A note on "Optimal stopping and perpetual options for Lévy processes"

Ernesto Mordecki
Facultad de Ciencias, Centro de Matemática
Iguá 4225. CP 11400. Montevideo. Uruguay

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Abstract

In this note some details of the proof of Theorem 1 in the article [3] are given[1].

For convenience of the reader we state the Theorem. For all unexplained notation we refer to [3].

Theorem 1 (Perpetual Call options) *Consider a Lévy market and M given in (3 of [3]).*
(a) *If $E(e^{X_1}) < e^r$ then $E(e^M) < \infty$, the price of a perpetual call option is given by*

$$V_c(S_0) = \frac{E[S_0 e^M - KE(e^M)]^+}{E(e^M)},$$

and is optimally exercised at the stopping time

$$\tau_c^* = \inf\{t \geq 0 \colon S_t \geq S_c^*\},$$

with $S_c^* = KE(e^M)$.
(b) *If $E(e^{X_1}) = e^r$, then $E(e^M) = \infty$, no optimal stopping time exists, and the cost function takes the form $V_c(S_0) = S_0$. For each $0 < \varepsilon < K/S_0$, the deterministic time*

$$\tau_c^\varepsilon = \frac{1}{r} \log\left(\frac{K}{S_0 \varepsilon}\right)$$

is ε-optimal in the sense that

$$S_0(1-\varepsilon) \leq E(e^{-r\tau_c^\varepsilon}(S_{\tau_c^\varepsilon} - K)^+) \leq S_0.$$

(c) *If $e^r < E(e^{X_1}) < \infty$ then $E(e^M) = \infty$ and $V_c(S_0) = \infty$. Given $H > 0$, the deterministic time*

$$\tau_c^H = \frac{1}{\log E(e^{X_1 - r})} \log\left(\frac{H + K}{S_0}\right)$$

[1] The author is indebted to Serguei Levendorskii for suggesting writing this note.

satisfies

$$E(e^{-r\tau_c^H}(S_{\tau_c^H}-K)^+)\geq H.$$

(d) *If* $E(e^{X_1})=\infty$ *then* $V_c(S_0)=\infty$ *and* $E(e^{-rt}(S_t-K)^+)=\infty$ *for each* $t>0$.

Proof of Theorem 1. In all cases, the finiteness or not of $E(e^M)$ follows from Lemma 1 in [3].

To prove that (a) in Theorem 1 provide the solution to the optimal stopping problem we verify the following two statements:

$$\sup_{\tau\in\mathcal{M}} E(e^{-r\tau}(e^{x+X_\tau}-K)^+)=\frac{E[e^{x+M}-KE(e^M)]^+}{E(e^M)} \tag{1}$$

$$\frac{E[e^{x+M}-KE(e^M)]^+}{E(e^M)}=E(e^{-r\tau_c^*}(e^{x+X_{\tau_c^*}}-K)^+). \tag{2}$$

The proof of (1) is done by approximation and given in [3].

Let us now see (2). Denote $A=\log S_c^*$, and observe that

$$\{x+M\geq A\}=\{\tau_c^*<\tau(r)\}. \tag{3}$$

Assume first that:

$$E(e^{x+M}\mathbf{1}_{\{x+M\geq A\}})=E(e^M)\times E(e^{x+X_{\tau_c^*}}\mathbf{1}_{\{x+M\geq A\}}). \tag{4}$$

Then, based on (4) we obtain

$$\begin{aligned}E[e^{x+M}-KE(e^M)]^+ &= E[e^{x+M}\mathbf{1}_{\{x+M\geq A\}}]-KE(e^M)E[\mathbf{1}_{\{x+M\geq A\}}]\\ &= E(e^M)\big[E(e^{x+X_{\tau_c^*}}\mathbf{1}_{\{x+M\geq A\}})-KE[\mathbf{1}_{\{x+M\geq A\}}]\big]\\ &= E(e^M)E[(e^{x+X_{\tau_c^*}}-K)\mathbf{1}_{\{x+M\geq A\}}].\end{aligned}$$

To obtain (2) from this, based on (3) and on the independence of the exponential random time $\tau(r)$, we verify:

$$\begin{aligned}&E[(e^{x+X_{\tau_c^*}}-K)\mathbf{1}_{\{x+M\geq A\}}]\\ &\quad= E[(e^{x+X_{\tau_c^*}}-K)\mathbf{1}_{\{\tau_c^*<\tau(r)\}}]=\int_\Omega\int_{\tau_c^*}^\infty(e^{x+X_{\tau_c^*}}-K)^+re^{-rt}dtdP\\ &\qquad=\int_\Omega e^{-r\tau_c^*}(e^{x+X_{\tau_c^*}}-K)^+dP=E[e^{-r\tau_c^*}(e^{x+X_{\tau_c^*}}-K)^+].\end{aligned}$$

This is essentially given in Darling et al. (1972) page 1368.

It remains to verify (4). In order to do this, we state the *strong Markov property* from Bertoin (Theorem 6, Chapter 1, 1996).

Proposition 1 *Let* τ *be a stopping time with* $P(\tau<\infty)>0$. *Then, conditionally on* $\{\tau<\infty\}$ *the process* $\{X_{\tau+t}-X_\tau\}_{t\geq0}$ *is independent of* $\mathcal{F}_\tau$ *and has the law* P.

More precisely, if $A \in \mathcal{F}_\tau$, $t_1, \dots, t_k$ are positive real numbers and $I_1, \dots, I_k$ are real intervals, the strong Markov property is the statement (see (4.24) in Skorokhod (1986)):

$$\begin{aligned} &P(A, \tau < \infty, X_{\tau+t_1} - X_\tau \in I_1, \dots, X_{\tau+t_k} - X_\tau \in I_k) \\ &\qquad = P(A, \tau < \infty) \times P(X_{\tau+t_1} - X_\tau \in I_1, \dots, X_{\tau+t_k} - X_\tau \in I_k). \end{aligned}$$

The verification of (4), for $r > 0$, is the following computation:

$$E(e^{x+M}\mathbf{1}_{\{\tau_c^* < \tau(r)\}}) = E\Big(\exp\big(\sup_{0 \le t < \tau(r)} (X_t - X_{\tau_c^*})\big)\exp(x + X_{\tau_c^*})\mathbf{1}_{\{\tau_c^* < \tau(r)\}}\Big) \tag{5}$$

$$= E\Big(\exp\big(\sup_{0 \le s < \tau(r) - \tau_c^*} (X_{\tau_c^*+s} - X_{\tau_c^*})\big)\exp(x + X_{\tau_c^*})\mathbf{1}_{\{\tau_c^* < \tau(r)\}}\Big) \tag{6}$$

$$= \int_\Omega \int_{\tau_c^*}^\infty e^{\sup_{0 \le s < t - \tau_c^*}(X_{\tau_c^*+s} - X_{\tau_c^*})} e^{x + X_{\tau_c^*}} r e^{-rt} dt dP \tag{7}$$

$$= \int_\Omega e^{-r\tau_c^*} \int_0^\infty e^{\sup_{0 \le s < v}(X_{\tau_c^*+s} - X_{\tau_c^*})} e^{x + X_{\tau_c^*}} r e^{-rv} dv dP \tag{8}$$

$$= E\Big(e^{-r\tau_c^*} e^{x + X_{\tau_c^*}} \big[\int_0^\infty e^{\sup_{0 \le s < v}(X_{\tau_c^*+s} - X_{\tau_c^*})} r e^{-rv} dv\big]\Big)$$

$$= E\Big(\int_0^\infty e^{\sup_{0 \le s < v}(X_{\tau_c^*+s} - X_{\tau_c^*})} r e^{-rv} dv\Big) \times E(e^{x + X_{\tau_c^*}}\mathbf{1}_{\{\tau_c^* < \tau(r)\}}) \tag{9}$$

$$= E\Big(\int_0^\infty e^{\sup_{0 \le s < v} X_s} r e^{-rv} dv\Big) \times E(e^{x + X_{\tau_c^*}}\mathbf{1}_{\{x + M \ge A\}}) \tag{10}$$

$$= E(e^M) \times E(e^{x + X_{\tau_c^*}}\mathbf{1}_{\{x + M \ge A\}}).$$

Here, in line (5) we add and substract $X_{\tau_c^*}$, in line (6) we observe that the supremum is realized when $t \ge \tau_c^*$, in line (7) we relay on the independence of $\tau(r)$, in line (8) we change variables according to $v = t - \tau_c^*$. Finally, to obtain (9) and (10), we apply the strong Markov property: first independence and second identity of distributions, as we integrate on the set $\{\tau_c^* < \infty\}$ (the integrand vanishes on the set $\{\tau_c^* = \infty\}$): the integral in square brackets is independent of τ_c^* and $X_{\tau_c^*}$, that are $\mathcal{F}_{\tau_c^*}$ measurable.

Case $r = 0$ of equation (4) is simpler and can be obtained taking limits as $r \to 0$; or directly, under similar arguments as the case $r > 0$, in the following way:

$$\begin{aligned} E(e^{x+M}\mathbf{1}_{\{\tau_c^* < \tau(r)\}}) &= E\Big(\exp\big(\sup_{t \ge 0}(X_t - X_{\tau_c^*})\big)\exp(x + X_{\tau_c^*})\mathbf{1}_{\{\tau_c^* < \infty\}}\Big) \\ &= E\Big(\exp\big(\sup_{s \ge 0}(X_{\tau_c^*+s} - X_{\tau_c^*})\big)\exp(x + X_{\tau_c^*})\mathbf{1}_{\{\tau_c^* < \infty\}}\Big) \\ &= E(e^M) \times E(e^{x + X_{\tau_c^*}}\mathbf{1}_{\{x + M \ge A\}}). \end{aligned}$$

For the rest of the proof of Theorem 1, that is, parts (b), (c) and (d) see [3].

References

[1] Bertoin, J.: Lévy Processes. Cambridge: Cambridge University Press 1996

[2] Darling, D.A., Ligget, T. Taylor, H.M.: Optimal stopping for partial sums. The Annals of Mathematical Statistics **43**, 1363–1368 (1972)

[3] Mordecki, E. Optimal stopping and perpetual options for Lévy processes, Finance and Stochastics. Volume VI (2002) 4, 473-493.

[4] Skorokhod, A. V.: Random processes with independent increments Dordrecht: Kluwer Academic Publishers 1991