_t\}_{t\geq 0}$, such that

$$B_t = B_0 e^{rt}, \qquad B_0 = 1.$$

  $r > 0$ is the interest rate.

- Stock $S = \{S_t\}_{t\geq 0}$ defined by

$$S_t = S_0 e^{X_t}$$

  with $X = \{X_t\}_{t\geq 0}$ is a random process on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ and

$$\{S_t/B_t\} \ \text{ is a martingale under } P.$$

Black-Scholes market model.

If

$$X_t = \sigma W_t + (r - \delta - \frac{\sigma^2}{2})t$$

and $W = \{W_t\}_{t\geq 0}$ is a Wiener process we have the well known Black-Scholes (1973) model (B-S), with dividends rate δ. In this case $S(r-\delta)$ denotes the stock.

2 Lévy market model.

Suppose that X is a Lévy process, i.e. we add a jump process in the B-S model,

$$X_t = at + \sigma W_t + J_t$$

where $J = \{J_t\}_{t\geq 0}$ is certain jump process (Poisson, compund Poisson, ...).

Lévy-Khinchine formula states $E(e^{zX_t}) = \exp(t\psi(z))$

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}})\Pi(dy),$$

ψ is the characteristic exponent, and the triplet (b, σ^2, Π) characterizes the law of the process.

Condition $\{S_t/B_t\}$ is a martingale reads $\psi(1) = r$, and in particular $E(e^{X_t})$ must be finite.

3 European and American options in a B-S

The value of the european option is given by the B-S formula

$$V(S_0,T) = E(e^{rT}(S(r)_T - K)^+) =$$
$$S_0\tilde{P}(S(r)_T > K) - Ke^{-rT}P(S(r)_T > K)$$

with

$$d\tilde{P} = e^{X_T - rT}dP$$

Perpetual Options in B-S.

The price of a *Perpetual put option* is (Mc. Kean, 1965)

$$\sup_{\tau\geq 0} Ee^{-r\tau}(K - S(r)_\tau)^+ = \begin{cases} K - S_0, & S_0 \leq S_p^* \\ AS_0^{-\alpha}, & S_0 > S_p^* \end{cases}$$

where

$$\alpha = \frac{2r}{\sigma^2}, \quad S_p^* = K\frac{\alpha}{1+\alpha}, \quad A = \alpha^\alpha(\frac{K}{1+\alpha})^{1+\alpha}.$$

Optimal stopping time is

$$\tau^* = \inf\{t \geq 0 \colon S_t \leq S_p^*\}$$

The price of a *Perpetual call option* with positive dividends rate $\delta > 0$ is (Merton, 1973)

$$\sup_{\tau \geq 0} Ee^{-r\tau}(S(r-\delta)_\tau - K)^+ = \begin{cases} BS_0^\beta, & S_0 < S_c^* \\ S_0 - K, & S_c^* \leq S_0 \end{cases}$$

where

$$\beta = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{(\frac{1}{2} + \frac{r-\delta}{\sigma^2})^2 + \frac{2\delta}{\sigma^2}} > 1$$

$$S_c^* = K\frac{\beta}{\beta - 1}, \quad B = \beta^{-\beta}(\frac{\beta-1}{K})^{\beta-1}.$$

Optimal stopping time is

$$\tau^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\}$$

Remark: Problems with positive dividens are equivalent to the pricing of *discounted* options at a constant rate γ withouth dividends, i.e.

$$\sup_{\tau \geq 0} Ee^{-(r+\gamma)\tau}(S(r)_\tau - K)^+$$

and for the put,

$$\sup_{\tau \geq 0} Ee^{-(r+\gamma)\tau}(K - S(r)_\tau)^+.$$

4 Generalizations of B-S

Problems:

(a) Observed stock data does not exhibit statistical properties of BS model (gaussian, independence, ...)

(b) Theoretical prices of options does not coincide with observed prices (smile phenomena)

We distinguish innovations in at least two direcctions (despite the popularity of B-S),

- Stochastic volatility models: forget independence, forget homogeneity of increments in time, preserve continuity

  $$dS_t = S_t(\mu dt + \sigma(S_t, S'_t)dW_t)$$

  with possible dependence of other sources of uncertainty S'

- Models with jumps: forget continuity, conserve independence and homogeneous increments

  $$S_t = S_0 e^{X_t}$$

  where X is a Lévy process.

5 Perpetual options in Lévy markets

Let X be a Lévy process, $S_t = S_0 e^{X_t}$ and $\gamma > 0$. Denote

$$V_C(S_0) = \sup_{\tau \geq 0} E(e^{-(r+\gamma)\tau}(S_\tau - K)^+).$$

Theorem 1 (Discounted perpetual calls) *Let*

$$M = \sup_{0 \leq t < \tau(r+\gamma)} X_t$$

with $\tau(r+\gamma)$ an exponential time with parameter $r+\gamma$, and independent of X. Then $E(e^M) < \infty$, and the optimal stopping problem has cost function

$$V_C(S_0) = \frac{E[S_0 e^M - KE(e^M)]^+}{E(e^M)},$$

and optimal stopping time

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\},$$

with $S_c^* = KE(e^M)$.

Remark: This Theorem is a direct generalization of Darling, Ligget and Taylor (1972) result for random walks.

Denote now, for $\gamma \geq 0$

$$V_P(S_0) = \sup_{\tau \geq 0} E(e^{-(r+\gamma)\tau}(K - S_\tau)^+).$$

Theorem 2 (Discounted perpetual puts) *Let*

$$I = \inf_{0 \leq t < \tau(r+\gamma)} X_t$$

where $\tau(\gamma + r)$ an exponential time with parameter $\gamma + r$ independent of X. Then $E(e^I) > 0$ and we have

$$V_P(S_0) = \frac{E[KE(e^I) - S_0 e^I]^+}{E(e^I)},$$

and optimal stopping time

$$\tau_p^* = \inf\{t \geq 0 \colon S_t \leq S_p^*\},$$

where $S_p^* = KE(e^I)$.

Corollary 1 (B-S type formula) *(a) If $\gamma > 0$*

$$V_C(S_0) = S_0 \tilde{P}(S_0 e^M \geq S_c^*) - KP(S_0 e^M \geq S_c^*),$$

and $\tilde{P}$ is the measure defined by

$$d\tilde{P} = \frac{e^M}{E(e^M)} dP.$$

(b) For $\gamma \geq 0$,

$$V_P(S_0) = KP(S_0 e^I \leq S_p^*)) - S_0 \tilde{P}(S_0 e^I \leq S_p^*)),$$

where $\tilde{P}$

$$d\tilde{P} = \frac{e^I}{E^*(e^I)} dP.$$

6 Distribution of Maximum and infimum of LP

Theorem 3 *(a) Let X be a Lévy process with $\sigma > 0$, and*

$$\Pi(y) = \begin{cases} \lambda \sum_{k=1}^{n} a_k \alpha_k e^{-\alpha_k y}, & y > 0, \\ \\ \pi(dy) & y < 0, \end{cases}$$

$\pi(dy)$ arbitrary, i.e. X has mixed exponential positive jumps. ψ denotes the analytical continuation to $\{\Re(z) \geq 0, z \neq \alpha_k, k = 1, \ldots, n\}$ of the characteristic exponent. Then M, when proper, has density

$$f_M(y) = \sum_{k=1}^{n+1} A_k p_k e^{-p_k y}, \quad y > 0,$$

where $p_1, \ldots, p_{n+1}$, with

$$0 < p_1 < \alpha_1 < p_2 < \ldots < \alpha_n < p_{n+1}$$

are the real and positive roots of $\psi(p) = r + \gamma$ and

$$A_j = \frac{\prod_{k=1}^{n}\left(\frac{p_j}{\alpha_k} - 1\right)}{\prod_{k=1, k\neq j}^{n+1}\left(\frac{p_j}{p_k} - 1\right)}, \qquad j = 1, \ldots, n+1.$$

(b) Let X be a Lévy process with $\sigma > 0$, and

$$\Pi(y) = \begin{cases} \pi(dy) & y > 0, \\ \\ \mu \sum_{k=1}^{n} b_k \beta_k e^{\beta_k y} dy & y < 0, \end{cases}$$

$\pi(dy)$ is arbitrary. ψ denotes now the analytical continuation to $\{\Re(z) \leq 0, z \neq -\beta_k, k = 1, \ldots, n\}$ of the characteristic exponent. Then I has density

$$f_I(y) = \sum_{k=1}^{n+1} B_k q_k e^{-q_k y}, \quad y < 0,$$

where $-q_{n+1} < -q_n < \ldots < -q_1 < 0$ are the real and negative roots of $\psi(p) = r + \gamma$, and the coefficients are given by

$$B_j = \frac{\prod_{k=1}^{n}(\frac{q_j}{\beta_k} - 1)}{\prod_{k=1,k\neq j}^{n+1}(\frac{q_j}{q_k} - 1)}, \qquad j = 1, \ldots, n+1,$$

7 Closed solutions: mixed exponential jumps

Plug in previous formulae to obtain

Theorem 4 (Perpetual Call) *Let X be a LP with positive mixed exponential jums. Denote, for $\gamma > 0$,*

$$S_c^* = K \sum_{k=1}^{n+1} A_k \frac{p_k}{p_k - 1},$$

with $0 < p_1 < \ldots < p_{n+1}$ the positive roots of $\psi(p) = r + \gamma$, and the coefficients $A_1, \ldots, A_{n+1}$ as before. The price of a perpetual call is

$$V_c(S_0) = \begin{cases} \sum_{k=1}^{n+1} \frac{A_k}{p_k - 1} \left(\frac{S_0}{S_c^*}\right)^{p_k} & S_0 \leq S_c^*, \\ \\ S_0 - K & S_0 > S_c^*. \end{cases}$$

The optimal exercise is given by the stopping time

$$\tau^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\}.$$

Remarks:

- When $\Pi = 0$ (no jumps), we obtain Merton's (1973) result.
- With $h(x) = \alpha e^{-\alpha x}$ for $x > 0$ and $\pi = 0$ we obtain our previous results in Finance and Stochastics (1999).

Theorem 5 (Perpetual Put) *Let X be a LP with negative mixed exponential jums. Denote, for $\gamma \geq 0$,*

$$S_p^* = K \sum_{k=1}^{n+1} B_k \frac{q_k}{q_k + 1},$$

with $-q_1, \ldots, -q_{n+1}$, the negative roots of $\psi(p) = r + \gamma$, and the coefficients $B_1, \ldots, B_{n+1}$ as before. The price of a perpetual put is

$$V_P(S_0) = \begin{cases} K - S_0 & S_0 \leq S_p^*, \\ \\ \sum_{k=1}^{n+1} \frac{B_k}{q_k+1} \left(\frac{S_0}{S_p^*}\right)^{q_k} & S_0 > S_p^*. \end{cases}$$

The optimal exercise is given by the stopping time

$$\tau_p^* = \inf\{t \geq 0\colon S_t \leq S_p^*\}.$$

Remarks:

- When $\Pi = 0$ (no jumps), we obtain Mc Kean's (1965) result.
- With $h(x) = 0$ and $\pi = \lambda F$, F a distribution we recover X.L. Zang's (1994) results.

8 Open questions and remarks

(a) How to choose the martingale measure P in an incomplete market. For Lévy processes, Gerber and Shiu (1994) proposed Escher transforms.

(b) $X_t = at + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ with $\{Y_i\}$ i.i.d.r.v. two-sided mixed exponentially distributed, gives closed prices for call and put, is enough rich, and remains with the same type distribution (changing parameters) under Esscher transforms.

(c) Details at `http://www.cmat.edu.uy/~mordecki`

9 References

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