Dedicada a *Gonzalo Pérez Iribarren*

Optimal stopping for for stochastic processes and applications in finance

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1 The model

Let $X = \{X_t\}_{t\geq 0}$ be a real valued stochastic process on $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, such that

- X is càdlàg and adapted,
- $X_0 = 0$, for $0 \leq s \leq t$ $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$
- The distribution of $X_t - X_s$ depends on the difference $t - s$.

X is called a Lévy process (LP). Lévy-Kinchine formula states

$$E(e^{izX_t}) = e^{t\psi(z)},$$

where $\psi\colon \mathbf{R} \to \mathbf{C}$, the charactierisic exponent is given by

$$\psi(z) = ibz - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{izy} - 1 - izy\mathbf{1}_{\{|y|<1\}})\Pi(dy)$$

where b and $\sigma \geq 0$ are real constants, and Π is a positive measure on $\mathbf{R} - \{0\}$ such that $\int(1 \wedge y^2)\Pi(dy) < +\infty$, called the Lévy measure.

2 Examples of LP

2.1 Wiener process with trend

$$X_t = \sigma B_t + at,$$

with $B = \{B_t\}_{t\geq 0}$ a Wiener process, a σ real constants, $\Pi \equiv 0$,

2.2 Poisson processes

Let $T_1, T_2, \ldots$ be a sequence of independent exponential random variables with parameter λ. Let

$$N_t = \inf\{k \colon T_1 + T_2 + \ldots T_k \leq t\}$$

$N = \{N_t\}_{t\geq 0}$ is a Poisson process with parameter λ, and a LP with $a = 0$, $\sigma = 0$, and

$$\Pi(dt, dx) = \lambda dt \delta_1(dx).$$

2.3 Compound Poisson process

$$X_t = at + \sum_{k=1}^{N_t} Y_k,$$

$Y = \{Y_k\}_{k\in\mathbf{N}}$ is a sequence of independent random variables with distrubution $F = F(y)$. The characteristic exponent given by

$$\psi(z) = iaz + \lambda \int_{-\infty}^{+\infty} (e^{izy} - 1) dF(y)$$

Remark: A linear combination of LP is a LP.

3 American perpetual option/optimal stopping

We have a financial market with two assets,

- The bonds $B = (B_t)_{t\geq 0}$, such that

$$\frac{dB_t}{B_t} = rdt, \quad B_0 = 1.$$

- The stock $S = (S_t)_{t\geq 0}$ defined by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dX_t, \quad S_0 = x.$$

with X a LP ($X = W$ is the usual Black-Scholes model).

Options

An option is a compromise to sell one unit of S at time τ at price K.

- If $\tau = T$ is fixed we have an european option.
- If $0 \leq \tau \leq T$ is random (without anticipating) we have an american option.
- If $0 \leq \tau \leq +\infty$ we have a perpetual american option (*our case*).

The problem is to find the price $V(x)$ of the contract, and the optimal exercise time τ^*. It is known that the solution to this problem is

$$V(x) = \sup_{\tau} E(S_\tau - K)^+ = E(S_{\tau^*} - K)^+.$$

Note: It can be seen, that $S_t = e^{Y_t}$ with Y a LP. We expose the results with $S = X$ and $K = 0$, (linear case) and make a final remark on how to obtain the exponential (or geometric) case.

4 Pricing

Our problem is: find

- The price $V(x)$ of the contract.
- The optimal exercise time τ^*.

such that

$$V(x) = \sup_{\tau} E(x + X_\tau)^+ = E(x + X_{\tau^*})^+.$$

The solution is expressed in terms of

$$M = \sup_{t \geq 0} X_t$$

the overall supremum of X, whenever $E(M)$ is finite. If

$$B(x) = P(M \geq x)$$

is the probability of crossing the barrier at level x, and $x^* = E(M)$.

Theorem 1 *Let X be a Lévy process such that*

$$E(M) = \int_0^{+\infty} B(y)dy < +\infty.$$

Then, the solution to the optimal stopping problem

$$V(x) = E(x + M - E(M))^+ = \int_{E(M)-x}^{+\infty} B(y)dy,$$

$$\tau^* = \inf\{t \geq 0 \colon x + X_t \geq E(M)\}.$$

Remarks:

In general, the optimal stopping problem for a Markov process leds to a free boundary problem.

The problem reduces to the barrier problem (i.e. finding $B(x)$) for X. that has fixed boundary conditions.

This is equivalent to find the *ruin probablilty* for the process $(x - X_t)$

A similar result for random walks is Darling, Ligget and Taylor (1972), (DLT).

It opens the question to find simple conditions in order to $E(M) < +\infty$.

5 Prophet vs gamblers

Theorem 2 *Let X be an arbitrary Lévy process, and $x \geq 0$.*

(i) *If $E(M) < +\infty$, the following inequality holds:*

$$V(x) \leq E(x + M) \leq eV(x).$$

(ii) *If $E(M) = +\infty$, then $V(x) = +\infty$.*

This result is called a prophet inequality in the sense that it compares the expected reward of a gambler that has complete foresight, or a "prophet" that can choose the moment at wich the maximum is attained, obtaining $x + M$, with the expectation of the best possible strategy in the class $\mathcal{M}$ of nonanticipating strategies.

DLT (1972) for random walks, Hill and Kertz (1990) for sequences of rnadom varibles.

In mathematical finance, it relates *lookback* with *vainila* options.

6 Ruin probablilities. Closed solutions

X has characteristic exponent

$$\psi(z) = iaz - \frac{1}{2}\sigma^2 z^2 + \lambda\frac{iz}{\alpha - iz} + \int_{-\infty}^{0}(e^{izy} - 1 - izy\mathbf{1}_{\{-1<y<0\}})\pi(dy)$$

where π is an arbitrary Lévy measure concentrated on $(-\infty, 0)$ and $\lambda > 0$. X has

- positive jumps with exponential distribution at poissonian moments with rate λ.
- arbitrary negative jumps.

Consider the *Laplace exponent* of X

$$\kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \lambda\frac{p}{\alpha - p} + \int_{-\infty}^{0}(e^{py} - 1 - py\mathbf{1}_{\{-1<y<0\}})\pi(dy).$$

Observe that, $\kappa(p) = \psi(-ip)$, and $E(e^{pX_t}) = e^{t\kappa(p)}$.

Under condition

$$\kappa'(0+) = \lim_{p\to 0+}\frac{1}{p}\kappa(p) = a + \frac{\lambda}{\alpha} + \int_{-\infty}^{-1} y\pi(dy) < 0,$$

(the integral can take the value $-\infty$) equation $\kappa(p) = 0$ has two positive roots p_1, p_2 that satisfy $0 < p_1 < \alpha < p_2$.

Theorem 3 *Under the stated conditions*

(i) *The process X drifts to $-\infty$, and the barrier problem for X has solution*

$$B(x) = \begin{cases} A_1 e^{-p_1 x} + A_2 e^{-p_2 x}, & x > 0, \\ 1, & x \leq 0, \end{cases}$$

where p_1 and p_2 are the two positive solutions of the equation $\kappa(p) = 0$, and the constants A_1 and A_2 are given by

$$A_1 = \frac{p_2(\alpha - p_1)}{\alpha(p_2 - p_1)}, \qquad A_2 = \frac{p_1(p_2 - \alpha)}{\alpha(p_2 - p_1)}.$$

(ii) *The random variable M has finite expectation,*

$$x^* = E(M) = A_1/p_1 + A_2/p_2,$$

and the optimal stopping problem for the process X with reward x^+ has solution

$$V(x) = \begin{cases} \frac{A_1}{p_1} e^{p_1(x - x^*)} + \frac{A_2}{p_2} e^{p_2(x - x^*)}, & x < x^*, \\ x, & x \geq x^*. \end{cases}$$

7 Conditions for $E(M) < +\infty$

It can be seen, that under

$$b + \int_{\{|y|\geq 1\}} y\Pi(dy) < 0.$$

we have

$$\lim_{t\to+\infty} X_t = -\infty \quad \text{i.e.} \quad X \quad \textit{drifts to} \quad -\infty.$$

Under the aditional condition

$$\int_1^{+\infty} y^2\Pi(dy) < +\infty,$$

we have

$$E(M) < +\infty.$$

Proof: a) Decompose X_t as

$$X_t = \sigma B_t + M_t + at + \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq 1\}}.$$

where

- $B = \{B_t\}_{t\geq 0}$ is a Brownian motion,
- $M_t = \int_0^t \int_{\{|y|<1\}} y * (\mu - \nu)$ is a martingale with bounded jumps,
- $\mu = \mu(\omega, dy, dt)$ is the random measure of the process X, $\nu = \nu(dy, dt) = \Pi(dy)dt$ its compensator,
- $b = E(X_1 - \int_0^1 \int_{\{|y|\geq 1\}} y * \mu)$.

b) Add a negative drift on each term B, M

c) Reduce the sum with big jumps to the random walk case. Theorem 5 in Kiefer and Wolfowitz (1956).

8 Examples

We are interested in

- Crucial point: Find $B(x) = P(M \geq x)$.
- Compute $x^* = E(M) = \int_0^{+\infty} B(y)dy$
- Establish whether the bound e is optimal.

8.1 Lévy Processes with no positive jumps

π has support on $(-\infty, 0)$. The distibution of M can be computed explicitely. ψ and κ are

$$\psi(z) = iaz - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{0} (e^{izy} - 1 - izy\mathbf{1}_{\{-1<y<0\}})\pi(dy).$$

$$\kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \int_{-\infty}^{0} (e^{py} - 1 - py\mathbf{1}_{\{-1<y<0\}})\pi(dy).$$

Under

$$\kappa'(0+) = a + \int_{-\infty}^{-1} y\pi(dy) < 0,$$

there exists an unique positive root p_0 of the equation $\kappa(p) = 0$,

$$B(x) = P(M \geq x) = e^{-p_0 x} \quad \text{and} \quad E(M) = \frac{1}{p_0}.$$

Observe that the bound in the prophet inequality is attained at $x = 0$, that is

$$E(M) = eV(0).$$

Wiener process with trend

$$X_t = \sigma B_t + at,$$

The condition is $a < 0$. The root is $p_0 = -\frac{2a}{\sigma^2}$

Risk processes

$$X_t = at - \sum_{k=1}^{N_t} Y_k,$$

The condition is

$$a < \lambda \int_0^{+\infty} yF(dy),$$

where the integral can take the value $+\infty$. p_0 is the positive root of the equation (Lundberg constant)

$$ap + \lambda \int_0^{+\infty} (e^{-py} - 1)F(dy) = 0.$$

- **Exponential jumps. (M, PMU)**

  If $F(x) = 1 - e^{-\beta x}$ we obtain $p_0 = \frac{\lambda}{a} - \beta$.

- **Negative Poisson process. (KM)**

  Taking $F(dx) = \delta_\rho(dx)$, the probability measure concentrated at $\rho > 0$, our process results

  $$X_t = at - N_t.$$

  The condition is $a < \rho\lambda$, and p_0 is the solution of the equation

  $$ap + \lambda(e^{-\rho p} - 1) = 0.$$

8.2 Processes with positive jumps

Poisson processes (KM)

$$X_t = N_t - at$$

$B(x)$ was obtained by Pyke (1959), and when $\lambda < a$ is

$$B(x) = 1 - (1 - \frac{\lambda}{a}) \sum_{k=0}^{[x]} (\frac{\lambda}{a})^k \frac{(k-x)^k}{k!} e^{-\lambda(k-x)/a}.$$

Then the problem is solved according to Theorem 1. It can be see

$$x^* = \frac{\lambda}{2(a-\lambda)}.$$

8.3 Signed exponential jumps

$$X_t = at + \sigma B_t + \sum_{k=1}^{N_t^\lambda} Y_k^\alpha - \sum_{k=1}^{N_t^\mu} Y_k^\beta$$

with

- $B = \{B_t\}_{t\geq 0}$ is a Wiener process,
- $N^\lambda = \{N_t^\lambda\}_{t\geq 0}$, $N^\mu = \{N_t^\mu\}_{t\geq 0}$ are Poisson processes with parameters λ and μ
- $Y^\alpha = \{Y_k^\alpha\}_{k\in\mathbf{N}}$ $Y^\beta = \{Y_k^\beta\}_{k\in\mathbf{N}}$ two sequences of independent exponentially distributed random variables with parameters α and β.
- B, N^λ, N^μ, Y^α, Y^β are independent.

X has exponentially distributed positive (negative) jumps with rate λ (μ) and parameter α (β).

$$\psi(z) = iaz - \frac{1}{2}\sigma^2 z^2 + \lambda \frac{iz}{\alpha - iz} - \mu \frac{iz}{\beta + iz}$$

In consequence, the Laplace exponent is

$$\kappa(p) = ap + \frac{1}{2}\sigma^2 p^2 + \frac{\lambda p}{\alpha - p} - \frac{\mu p}{\beta + p}.$$

The conclusion follows from Theorem 3. This example is motivated by example VI.8 (b) in Feller (1966). The case with only positive jumps and was considered in (M BB).

Compound Poisson process with exponential jumps

In the following example we have $P(M = 0) > 0$. This fact makes the bound in Theorem 2 not to be attained.

$$X_t = \sum_{k=1}^{N_t} Y_k - at,$$

$Y = \{Y_k\}_{k \in \mathbf{N}}$ i.i.d.r.v. exponential(α).

$$\psi(z) = -iaz + \lambda \frac{iz}{\alpha - iz}.$$

The condition reads $\lambda < a\alpha$. It is known that for $x > 0$

$$B(x) = \frac{\lambda}{a\alpha} e^{-\rho x}$$

with $\rho = \alpha - \frac{\lambda}{a} > 0$. Simple computations show, that

$$x^* = E(M) = \frac{\lambda}{\alpha(a\alpha - \lambda)}$$

In particular, taking $\frac{\lambda}{a\alpha} = \frac{1}{2}$, follows that

$$\sup_{x \geq 0} \frac{E(x + M)}{V(x)} = 2 < e.$$

and the bound in Theorem 2 is not attained.

9 Proof of Theorem 1.

Let $X^n = \{X_{k2^{-n}}: k \in \mathbf{N}\}$, $n \in \mathbf{N}$.

Denote $g(x) = x^+$ and $T_t g(x) = Eg(x + X_t)$.

$$Q_n g(x) = \max\{g(x), T_{2^{-n}} g(x)\}, \qquad M^n = \sup_k X_{k2^{-n}}.$$

Observe that $M^n \to M = \sup_t X_t$ P-$a.s.$. Denote $\mathbf{F}^n = (\mathcal{F}^n_k)$ the filtration of X^n. $\mathcal{M}^n$ the class of stopping times relative to $\mathbf{F}^n$, that take values $k2^{-n}$, $k \in \mathbf{N}$. Now

$$\sup_{\tau \in \mathcal{M}} Eg(x + X_\tau) = \lim_n (\lim_N Q_n^N g(x)) = \lim_n (\sup_{\tau \in \mathcal{M}_n} Eg(x + X_\tau))$$

$$= \lim_n E(x + M^n - E(M^n))^+ = E(x + M - E(M))^+.$$

The first equality follows from Lemma III.3 and Theorem II.3, in Shiryaev (1978). The second from Theorem II.3 also in Shiryaev (1978). The third from Darling, Ligget and Taylor (1972) and the fourth by dominated convergence, because

$$-E(M) \le M^n - E(M^n) \le M.$$

(ii) is proved as in DLT (1972).

Remark: This proof also apply to any result on random walks, for instance the geometric case, also given in this context by DLT.

10 Proof of Theorem 2.

Let $E(M) < +\infty$. We have to prove, that for $x \geq 0$

$$E(x + M) \leq eV(x).$$

First we prove based on the independence of the increments

$$B(x + y) \geq B(x)B(y).$$

Second, we obtain that for $x \geq 0$,

$$\int_x^{+\infty} B(y)dy \geq x^* e^{-\frac{x}{x^*}}.$$

This gives a relation on the tail of M with the tail of an exponential r.v. with the same expected value.

Finally, as $V(x) = E(x + M - E(M))^+$, this amounts to say, that for $x \geq 0$

$$\frac{E(x + M)}{E(x + M - E(M))^+} = \frac{x + x^*}{\int_{x^*-x}^{+\infty} B(y)dy}$$
$$\leq \frac{x + x^*}{x^*} \exp\left(\frac{x^* - x}{x^*}\right) = c(x).$$

It is direct to see that the function $c(x)$ is decreasing, and that $c(0) = e$, concluding the proof of (i).

$E(M) = +\infty$ is deduced from the first case.

11 Proof of Theorem 3.

Fix $x > 0$, and $A(z) = B(x - z)$. Apply Meyer-Itô's formula to X and A, at time $t \wedge \tau_x$:

$$A(X_{t\wedge\tau_x}) - A(0) = \int_0^{t\wedge\tau_x} (L^X A)(X_{s-})ds + M(A)_{t\wedge\tau_x}$$

where

$$(L^X A)(x) = \frac{1}{2}\sigma^2 A''(x) + aA'(x)$$
$$+ \int_{\mathbf{R}} (A(x+y) - A(x) - yA'(x)\mathbf{1}_{\{-1<y<0\}})\Pi(dy)$$

is the infinitesimal generator of X, and $M(A)$ is a local martingale.

Next

$$L^X A(z) = 0 \quad \text{for} \quad z < x.$$

As $X_{s-} < x$ on the set $[0, t \wedge \tau_x]$,

$$\int_0^{t\wedge\tau_x} (L^X A)(X_{s-})ds = 0.$$

Taking expected values and limit as $t \to +\infty$ in

$$B(x) = A(0) = EA(X_{\tau_x}) = P(\tau_x < +\infty) = P(M \geq x).$$

because $X_{\tau_x} = -\infty$ on the set $\{\tau_x = +\infty\}$ and $\lim_{x\to-\infty} A(y) = 0$.