Optimal Stopping and Perpetual Options for Lévy processes

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Abstract

Consider a model of a financial market with a stock driven by a Lévy process and constant interest rate. A closed formula for prices of perpetual American call options in terms of the overall supremum of the Lévy process, and a corresponding closed formula for perpetual American put options involving the infimum of the after-mentioned process are obtained. As a direct application of the previous results, a Black-Scholes type formula is given. Also as a consequence, simple explicit formulas for prices of call options are obtained for a Lévy process with positive mixed-exponential and arbitrary negative jumps. In the case of put options, similar simple formulas are obtained under the condition of negative mixed-exponential and arbitrary positive jumps. Risk-neutral valuation is discussed and a simple jump-diffusion model is chosen to illustrate the results.

Key words: Optimal stopping, Lévy processes, mixtures of exponential distributions, American options, jump-diffusion models.

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1 Introduction

Let $X = \{X_t\}_{t\geq 0}$ be a real valued stochastic process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, P)$. Assume that X is càdlàg, adapted, $X_0 = 0$, and for $0 \leq s < t$ the random variable $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$ with a distribution that only depends on the difference $t - s$. Assume that the stochastic basis $\mathcal{B}$ satisfies the usual conditions. The process X is a process with stationary independent increments (PIIS), or a Lévy process.

Consider a model of a financial market with two assets, a savings account $B = \{B_t\}_{t\geq 0}$, and a stock $S = \{S_t\}_{t\geq 0}$. The evolution of B is deterministic, with

$$B_t = e^{rt}, \qquad r \geq 0,$$

(where $B_0 = 1$ for simplicity), and the stock is random, and evolves according to the formula

$$S_t = S_0 e^{X_t}, \qquad S_0 > 0, \tag{1}$$

where $X = \{X_t\}_{t\geq 0}$ is a Lévy process. We call this model a Lévy market. When the process X has continuous paths, we obtain the classical Black-Scholes model (Black-Scholes (1973)).

In this model a derivative asset, namely an American option is introduced. This is a contract between two parts, in which one part, the holder, buys the right to receive at time τ, that he chooses, from the other part, the seller, an amount $G(S_\tau)$. Call and put options have reward functions given by $G_c(S) = (S-K)^+$ and $G_p(S) = (K-S)^+$ respectively. In the financial practice the contract includes an exercise time T (maturity) such that $0 \leq \tau \leq T$ and the mathematical problem consists in finding a price for this contract and an optimal exercise time, the optimal stopping time. We are able to find closed solutions only with $T = \infty$, the perpetual case.

In order to obtain prices and optimal exercise times, we solve an optimal stopping problem, by analogy with the Black-Scholes model. Consider then $\mathcal{M}$ the class of all stopping times relative to $\mathbf{F}$ (τ is a stopping time if $\tau\colon \Omega \to [0, +\infty]$ and $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$). Given a Borel function $G\colon \mathbf{R} \to \mathbf{R}$, the reward function, a process X as above, and a discount rate $r \geq 0$, the optimal stopping problem consists in finding a real function V and a stopping time τ^* such that

$$V(S_0) = \sup_{\tau\in\mathcal{M}} E(e^{-r\tau}G(S_\tau)) = E(e^{-r\tau^*}G(S_{\tau^*})). \tag{2}$$

The cost function V is the price of the perpetual option, and the optimal stopping time τ^*, (i.e. the stopping time that realizes the supremum), gives the optimal exercise time. As usual, we assume that $e^{-r\tau}G(S_\tau)\mathbf{1}_{\{\tau=\infty\}} = \limsup_{t\to\infty} e^{-rt}G(S_t)$.

The main purpose of this paper is to find solutions for optimal stopping problems with call and put rewards and Lévy processes as log-prices. In order to introduce our first results, given X and $r \geq 0$ consider

$$M = \sup_{0\leq t<\tau(r)} X_t \quad \text{and} \quad I = \inf_{0\leq t<\tau(r)} X_t, \tag{3}$$

where $\tau(r)$ is an exponential random variable with parameter $r > 0$, independent of X, and $\tau(0) = \infty$. M and I will be called the "supremum" and the "infimum" of X in both cases $r = 0$ and $r > 0$. When $r > 0$ we say that the process X is killed at rate r.

The first result of the paper gives closed solutions for prices and optimal exercise times of perpetual call and put option under the given probability measure for general Lévy processes, in terms of the random variables M and I in (3) respectively. These are prices for an investor who wishes to buy an option for its expected value. A first consequence of the results obtained, is a Black-Scholes type formula for perpetual options. This is the content of section 2. As a second result, simple explicit formulae are obtained under the assumption of positive mixed-exponentially distributed and arbitrary negative jumps for call options, and negative mixed-exponentially distributed and arbitrary positive jumps for put options. These results, that generalize the well known Mc. Kean (1965) and Merton (1973) closed formulae for Brownian motion and other posterior results, are presented in section 3. The section 4 discusses applications of these results to risk-neutral valuation, and a simple model with Brownian component and positive and negative jumps exponentially distributed is considered in order to illustrate the results. The section 5 contains the proofs, and the section 6 a conclusion.

Models including risky assets with jumps were considered by Merton (1976) in the case of adding to the Brownian component, a compound Poisson process. Pure jump Lévy process appear in the work of Mandelbrot (1963 a,b) and Fama (1963, 1965) with a Lévy measure of Pareto-Lévy (stable) type. More recent proposals, including statistical work in order to fit empirical data include Generalized Hyperbolic models introduced by Barndorff-Nielsen (1977), considered also by Eberlein and Keller (1995). Generalized Hyperbolic distributions are obtained as the result of a variance-mean mixture of normal and generalized inverse Gaussian distributions. They include Hyperbolic distributions and normal inverse Gaussian distributions (Eberlein and Prause (1998)). Other works propose the use of the Variance Gamma processes (Madan et al. (1998)), a process obtained as the result of a time change of a Brownian motion with drift by a Gamma process (subordination), and more recently, a generalization of the previous models was studied by Carr et al. (2000). Other empirical proposals include Truncated Lévy processes (Cont et al. (1997), Matacz (2000)). Leblanc and Yor (1998) propose the use of Lévy processes to model financial assets based on theoretical considerations. The problem of pricing American options in a jump-diffusion framework was considered by many authors, including Zhang (1994), Pham (1997), Mulinacci (1996), Gerber and Landry (1998), Gerber and Shiu (1999), Mordecki (1997, 1999). Recently, Boyarchenko and Levendroskiĭ (2000), based on the Wiener-Hopf's factorization obtained results related to ours under additional conditions on the characteristic exponent of the Lévy process. See also Boyarchenko and Levendroskiĭ, (2001). Similar results for perpetual options in the Bachelier model (i.e. the stock is a Lévy process) have been obtained in Mordecki (2001). Based on the leptokurtic feature, volatility smile and analytical tractability, Kou (2000) proposes a jump diffusion model similar to the model considered in section 4, and in Kou and Wang (2001), the method of the infinitesimal generator used in Mordecki (1999) is adapted in order to obtain (b) in our Theorem 4, within other results. Chan (2000) studies the pricing of a perpetual put option under a Lévy process with no positive jumps, obtaining an explicit formula for the optimal stopping time and a closed formula numerically tractable for perpetual put option prices.

For general reference on Lévy processes see Jacod and Shiryaev (1987), Skorokhod (1991), Bertoin (1996), or Sato (1999). For general reference on pricing perpetual options and related questions see Shiryaev et al. (1994), Karatzas and Shreve (1998), Merton (1990) or Shiryaev (1999).

2 General results on pricing and optimal stopping

In this section we consider the problem of pricing perpetual call and put options by its expected value, i.e. the pricing problem under the historical probability measure P. Risk neutral valuation is discussed in the section 4.

Theorem 1 (Perpetual Call options) *Consider a Lévy market and M given in (3).*

(a) *If $E(e^{X_1}) < e^r$ then $E(e^M) < \infty$, the price of a perpetual call option is given by*

$$V_c(S_0) = \frac{E[S_0 e^M - KE(e^M)]^+}{E(e^M)}, \tag{4}$$

and is optimally exercised at the stopping time

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\}, \tag{5}$$

with optimal level

$$S_c^* = KE(e^M). \tag{6}$$

(b) *If $E(e^{X_1}) = e^r$, then $E(e^M) = \infty$, no optimal stopping time exists, and the cost function takes the form $V_c(S_0) = S_0$. For each $0 < \varepsilon < K/S_0$, the deterministic time*

$$\tau_c^\varepsilon = \frac{1}{r}\log\Big(\frac{K}{S_0\varepsilon}\Big) \tag{7}$$

is ε-optimal in the sense that

$$S_0(1-\varepsilon) \leq E(e^{-r\tau_c^\varepsilon}(S_{\tau_c^\varepsilon} - K)^+) \leq S_0.$$

(c) *If $e^r < E(e^{X_1}) < \infty$ then $E(e^M) = \infty$ and $V_c(S_0) = \infty$. Given $H > 0$, the deterministic time*

$$\tau_c^H = \frac{1}{\log E(e^{X_1 - r})}\log\Big(\frac{H+K}{S_0}\Big)$$

satisfies

$$E(e^{-r\tau_c^H}(S_{\tau_c^H}-K)^+) \geq H. \tag{8}$$

(d) *If* $E(e^{X_1})=\infty$ *then* $V_c(S_0)=\infty$ *and* $E(e^{-rt}(S_t-K)^+)=\infty$ *for each* $t>0$.

Theorem 2 (Perpetual put options) *Consider a Lévy market and* I *given in (3).*
(a) *Assume either* $r>0$, *or* $r=0$ *and* $P(\lim_{t\to\infty}X_t=\infty)=1$. *Then* $E(e^I)>0$, *the price of a perpetual put option is given by*

$$V_p(S_0)=\frac{E[KE(e^I)-S_0e^I]^+}{E(e^I)}, \tag{9}$$

and is optimally exercised at the stopping time

$$\tau_p^*=\inf\{t\geq 0\colon S_t\leq S_p^*\}, \tag{10}$$

where

$$S_p^*=KE(e^I). \tag{11}$$

(b) *Assume* $r=0$ *and* $P(\liminf_{t\to\infty}X_t=-\infty)=1$. *Then* $E(e^I)=0$ *and* $V_p(S_0)=K$. *For each* $0<\varepsilon<K$, *the stopping time*

$$\tau_p^\varepsilon=\inf\{t\geq 0\colon S_t\leq\varepsilon\}$$

is ε*-optimal in the sense that*

$$K-\varepsilon\leq E(K-S_{\tau_p^\varepsilon})^+\leq K.$$

Remark: It is interesting to note, that the optimal exercise levels given in (6) and (11) for perpetual call and put options respectively, when assumptions of (a) in Theorem 1 and $r>0$ in Theorem 2 hold, satisfy the *duality* relation

$$\frac{rK^2}{r-\log E(e^{X_1})}=S_c^*S_p^*. \tag{12}$$

This is due to the Wiener-Hopf factorization for Lévy processes, obtained by Rogozin (1966), that states

$$\frac{r}{r-\log E(e^{zX_1})}=E(e^{zM})E(e^{zI}), \tag{13}$$

for $z=iq$ and real q. In our case, due to the assumption $E(e^{X_1})<e^r$, the functions in (13) can be extended to the complex strip $0\leq\Im(z)\leq 1$ giving the result.

Remark: Part (a) of Theorem 1 is a consequence of Darling et al. (1972), where similar results for random walks were obtained. Theorem 2 has an obvious "dual" form with respect to Theorem 1. Its proof is based on similar results for random walks stated in section 5.

2.1 Black-Scholes type formula

A simple transformation of formulae (4) and (9) in Theorems 1 and 2 respectively allows to present prices of perpetual American options in a way similar in certain extent to the classical Black-Scholes formula for European options (compare (14) and (15) with (16) and (17)). To this end, let us write the Black-Scholes formula in a convenient way. Take $X_t = \sigma W_t + (a - \sigma^2/2)t$ in the stock (1), with $W = \{W_t\}_{t \geq 0}$ a standard Wiener process. The case $a = r$ corresponds to the risk neutral situation without dividends. The price of an European call option with maturity T and exercise price K under the historical probability measure is

$$\begin{aligned} BS_c = E(e^{-rT}(S_T - K)^+) &= S_0 E(e^{X_T - rT}\mathbf{1}_{\{S_T > K\}}) - Ke^{-rT}P(S_T > K) \\ &= S_0 e^{(a-r)T}\tilde{P}(S_T > K) - Ke^{-rT}P(S_T > K). \end{aligned} \tag{14}$$

where the measure $\tilde{P}$ is defined by

$$d\tilde{P} = e^{\sigma W_T - (\sigma^2/2)T} dP.$$

In the put case, with the same maturity and exercise price,

$$BS_p = Ke^{-rT}P(S_T < K) - S_0 e^{(a-r)T}\tilde{P}(S_T < K). \tag{15}$$

Theorem 3 (Black-Scholes type formula) *Consider a Lévy market.*
(a) *If $E(e^{X_1}) < e^r$, the price of the perpetual call option is*

$$V_c(S_0) = S_0 \tilde{P}_c(S_0 e^M > S_c^*) - KP(S_0 e^M > S_c^*) \tag{16}$$

where M is given in (3), $S_c^* = KE(e^M)$ and $\tilde{P}_c$ is the measure defined by

$$d\tilde{P}_c = \frac{e^M}{E(e^M)} dP.$$

The optimal strategy is given by (5).
(b) *Under the conditions of* (a) *in Theorem 2 the price of the perpetual put is given by*

$$V_p(S_0) = KP(S_0 e^I < S_p^*) - S_0 \tilde{P}_p(S_0 e^I < S_p^*), \tag{17}$$

where I is given in (3), $S_p^* = KE(e^I)$, and $\tilde{P}_p$ is the measure defined by

$$d\tilde{P}_p = \frac{e^I}{E(e^I)} dP.$$

The optimal strategy is given by (10).

3 Explicit solutions for mixed exponential jumps

In order to obtain explicit expressions for prices of perpetual options, we specify the jump distribution of the underlying process. If $q \in \mathbf{R}$, the Lévy-Khinchine formula states

$$E(e^{iqX_t}) = \exp\Big\{t\Big[ibq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbf{R}} (e^{iqy} - 1 - iqh(y))\Pi(dy)\Big]\Big\}, \tag{18}$$

with $h(y) = y\mathbf{1}_{\{|y|<1\}}$ a fixed truncation function, b and $\sigma \geq 0$ real constants, and Π a positive measure on $\mathbf{R} - \{0\}$ such that $\int(1 \wedge y^2)\Pi(dy) < +\infty$, called the Lévy measure. The triplet (b, σ^2, Π) is the characteristic triplet of the process, and completely determines its law. We always assume that the process does not degenerate, i.e. $\sigma \neq 0$ or $\Pi \neq 0$. Consider the set

$$\mathbf{C_0} = \Big\{c \in \mathbf{R}\colon \int_{\{|y|>1\}} e^{cy}\Pi(dy) < \infty\Big\}. \tag{19}$$

The set $\mathbf{C_0}$ is convex, contains the origin, and consists of all $c \in \mathbf{R}$ such that $E(e^{cX_t}) < \infty$ for some $t > 0$. Furthermore, if $z \in \mathbf{C}$ and $\Re(z) \in \mathbf{C_0}$ we can define the characteristic exponent

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - izh(y))\Pi(dy) \tag{20}$$

having $E|e^{zX_t}| < \infty$ for all $t \geq 0$, and $E(e^{zX_t}) = e^{t\psi(z)}$. Observe that if $z = iq$ the preceding formula gives (18).

Given $a = (a_1, \dots, a_n)$ with $\sum_{k=1}^n a_k = 1$, $a_k > 0$ for $k = 1, \dots, n$, and $\alpha = (\alpha_1, \dots, \alpha_n)$, with $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$, denote by

$$h_n(y; a, \alpha) = \sum_{k=1}^n a_k \alpha_k e^{-\alpha_k y}, \qquad y \geq 0,$$

the density of a mixture of n exponential random variables with parameters $\alpha_1, \dots, \alpha_n$ and mixture coefficients $a_1, \dots, a_n$. A random variable with this density will be called a mixed-exponential random variable.

3.1 Call options

Consider a Lévy process X with Lévy measure given by

$$\Pi(y) = \begin{cases} \lambda h_n(y; a, \alpha)dy, & y > 0, \\ \\ \pi(dy), & y < 0, \end{cases}$$

where $\pi(dy)$ is an arbitrary Lévy measure with support on $(-\infty, 0)$, and a, α are as before. The process X has mixed-exponentially distributed positive jumps, and arbitrary negative jumps. In this case, the set $\mathbf{C_0}$ in (19) contains the interval $[0,\alpha_1)$ and the characteristic exponent has an analytical continuation to the set $\{z\in\mathbf{C}\colon \Re(z)\geq 0, z\neq\alpha_k, k=1,\ldots,n\}$ given by

$$\Psi(z)=az+\frac{1}{2}\sigma^2z^2+\int_{-\infty}^{0}(e^{zy}-1-zh(y))\pi(dy)+\lambda\sum_{k=1}^{n}a_k\frac{z}{\alpha_k-z}. \tag{21}$$

The random variable M in (3) for this class of processes, is also mixed exponential (see Mordecki (1999b)), giving the following result.

Corollary 1 *Consider a Lévy market whose driving process X has characteristic exponent given in (21) with $\sigma>0$.*
(a) *Under the conditions $\alpha_1>1$, and $E(e^{X_1})<e^r$, i.e.*

$$\psi(1)=a+\frac{1}{2}\sigma^2+\int_{-\infty}^{0}(e^{y}-1-h(y))\pi(dy)+\lambda\sum_{k=1}^{n}\frac{a_k}{\alpha_k-1}<r, \tag{22}$$

we have

$$E(e^M)=\sum_{k=1}^{n+1}A_k\frac{p_k}{p_k-1}, \tag{23}$$

where p_k $(k=1,\ldots,n+1)$, are the positive roots of $\Psi(p)=r$, and satisfy

$$1<p_1<\alpha_1<p_2<\ldots<\alpha_n<p_{n+1}. \tag{24}$$

The coefficients A_j are given by

$$A_j=\frac{\prod_{k=1}^{n}(\frac{p_j}{\alpha_k}-1)}{\prod_{k=1,k\neq j}^{n+1}(\frac{p_j}{p_k}-1)},\qquad j=1,\ldots,n+1. \tag{25}$$

Furthermore, the price of a perpetual call option is given by

$$V_c(S_0)=\begin{cases}\sum_{k=1}^{n+1}\frac{A_k}{p_k-1}\left(\frac{S_0}{S_c^*}\right)^{p_k} & S_0\leq S_c^*,\\[2ex] S_0-K & S_c^*<S_0,\end{cases}$$

with $S_c^*=K\sum_{k=1}^{n+1}A_k\frac{p_k}{p_k-1}$. The optimal stopping time given in (5).
(b) *If $\alpha_1>1$ and $\psi(1)=r$, conclusion of* (b) *in Theorem 1 holds. If $\alpha_1>1$ and $\psi(1)>r$, conclusion of* (c) *in Theorem 1 holds. If $\alpha_1\leq 1$ conclusion of* (d) *in Theorem 1 holds.*

Remark: In view of the duality relation (12) and (23) we also obtain the optimal exercise level for the perpetual put option in the Lévy market of Corollary 1 when $\psi(1)<r$. Namely $S_p^*=\frac{K}{E(e^M)}\frac{r}{r-\psi(1)}$. If $\psi(1)=r$, the smallest real root

in (24) is $p_1 = 1$, and the distribution of M obtained in Mordecki (1999b) gives, for $\Re(z) < 1$

$$E(e^{zM}) = A_1 \frac{1}{1-z} + \sum_{k=2}^{n+1} A_k \frac{p_k}{p_k - z} = \frac{1}{1-z}\Big(A_1 + \sum_{k=2}^{n+1} A_k \frac{p_k(1-z)}{p_k - z}\Big).$$

As $r = \psi(1)$, from (13) we obtain

$$E(e^{zI}) = \frac{r}{A_1 + \sum_{k=2}^{n+1} A_k \frac{p_k(1-z)}{p_k - z}} \frac{1-z}{\psi(1) - \psi(z)}.$$

As the limit exists as $z \to 1$ in the l.h.s. of the last expression, we obtain the optimal stopping exercise level for the perpetual put, as

$$S_p^* = KE(e^I) = \frac{Kr}{A_1 \psi'(1)}. \tag{26}$$

Formula (26) was obtained in Chan (2000) for Lévy process with no positive jumps. In this case $E(e^{zM}) = (1-z)^{-1}$ and $A_1 = 1$.

3.2 Put options

In order to obtain closed form solutions for prices of perpetual put options the negative jumps of the driving process are specified. Let X be a Lévy process with Lévy measure

$$\Pi(y) = \begin{cases} \pi(dy) & y > 0, \\ \mu h_n(-y; b, \beta) dy, & y < 0, \end{cases}$$

where $\pi(dy)$ is an arbitrary Lévy measure with support on the set $(0, \infty)$, $b = (b_1, \dots, b_n)$, with $\sum_{k=1}^n b_k = 1$, $b_k > 0$, and $\beta = (\beta_1, \dots, \beta_n)$, with $0 < \beta_1 < \beta_2 < \dots < \beta_n$. The set $\mathbf{C_0}$ in (19) contains the interval $(-\beta_1, 0]$ and the characteristic exponent has an analytical continuation to the set $\{z \in \mathbf{C} \colon \Re(z) \le 0, z \ne -\beta_k, k = 1, \dots, n\}$ given by

$$\Psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (e^{zy} - 1 - zh(y))\pi(dy) - \mu \sum_{k=1}^n b_k \frac{z}{\beta_k + z}. \tag{27}$$

The random variable I in (3) in this case is also mixed exponential, as follows by considering the distribution of the maximum of the dual process $-X$.

Corollary 2 *Consider a Lévy market whose driving process X has characteristic exponent given in (27) with $\sigma > 0$.*

(a) *Assume* $r > 0$, *or* $r = 0$ *and*

$$\Psi'(0-) = \lim_{p \to 0-} \frac{\Psi(p)}{p} = a + \int_1^\infty y\pi(dy) - \mu \sum_{k=1}^n \frac{b_k}{\beta_k} > 0. \tag{28}$$

Then $P(\lim_{t\to\infty} X_t = \infty) = 1$ *and*

$$E(e^I) = \sum_{k=1}^{n+1} B_k \frac{r_k}{r_k + 1}, \tag{29}$$

where $-r_k$ $(k = 1, \ldots, n+1)$, *are the real and negative roots of* $\Psi(p) = r$, *and satisfy*

$$0 < r_1 < \beta_1 < r_2 < \ldots < r_n < \beta_n < r_{n+1}.$$

The coefficients B_k *are given by*

$$B_j = \frac{\prod_{k=1}^n (\frac{r_j}{\beta_k} - 1)}{\prod_{k=1, k\neq j}^{n+1} (\frac{r_j}{r_k} - 1)}, \qquad j = 1, \ldots, n+1. \tag{30}$$

Furthermore, the price of a perpetual put option is given by

$$V_p(S_0) = \begin{cases} K - S_0 & S_0 \leq S_p^*, \\ \sum_{k=1}^{n+1} \frac{B_k}{r_k+1} \left(\frac{S_p^*}{S_0}\right)^{r_k} & S_p^* < S_0, \end{cases} \tag{31}$$

with $S_p^* = K \sum_{k=1}^{n+1} B_k \frac{r_k}{r_k+1}$. *The optimal stopping time is given in (10).*

(b) *Otherwise, the conclusion of* (b) *in Theorem 1 hold.*

Remark: As in Corollary 1, in view of (12) and (29) we obtain the optimal exercise level for the perpetual call option in the model of Corollary 2 when $\psi(1) < r$, given by $S_c^* = \frac{K}{E(e^I)} \frac{r}{r - \psi(1)}$. If $\psi(1) = r$ see (b) in Theorem 1.

Remark: The formulas (22) and (28) correct respectively the formulas (11) and (8) in Mordecki (1999).

4 Risk neutral valuation

Consider a Lévy market as in section 1, where the stock pays a continuous dividend at a fixed rate $\delta \geq 0$. If the market is complete, rational pricing is equivalent to the solution of an optimal stopping problem under the risk neutral martingale measure. In our context, this situation is verified only in two extreme cases: when $\Pi = 0$, or when $\sigma = 0$ and $\Pi(dx) = \lambda\delta_a(dx)$, corresponding respectively to the Black Scholes model and to the pure jump Poisson model. If the completeness does not hold, a variety of prices can be obtained, depending on the election of the risk-neutral martingale measure, and some criteria must be used in order to choose a convenient (in some sense) martingale measure. Any of these measures is called an *admissible measure*, and the resulting price an *admissible price*. Observe, that as follows from III.3.24 in Jacod and Shiryaev (1987), condition of X being a Lévy process under an admissible measure does not require X being a Lévy process under P, nor is a consequence of it. Let us also notice, that in view of results in Eberlein and Jacod (1997), Jakubenas (1998), and Gushchin and Mordecki (2001), the range of admissible prices for European options with convex payoff under Lévy processes, or more generally, in the presence of jumps in the stock, is in general very large.

In accordance with our discussion, we suppose that a measure P^* is chosen such that

- The process X is a Lévy process under P^* with characteristic exponent ψ^*,
- The process $(e^{-(r-\delta)t}S_t)_{t\geq 0}$ is a martingale under P^*.

Observe that

$$E^*(e^{-(r-\delta)t}S_t|\mathcal{F}_s) = \frac{S_s}{B_s}\exp\{[\psi^*(1)-(r-\delta)](t-s)\},$$

so our second condition is equivalent to $\psi^*(1) = r-\delta$. The P^*-admissible price of the perpetual option with reward $\{G(S_t)\}$ is then

$$V(S_0) = \sup_{\tau\in\mathcal{M}} E^*\big(e^{-r\tau}G(S_\tau)\big).$$

and results in sections 2 and 3 apply.

4.1 Lévy market with exponential jumps

In order to illustrate our results, consider a Lévy market with X in (1) given by

$$X_t = at + \sigma W_t + \sum_{i=1}^{N_t} Y_k, \tag{32}$$

where $W = \{W_t\}_{t\geq 0}$ is a standard Wiener process, a, $\sigma > 0$ are real constants, $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda+\mu$, and $Y = \{Y_k\}_{k\in\mathbf{N}}$ is a sequence of independent random variables with identical distribution given by the density

$$h(y) = \frac{\lambda}{\lambda+\mu}\alpha e^{-\alpha y}\mathbf{1}_{\{y>0\}} + \frac{\mu}{\lambda+\mu}\beta e^{\beta y}\mathbf{1}_{\{y<0\}}$$

for $\lambda \geq 0$, $\mu \geq 0$, and α, β real and positive. The process X jumps upwards at rate λ, downwards at rate μ, and the magnitude of the jumps is exponentially distributed with parameters α and β respectively. The processes $W = \{W_t\}_{t\geq 0}$, $N = \{N_t\}_{t\geq 0}$ and $Y = \{Y_n\}_{n\in\mathbf{N}}$ are defined on a stochastic basis $\mathcal{B}$, and are mutually independent.

In order to choose the risk-neutral martingale measure in the model considered, we apply the Esscher transform, as proposed by Gerber and Shiu (1994). This means that P^* is defined by its restrictions to $\mathcal{F}_t$ by

$$dP_t^* = e^{\gamma X_t - t\psi(\gamma)} dP_t,$$

where γ is the unique root of the equation

$$\psi(\gamma + 1) - \psi(\gamma) = r - \delta.$$

The characteristic exponent of X under P^* is given by $\psi^*(z) = \psi(z+\gamma) - \psi(\gamma)$, and from this follows that the log-price of the stock is a jump-diffusion process as in (32) with risk-neutral parameters given by

$$a^* = a + \gamma\sigma^2, \quad \sigma^* = \sigma, \quad \lambda^* = \frac{\lambda\alpha}{\alpha - \gamma}, \quad \mu^* = \frac{\mu\beta}{\beta + \gamma},$$
$$\alpha^* = \alpha - \gamma, \quad \beta^* = \beta + \gamma.$$

According to our previous results, risk-neutral prices of perpetual options will be given.

Theorem 4 *Consider a Lévy market with X the Lévy process given in (32). Then*
(a) *For positive dividend rate $\delta > 0$, and $r \geq 0$ denote*

$$S_c^* = K\Big(\frac{A_1 p_1}{p_1 - 1} + \frac{A_2 p_2}{p_2 - 1}\Big)$$

where p_1, p_2 $(1 < p_1 < \alpha^* < p_2)$ are the positive roots of $\psi^*(p) = r$, and the coefficients A_1, A_2 are given by

$$A_1 = \frac{p_2(\alpha^* - p_1)}{\alpha^*(p_2 - p_1)}, \qquad A_2 = \frac{p_1(p_2 - \alpha^*)}{\alpha^*(p_2 - p_1)}.$$

The price of a perpetual call option under the risk neutral martingale measure is

$$V_c(S_0) = \begin{cases} \frac{A_1}{p_1 - 1}\left(\frac{S_0}{S_c^*}\right)^{p_1} + \frac{A_2}{p_2 - 1}\left(\frac{S_0}{S_c^*}\right)^{p_2} & S_0 \leq S_c^*, \\ S_0 - K & S_0 > S_c^*, \end{cases}$$

The optimal exercise is given by the stopping time in (5).
(b) *Denote for $\delta \geq 0$, and positive interest rate $r > 0$*

$$S_p^* = K\Big(\frac{B_1 r_1}{r_1 + 1} + \frac{B_2 r_2}{r_2 + 1}\Big),$$

where $-r_1, -r_2$ $(0 < r_1 < \beta^* < r_2)$, are the negative roots of $\psi^*(p) = r$, and the coefficients B_1, B_2 are given by

$$B_1 = \frac{r_2(\beta^* - r_1)}{\beta^*(r_2 - r_1)}, \qquad B_2 = \frac{r_1(r_2 - \beta^*)}{\beta^*(r_2 - r_1)}.$$

The price of a perpetual put option under the risk neutral martingale measure is

$$V_p(S_0) = \begin{cases} K - S_0 & S_0 \le S_p^*, \\ \frac{B_1}{r_1+1}\left(\frac{S_p^*}{S_0}\right)^{r_1} + \frac{B_2}{r_2+1}\left(\frac{S_p^*}{S_0}\right)^{r_2}, & S_0 > S_p^*. \end{cases}$$

The optimal exercise is given by the stopping time in (10).

5 Proofs

In the proof of the Theorem 1, we use the following auxiliary result, related to a similar result for random walks in Darling et al. (1972).

Lemma 1 *Consider a Lévy process X with characteristic exponent ψ in (20), $r \ge 0$, and $\tau(r)$, M as in (3). The following two conditions are equivalent.*

(a) $E(e^M) < \infty$.

(b) $E(e^{X_1}) < e^r$.

Proof. (a)$\Longrightarrow$(b). If $r > 0$, as $E(e^{X_1}) = e^{\psi(1)}$ we can write

$$e^{(\psi(1)-r)t} = E(e^{X_t}\mathbf{1}_{\{\tau(r)>t\}}) \le E(e^M\mathbf{1}_{\{\tau(r)>t\}}) \to 0$$

if $t \to \infty$, giving $\psi(1) < r$. If $r = 0$, $P(M < \infty) = 1$, implies $\lim_{t\to\infty} X_t = -\infty$ P-a.s., as only three possible asymptotic behaviours for X are possible. As $e^{X_t} \le e^M$, we obtain by dominated convergence

$$e^{\psi(1)t} = E(e^{X_t}) \to 0 \text{ as } t \to \infty,$$

and in conclusion $\psi(1) < 0$.

(b)$\Longrightarrow$(a) We know $\psi(1) < r$. Choose $c > 1$ and $\varepsilon > 0$ such that

(i) $\psi(1) + \varepsilon + \int_{\{y\ge c\}}(e^y - 1)\Pi(dy) < r$,

(ii) $\delta = \psi(1) + \varepsilon > 0$ if $r > 0$.

Denote

$$\psi^1(z) = (b - \delta)z + \tfrac{1}{2}\sigma^2 z^2 + \int_{\{y<c\}}(e^{yz} - 1 - zh(y))\Pi(dy),$$

$$\psi^2(z) = \delta z + \int_{\{y\ge c\}}(e^{yz} - 1)\Pi(dy),$$

and consider a pair of independent Lévy processes X^1 and X^2 with characteristic exponents ψ^1 and ψ^2 respectively. As $\psi = \psi^1 + \psi^2$ we have $X =^d X^1 + X^2$. Furthermore

$$\psi^1(1) = (b-\delta) + \tfrac{1}{2}\sigma^2 + \textstyle\int_{\{y<c\}}(e^y - 1 - h(y))\Pi(dy) < 0,$$

$$\psi^2(1) = \delta + \textstyle\int_{\{y\geq c\}}(e^y - 1)\Pi(dy) < r.$$

Denoting $M^1 = \sup_{t\geq 0} X^1_t$ and $M^2 = \sup_{0\leq t<\tau(r)} X^2_t$ (with $\tau(r)$ independent of X^1, X^2) we have

$$E(e^M) \leq E(e^{M^1})E(e^{M^2}).$$

In order to verify that $E(e^{M^1})$ is finite, see that the function ψ^1 has an analytical continuation in $\Re(z) \geq 0$ due to the fact that X^1 has bounded by c positive jumps. As $\psi^1(1) < 0$ we deduce the existence of a real constant $\omega > 1$ such that $\psi(\omega) = 0$ and in consequence $\lim_{x\to\infty} e^{\omega x} P(M^1 > x) = C$ with $0 \leq C \leq 1$ (see Bertoin and Doney (1994)), concluding that $E(e^{M^1})$ is finite. Let us see that $E(e^{M^2})$ is finite. If $r > 0$, as $\delta > 0$ the process X^2 is nondecreasing a.s., and

$$E(e^{M^2}) \leq E(e^{X^2{}_{\tau(r)}}) = \frac{r}{r - \psi^2(1)} < \infty.$$

Consider now $r = 0$. By (i) we have $\delta < 0$ and the finiteness of $E(e^{M^2})$ reduces to the corresponding result for in Darling et al. (1972) for random walks in the following way. Define the sequence of stopping times

$$T_0 = 0, \qquad T_{k+1} = \inf\{t > T_k \colon \Delta X_t \geq 1\}.$$

The process $N = \{N_t\}_{t\geq 0}$ defined by $N_t = \sup\{k \colon T_k \leq t\}$ is a Poisson process with parameter $\lambda = \int_{\{y\geq c\}} \Pi(dy)$ that we assume to be positive (in case $\lambda = 0$, $X^2_t = \delta t \leq 0$). X^2_t can be represented as

$$X^2_t = \sum_{k=1}^{N_t} \Delta X_{T_k} + \delta t.$$

If $Z_k = \Delta X_{T_k} + \delta(T_k - T_{k-1})$ for $k = 1, 2, \ldots$, and $Z_0 = 0$, as $\delta < 0$,

$$M^2 = \sup_{t\geq 0} X^2_t = \sup_{n\geq 0} X^2_{T_n} = \sup \sum_{k=0}^{n} Z_k.$$

By Theorem 18§4.3 in Skorokhod (1991)

$$E(e^{Z_1}) = E(e^{\Delta X_{T_1}} e^{\delta T_1}) = \frac{1}{\lambda - \delta} \int_{\{y\geq c\}} e^y \Pi(dy) < 1,$$

and, according to Darling et al. (1972) $E(e^{M^2}) < \infty$ follows concluding the proof.

Remark: An alternative shorter proof can be based on (13).

Proof of Theorem 1. In all cases, the finiteness or not of $E(e^M)$ follows from the Lemma 1. Consider the case (a). Putting $S_0 = e^x$, and denoting $v(x) = V_c(e^x) = V_c(S_0)$, $g(x) = (e^x - K)^+$, (4) becomes

$$v(x) = \sup_{\tau \in \mathcal{M}} E(e^{-r\tau} g(x + X_\tau)) = \frac{E[e^{x+M} - KE(e^M)]^+}{E(e^M)}$$

In order to verify this formula, on one side we recall the fact that the cost function of the optimal stopping problem considered is the minimal r-excessive majorant function of the reward g, and satisfies

$$v(x) = \lim_{n\to\infty} \lim_{N\to\infty} Q_n^N g(x) \tag{33}$$

where the operator Q_n is given by

$$Q_n g(x) = \max\{g(x), e^{-r2^{-n}} E(g(x + X_{2^{-n}}))\},$$

and Q_n^N is the N^{th} iteration of Q_n. (see Shiryaev (1978) and VIII.2.4 in Shiryaev (1999)) Fix now n. The limit $\lim_{N\to\infty} Q_n^N g(x)$ is the cost function for the optimal stopping problem for the random walk $X^n = \{X_{k2^{-n}} \colon k \in \mathbf{N}\}$ discounted at rate $\alpha_n = e^{-r2^{-n}}$ with reward g, (Theorem II.23 in Shiryaev (1978)).

Let $\tau(r)$ be as in (3). For fixed n, and $r > 0$

$$\tau_n = \frac{[2^n \tau(r)] + 1}{2^n}$$

(with $[x]$ the integer part of x) is a geometric random variable taking values on the set $\{\frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{k}{2^n}, \dots\}$ with parameter $\alpha_n = e^{-r2^{-n}}$, i.e. $P(\tau_n > k2^{-n}) = (e^{-r/2^n})^k$. If $r = 0$ put $\tau_n = \infty$. On the other side Darling et al. (1972) give the solution to this discrete time optimal stopping problem. In conclusion

$$\lim_{N\to\infty} Q_n^N g(x) = \frac{E[e^{x+M^n} - KE(e^{M^n})]^+}{E(e^{M^n})} \tag{34}$$

where the r.h.s. is the formula in Darling et al. (1972), and

$$M^n = \sup\{X_{k2^{-n}} \colon 0 \le k2^{-n} < \tau_n\},$$

$M^n \le M$ and $E(e^{M^n}) < \infty$. Now we make $n \to \infty$. First

$$M_n = \sup_{0 \le k < \tau_n} X_{k2^{-n}} = \sup_{0 \le k < \tau(r)} X_{k2^{-n}} \to M = \sup_{0 \le t < \tau(r)} X_t \qquad P-a.s. \tag{35}$$

because $\tau_n > \tau(r)$, $\tau_n \to \tau(r)$ and X is càdlàg. The result then follows taking limits in (34). The l.h.s. has limit $v(x)$ by (33) and

$$\frac{E[e^{x+M^n} - KE(e^{M^n})]^+}{E(e^{M^n})} \to \frac{E[e^{x+M} - KE(e^M)]^+}{E(e^M)}$$

in view of (35) and dominated convergence since $E(e^M) < \infty$. The fact that τ_c^* is the optimal stopping rule follows exactly as in Darling et al. (1972).

Proof of (b). We follow the proof of Theorem 1(a) in §8 of Shiryaev et al. (1984). First, as X is non-degenerate, $E(X_t - rt) < E(e^{X_t - rt} - 1) = 0$ and in consequence $\lim_{t\to\infty} X_t - rt = -\infty$. Take now $\tau \in \mathcal{M}$.

$$E(e^{-r\tau}(e^{x+X_\tau} - K)^+) \leq E(e^{x+X_\tau - r\tau}) = E(\lim_{t\to\infty} e^{x+X_{\tau\wedge t} - r(\tau\wedge t)})$$

$$\leq \lim_{t\to\infty} E(e^{x+X_{\tau\wedge t} - r(\tau\wedge t)}) = e^x$$

where we used Fatou's Lemma. Denote now $T = \tau_c^\varepsilon$ in (7). $Ke^{-rT} = \varepsilon e^x$ and, as $x^+ \geq x$

$$E(e^{-rT}(e^{x+X_T} - K)^+) \geq E(e^{-rT}(e^{x+X_T} - K)) = e^x - Ke^{-rT} = e^x(1-\varepsilon)$$

completing the proof of (b).

Proof of (c). For any $t \geq 0$

$$v(x) \geq E(e^{-rt}(e^{x+X_t} - K)^+) \geq E(e^{x+X_t - rt}) - K = e^x e^{(\psi(1)-r)t} - K. \qquad (36)$$

Taking supremum over $t \geq 0$ in (36), as $\psi(1) = \log E(e^{X_1}) > r$ we obtain $v(x) = \infty$. Formula (8) is obtained replacing $t = \tau_c^H$ in (36), and observing that $(S-K)^+ \geq S - K$.

Proof of (d). If $\psi(1) = \infty$, (36) shows that $E(e^{-rt}(e^{x+X_t} - K)^+) = \infty$. This concludes the proof of Theorem 1.

The proof of Theorem 2 goes as the proof of Theorem 1, based on a result on random walks of independent interest, that follows. Introduce a sequence $X, X_1, X_2, \dots$ of i.i.d. random variables with partial sums denoted by

$$U_0 = 0, \quad U_n = \sum_{k=1}^n X_k, \quad n = 1, 2, \dots$$

and τ denotes a stopping time with respect to the filtration generated by the random walk $U = \{U_n\}_{n\in\mathbf{N}}$. The optimal stopping problem for this random walk with reward g and discount rate $\alpha \in (0,1]$ consists in finding a pair (V, τ^*) such that

$$v(x) = \sup_\tau E(\alpha^\tau g(x + U_\tau)) = E(\alpha^{\tau^*} g(x + U_{\tau^*})). \qquad (37)$$

Let $\tau(\alpha)$ be a geometric random variable with $P(\tau(\alpha) > k) = \alpha^k$, independent of U with $\alpha \in (0,1)$, and denote $\tau(1) = \infty$. The infimum of the random walk killed at rate α is

$$I_U = \inf_{0 \le n < \tau(\alpha)} U_n. \tag{38}$$

Proposition 1 *Assume $0 < \alpha < 1$, or $\alpha = 1$ and $P(\lim_n U_n = \infty) = 1$. Then $E(e^{I_U}) > 0$, and the solution to (37) with reward function $g_p(x) = (K - e^x)^+$ is given by the cost function*

$$v(x) = \frac{E[KE(e^{I_U}) - e^{x+I_U}]^+}{E(e^{I_U})}, \tag{39}$$

and optimal stopping rule

$$\tau^* = \inf\{n \ge 0 \colon x + U_n \le \log KE(e^{I_U})\}. \tag{40}$$

(b) *Under the condition $\alpha = 1$ and $P(\liminf_n U_n = -\infty) = 1$, we have $E(e^{I_U}) = 0$, and the solution to (37) is given by the cost function $v(x) = K$, no optimal stopping time exists, and an ε-optimal stopping time τ^ε, in the sense*

$$K - \varepsilon \le E(K - e^{x+U_{\tau^\varepsilon}})^+ \le K \tag{41}$$

is given by

$$\tau^\varepsilon = \inf\{n \ge 0 \colon x + U_n \le \log \varepsilon\}. \tag{42}$$

Remark: Observe that, by Theorem 1 in XII.2 of (Feller, (1966) II) all possible limit behaviors for the random walk are considered.

Proof. (a) The random variable in (38) is proper, so $E(e^{I_U}) > 0$. Following Darling et al. (1972), in order to obtain the results it is enough to verify that v and τ^* in (39) and (40) respectively satisfy the following three conditions, for each $x \in \mathbf{R}$:

(i) $v(x) \ge (K - e^x)^+$,

(ii) $v(x) \ge \alpha E(v(x + X))$,

(iii) $v(x) = E(\alpha^{\tau^*}(K - e^{x+U_{\tau^*}})^+)$.

In order to verify (i), as the function $\phi(y) = (KE(e^{I_U}) - y)^+$ is convex in y,

$$\begin{aligned} v(x)E(e^{I_U}) &= E[KE(e^{I_U}) - e^{x+I_U}]^+ \ge [KE(e^{I_U}) - e^x E(e^{I_U})]^+ \\ &= E(e^{I_U})(K - e^x)^+, \end{aligned}$$

by Jensen's inequality.

To see (ii) introduce a binomial random variable J independent of $\{U_n\}$ taking values 0 or 1 with probability $1-\alpha$ and α respectively. Taking into account that $\alpha P(\tau = k) = P(\tau = k+1)$ and

$$-(X+I_U)^- = \inf(0, X+I_U) =^d \inf_{0\le n<\tau(\alpha)+1} U_n$$

where $x^- = \max(-x, 0)$, we verify that the random variables I_U and $-J(X+I_U)^-$ have the same distribution. For $z = iq$, with $q \in \mathbf{R}$

$$E(e^{-zJ(X+I_U)^-}) = 1-\alpha+\alpha E(e^{-z(X+I_U)^-})$$

$$= P(\tau=1) + \alpha E(e^{z\inf_{0\le n<\tau(\alpha)+1} U_n}) = E(e^{zI_U}\mathbf{1}_{\{\tau=1\}})$$

$$+\sum_{k=1}^{\infty} E(e^{z\inf_{0\le n<k+1} U_n}\mathbf{1}_{\{\tau=k+1\}}) = E(e^{zI_U}).$$

As $-x^- \le x$,

$$v(x)E(e^{I_U}) = E[KE(e^{I_U}) - e^{x-J(X+I_U)^-}]^+ \ge \alpha E[KE(e^{I_U}) - e^{x-(X+I_U)^-}]^+$$

$$\ge \alpha E[KE(e^{I_U}) - e^{x+X+I_U}]^+ = \alpha E(e^{I_U})Ev(x+X),$$

concluding (ii).

Let us finally see (iii). Denote $A = \log KE(e^{I_U})$. Based on the spatial homogeneity and the strong Markov property of the random walk, and on the lack of memory of $\tau(\alpha)$

$$E(e^{x+I_U}\mathbf{1}_{\{x+I_U\le A\}}) = E(e^{x+U_{\tau^*}}e^{I_U-U_{\tau^*}}\mathbf{1}_{\{x+I_U\le A\}})$$

$$= E(e^{I_U})E(e^{x+U_{\tau^*}}\mathbf{1}_{\{x+I_U\le A\}}).$$

In conclusion

$$v(x) = \frac{E[KE(e^{I_U}) - e^{x+I_U}]\mathbf{1}_{\{x+I_U\le A\}}}{E(e^{I_U})} = E(K-e^{x+U_{\tau^*}})\mathbf{1}_{\{x+I_U\le A\}}$$

$$= E(K-e^{x+U_{\tau^*}})\mathbf{1}_{\{\tau^*<\tau\}} = E\alpha^{\tau^*}(K-e^{x+U_{\tau^*}})^+$$

concluding the proof of (a).

(b) We have $P(\liminf_n U_n = -\infty) = 1$, so for arbitrary H the stopping time

$$\tau_H = \inf\{n\ge 0 : x+U_n \le H\}$$

is finite with probability one. So, given $\varepsilon > 0$ (41) holds for τ^ε in (42).

Proof of Theorem 2. (a) In this case, the random variable I is proper, and $E(e^I) > 0$ follows. (9) and (10) are obtained considering the auxiliary random walks as in Theorem 1. The proof of (b) follows as the respective proof of (b) in Proposition 1.

Proof of Corollary 1 (a) Observe that the condition $\alpha_1 > 1$ ensures $1 \in \mathbf{C_0}$, and in consequence $E(e^{X_1}) = e^{\psi(1)} < \infty$. Furthermore, as $E(e^{X_1}) < e^r$, (a) in Theorem 1 applies. As seen in Mordecki (1999b), M in (3) is a mixture of exponential random variables, with density given by

$$f_M(y) = h_{n+1}(y; A, p) \tag{43}$$

where $p = (p_1, \dots, p_{n+1})$ is the set of positive roots of $\Psi(p) = r$, and $A = (A_1, \dots, A_{n+1})$ are given by (25).

In order to use (43) we observe that if $r = 0$ condition $\psi'(0+) < 0$ follows from (22) by convexity of $\psi(p)$ for $p \in [0, \alpha_1)$. In both cases $r = 0$ or $r > 0$ condition (22) ensures $p_1 > 1$ and M has the density given in (43). Then

$$E(e^M) = \int_0^\infty e^y h_{n+1}(y, A, p) dy = \sum_{k=1}^{n+1} A_k \frac{p_k}{p_k - 1}.$$

Furthermore, from (4) and (43), with $S_c^* = KE(e^M)$, for $S_0 \le S_c^*$,

$$V_c(S_0) = \frac{E[S_0 e^M - KE(e^M)]^+}{E(e^M)} = \sum_{k=1}^{n+1} \frac{A_k}{p_k - 1} \Big(\frac{S_0}{S_c^*}\Big)^{p_k}.$$

For $S_0 > S_c^*$, $S_0 e^M - KE(e^M) \ge 0$ and

$$V_c(S_0) = \frac{E[S_0 e^M - KE(e^M)]^+}{E(e^M)} = S_0 - K.$$

(b) If $\alpha_1 > 1$ then $E(e^{X_1}) < \infty$. Condition $\psi(1) = r$ is $E(e^{X_1}) = e^r$ and conclusion in (b) of Theorem 1 follows. The case $\alpha_1 > 1$ and $\psi(1) > r$ corresponds to $e^r < E(e^{X_1}) < \infty$ and conclusion in (c) of Theorem 1 holds. Finally, $\alpha_1 \le 1$ implies $E(e^{X_1}) = \infty$ and conclusion in (d) of Theorem 1 follows.

Proof of Corollary 2 (a) From (28) follows $P(\lim_t X_t = \infty) = 1$, see for instance Mordecki (1999a). Then I is a proper random variable. From (43) applied to the dual process $-X$, we obtain that the negative of the infimum of the process X has a mixed exponential distribution with parameters r_k, with $-r_k$ the real and negative roots of $\Psi(p) = r$ and mixture coefficients B_k given by 30. From this, (29) and the form of $V_p(S_0)$ in (31) follows as in the proof of the Corollary 1.

(b) Observe that if $\psi'(0-) \le 0$, this quantity is necessarily finite. If $\psi'(0-) < 0$ then $EX_1 = \psi'(0-) < 0$ and $\lim_{t\to\infty} X_t = -\infty$. If $\psi'(0-) = 0$ we have $EX_1 = 0$, and in consequence $\liminf_{t\to\infty} X_t = -\infty$ a.s. (see Theorem 2 in Rogozin (1966)). In both cases $I = -\infty$ and $E(e^I) = 0$.

Proof of Theorem 3 (a) As $E(e^{X_1}) < e^r$, (a) in Theorem 1 holds, giving

$$V_c(S_0) = E\big[S_0 \frac{e^M}{E(e^M)} - K\big]^+ = S_0 E \frac{e^M}{E(e^M)} \mathbf{1}_{\{S_0 e^M \geq S_c^*\}} - KP(S_0 e^M \geq S_c^*)$$

$$= S_0 \tilde{P}_c(S_0 e^M \geq S_c^*) - P(S_0 e^M \geq S_c^*).$$

(b) We apply (a) in Theorem 2

$$V_p(S_0) = E\big[K - S_0 \frac{e^I}{E(e^I)}\big]^+ = KP(S_0 e^I \leq S_p^*) - \tilde{P}_p(S_0 e^I \leq S_p^*),$$

concluding the proof.

6 Conclusion

The presented paper gives closed formulae for prices of perpetual call and put options on a stock driven by a general Lévy process, respectively in terms of the distribution of the supremum and the infimum of the process, killed at a constant rate, the interest rate in the market. From a mathematical point of view, the optimal stopping problem for a Lévy process X, with the classical rewards of call and put options, is reduced to the computation of the distribution of the supremum M and infimum I of X. It is interesting to note, that the optimal stopping problem in the case of a Markov process, leads to the solution of a free boundary problem, and the computation of the distribution of the supremum of a process is a problem with fixed boundary conditions. The closed formulae allows to obtain a Black Scholes type formula for prices of perpetual options.

Second, recent results on exact distributions of the supremum of a Lévy process when positive jumps are distributed according a mixture of exponentials are used, in order to obtain: (a) an explicit formula for prices of perpetual call options when the driving Lévy process has mixed exponential positive and arbitrary negative jumps generalizing Merton's (1973) classical result for Brownian motion, and (b) an explicit formula for perpetual put options when the driving process has mixed exponential negative jumps and arbitrary positive jumps generalizing Mc. Kean's (1965) result.

Third, on the assumption that X is a diffusion with jumps, risk-neutral prices of perpetual call and put options are obtained in a simple case by choosing the risk-neutral martingale measure by the Esscher transform. Proofs are based on general results of approximation of optimal stopping problems for continuous time Markov process by discrete time Markov chains (random walks in our case).

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