Optimal Stopping and Maximal Inequalities for Poisson Processes

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Abstract

Closed form solutions for some optimal stopping problems for stochastic processes driven by a Poisson processes $N = (N_t)_{t \geq 0}$ are given.

First, cost functions and optimal stopping rules are described for the problems

$$s(x) = \sup_{\tau} E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau),$$

$$v(x) = \sup_{\tau} E(\max[x, \sup_{0 \leq t \leq \tau} (bt - N_t)] - c\tau),$$

with a, b, c positive constants and τ a stopping time.

Based on the obtained results, maximal inequalities in the spirit of [1] are obtained.

To complete the picture, solutions to the problems

$$c(x) = \sup_{\tau} E(x + N_\tau - a\tau)^+, \qquad p(x) = \sup_{\tau} E(x + b\tau - N_\tau)^+$$

are given.

1 Introduction and main results

1.1. Let be given a Poisson process $N = (N_t)_{t \geq 0}$ with intensity $\lambda > 0$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Denote by $\overline{\mathcal{M}}$ the class of all stopping times (that can take the value $+\infty$), by $\mathcal{M}$ the class of finite valued stopping times, and by $\mathcal{M}_0$ the class of stopping times with finite expectation. All stopping times are considered with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

In this paper we face the problem of giving the cost functions and optimal stopping times in the following optimal stopping problems:

$$s(x) = \sup_{\tau} E(\max[x, \sup_{0\leq t\leq \tau}(N_t - at)] - c\tau), \tag{1}$$

$$v(x) = \sup_{\tau} E(\max[x, \sup_{0\leq t\leq \tau}(bt - N_t)] - c\tau), \tag{2}$$

where a, b and c are positive constants, (c is the "price" of one unity of observation), and $x \in \mathbf{R}$.

Problems (1) and (2) are related to the pricing of "Russian Options" introduced by L. Shepp and A.N. Shiryaev in [2]. In our case, the price process for the risky asset is driven by a Poisson process instead of a geometric Brownian motion.

The solutions to these problems are then used to obtain maximal inequalities of the form

$$\sup_{\tau\in\mathcal{M}_0} E[\sup_{0\leq t\leq \tau}(N_t - at))] \leq \phi(E\tau) \tag{3}$$

$$\sup_{\tau\in\mathcal{M}_0} E[\sup_{0\leq t\leq \tau}(bt - N_t))] \leq \psi(E\tau) \tag{4}$$

where $\phi = \phi(x)$, $\psi = \psi(x)$, $x > 0$ are the minimal possible functions, that satisfy (3) and (4).

In relation with this, we refer to the paper of L. Dubbins, L.A. Shepp and A.N. Shiryaev [1] devoted to the study of optimal stopping rules and maximal inequalities for Bessel Processes, the work of L. Dubbins and G. Schwartz [2], and to some recent results for linear diffusions of S.E. Graversen and G. Peškir [4].

Finally, to complete the picture, we give the cost function and optimal stopping time in the following problems

$$c(x) = \sup_{\tau\in\overline{\mathcal{M}}} E(x + N_\tau - a\tau)^+ \tag{5}$$

(6) $$p(x) = \sup_{\tau \in \overline{\mathcal{M}}} E(x + b\tau - N_\tau)^+$$

where $x \in \mathbf{R}$, z^+ denotes $\max(z, 0)$.

1.2. In order to formulate our results, let us introduce the function $u = u(x) = u(x; d), x \geq 0$ with d a positive constant, defined by

(7) $$u(x; d) = \sum_{k=0}^{+\infty} [e^{\frac{x-k}{d}} P_k(\frac{x-k}{d}) - (k+1)] \mathbf{I}_{[k,k+1)}(x)$$

where $P_k = P_k(x)$ is a polynomial of order k:

(8) $$P_k(x) = \sum_{l=0}^{k} d_{k-l} \frac{(-1)^l x^l}{l!},$$

and the coefficients $d_k, k \geq 0$ are defined by the following recurrence relation:

(9) $$d_0 = 1, \quad d_{k+1} = 1 + e^{1/d} \sum_{l=0}^{k} d_{k-l} \frac{(-1)^l}{d^l l!} = 1 + e^{1/d} P_k(1/d).$$

Some properties of the function $u = u(x)$ are revisted in section 2.

Denote now by τ_s^* the optimal stopping time in the problem (1), that is, the stopping time for which the supremum is realized. The price function $s = s(x)$ and the optimal stopping time τ_s^* are given in the following

Theorem 1

(i) *If $a + c \leq \lambda$, then $s(x) = +\infty$, for all $x \geq 0$.*

(ii) *If $a > \lambda$ and $c = 0$, then*

(10) $$s(x) = E(\max[x, \sup_{0 \leq t \leq +\infty} (N_t - at)]) = \frac{\lambda}{2(a-\lambda)} + \frac{a-\lambda}{\lambda} u(x; \frac{a}{\lambda})$$

(iii) *If $a > 0$ and $c \geq \lambda$, then $s(x) = x$ and $\tau_s^* = 0$.*

(iv) *If $c < \lambda < a + c$, then*

(11) $$s(x) = \begin{cases} x, & \text{if } x \geq x^* \\ x_s^* + \frac{a+c-\lambda}{\lambda}[u(x, \frac{a}{\lambda}) - u(x_s^*; \frac{a}{\lambda})], & \text{if } 0 \leq x < x^* \end{cases}$$

and

(12) $$\tau_s^* = \inf\{t \geq 0 \colon X_t \geq x_s^*\},$$

where $X = (X_t)_{t\geq 0}$ *is a stochastic process defined by*

$$X_t = \max[x, \sup_{0\leq r\leq t}(N_r - ar)] - (N_t - at), \tag{13}$$

and the positive constant x_s^* *is the solution of the equation*

$$\frac{a+c-\lambda}{\lambda}u'(x; \frac{a}{\lambda}) = 1. \tag{14}$$

We also have, when $x < x_s^*$

$$E(\tau_s^*) = \frac{1}{\lambda}[u(x_s^*, \frac{a}{\lambda}) - u(x; \frac{a}{\lambda})]. \tag{15}$$

Denote now by τ_v^* the optimal stopping time for the problem (2).The price function $v = v(x)$ and the optimal stopping time are given in the following

Theorem 2

(i) *If* $c + \lambda \leq b$*, then* $v(x) = +\infty$*, for all* $x \geq 0$.

(ii) *If* $\lambda > b$ *and* $c = 0$*, then*

$$v(x) = E(\max[x, \sup_{0\leq t<+\infty}(bt - N_t)]) = x + \frac{1}{\alpha^*}e^{-\alpha^* x}, \tag{16}$$

where the constant α^* *is the unique positive root of the equation*

$$\frac{b}{\lambda}\alpha + e^{-\alpha} - 1 = 0. \tag{17}$$

(iii) *If* $c \geq b$*, then* $v(x) = x$ *and* $\tau_v^* = 0$.

(iv) *If* $c < b < c + \lambda$*, then*

$$v(x) = \begin{cases} x, & \text{if } x \geq x_v^*, \\ x + \frac{c}{\lambda}u(x_v^* - x, \frac{b}{\lambda}), & \text{if } 0 \leq x < x_v^*, \end{cases} \tag{18}$$

and

$$\tau_v^* = \inf\{t \geq 0 \colon Y_t \geq x_v^*\},$$

where $Y = (Y_t)_{t\geq 0}$ *is a stochastic process defined by*

$$Y_t = \max[x, \sup_{0\leq r\leq t}(br - N_r)] - (bt - N_t), \tag{19}$$

and the positive constant x_v^* *is the solution of the equation*

$$\frac{c}{\lambda}u'(x;\frac{b}{\lambda}) = 1. \tag{20}$$

We also have, when $0 \leq x < x_v^*$

$$E(\tau_v^*) = \frac{1}{c+\lambda-b}[x_v^* - x - \frac{c}{\lambda}u(x_v^* - x, \frac{b}{\lambda})]. \tag{21}$$

1.3. Let $\mathcal{M}_T = \{\tau \in \mathcal{M}_0 : E(\tau) \leq T\}$ denote the set of all the stopping times τ with expected value less or equal than $T > 0$. Functions ϕ and ψ in (3) and (4) can be defined in the following way

$$\phi(T) = \sup_{\tau \in \mathcal{M}_T} E(\sup_{0 \leq t \leq \tau}(N_t - at)), \tag{22}$$

$$\psi(T) = \sup_{\tau \in \mathcal{M}_T} E(\sup_{0 \leq t \leq \tau}(bt - N_t)). \tag{23}$$

In what follows, the stopping times such that the supremum in (22) and (23) is realized will be called $\mathcal{M}_T$-optimal.

Theorem 3 *Let* $0 < T < +\infty$.

(i) *Denote by* $x_s^*(T)$ *the root of the equation*

$$u(x;\frac{a}{\lambda}) = \lambda T. \tag{24}$$

Then

$$\phi(T) = x_s^*(T) + (\lambda - a)T \tag{25}$$

and the stopping time

$$\tau_s^*(T) = \inf\{t \geq 0 : X_t \geq x_s^*(T)\},$$

is $\mathcal{M}_T$*-optimal for the problem (22), where in the process* $X = (X_t)_{t\geq 0}$ *defined in (13) we take* $x = 0$.

Furthermore, when $\lambda = a$ *we have*

$$\frac{\sqrt{4T+1}-1}{2} \leq \phi(T) \leq \sqrt{T} \tag{26}$$

and in consequence

$$\lim_{T\to+\infty}\frac{\phi(T)}{\sqrt{T}}=1. \tag{27}$$

(ii) *Denote by $x_v^*(T)$ the positive root of the equation*

$$\frac{xu'(x;\frac{b}{\lambda})-u(x;\frac{b}{\lambda})}{1+(1-\frac{b}{\lambda})u'(x;\frac{b}{\lambda})}=\lambda T, \tag{28}$$

Then

$$\psi(T)=x_v^*(T)+(b-\lambda)T, \tag{29}$$

and the stopping time

$$\tau_v^*(T)=\inf\{t\geq 0\colon Y_t\geq x_v^*(T)\},$$

is $\mathcal{M}_T$-optimal for the problem (23), where in the process $Y=(Y_t)_{t\geq 0}$ defined in (19) we take $x=0$.

Furthermore, when $\lambda=b$ we have

$$\lim_{T\to+\infty}\frac{\psi(T)}{\sqrt{T}}=1. \tag{30}$$

1.4. Denote by τ_c^* the optimal stopping time in the problem (5). The price function and the optimal stopping time for this problem are given in the following

Theorem 4

(i) *If $a\leq\lambda$, then $c(x)=+\infty$ for all $x\in\mathbf{R}$.*

(ii) *If $\lambda<a$, then*

$$c(x)=\begin{cases} x, & \textit{if } x\geq x_c^*,\\ x+(\frac{a}{\lambda}-1)u(x_c^*-x,\frac{a}{\lambda}), & \textit{if } x<x_c^*,\end{cases} \tag{31}$$

with $x_c^*=\frac{\lambda}{2(a-\lambda)}$. The optimal stopping time is

$$\tau_c^*=\inf\{t\geq 0\colon x+N_t-at\geq x_c^*\}. \tag{32}$$

Denote finally by τ_p^* the optimal stopping time in the problem (6). The cost function and the optimal stopping time for this problem are given in the following

Theorem 5

(i) *If* $b \geq \lambda$, *then* $p(x) = +\infty$ *for all* $x \in \mathbf{R}$.

(ii) *Assume* $b < \lambda$. *Denote by* α^* *the positive root of the equation (17). Then, the cost function is*

$$p(x) = \begin{cases} x, & \text{if } x \geq x_p^*, \\ x_p^* e^{\alpha^*(x - x_p^*)}, & \text{if } x < x_p^*, \end{cases} \tag{33}$$

with $x_p^* = \frac{1}{\alpha^*}$, *and the optimal stopping time is*

$$\tau_p^* = \inf\{t \geq 0 \colon x + bt - N_t \geq x_p^*\}. \tag{34}$$

2 Some auxiliar results

2.1. In this section we formulate some technical results concerning the function $u = u(x), x \geq 0$ defined in (7). Let us introduce the operator $\mathcal{K} = \mathcal{K}(d)$ defined on the set of continuously differentable functions by the following relation

$$\mathcal{K}w(x) = dw'(x) + w(x - x \wedge 1) - w(x), \qquad x \geq 0. \tag{35}$$

As will be shown in section 3, the operator $\lambda\mathcal{K}$ is the infinitesimal operator associated to the process $X = (X_t)_{t \geq 0}$ defined in (13) with $d = \frac{a}{\lambda}$.

Lemma 1 *The function* $u = u(x)$ *defined in (7) is continously differentable, and satisfies the following relation*

$$\mathcal{K}u(x) = du'(x) - u(x) + u(x - x \wedge 1) = 1, \qquad x \geq 0. \tag{36}$$

Proof. From the definition of the function $u = u(x)$ (see (7), (8) and (9)) follows, that this function is smooth on the intervals $(k, k+1)$ for any non-negative integer k.

If $x \in [0, 1)$, we have $u(x) = e^{x/d} - 1$, giving that $u(0) = 0$ and

$$\mathcal{K}u(x) = du'(x) - u(x) = 1.$$

If $k \geq 1$ and $x \in (k, k+1)$, equation (36) is

$$du'(x) - u(x) = 1 - u(x - 1).$$

In view of (7) for $x \in (k, k+1)$

$$u(x) = e^{\frac{x-k}{d}} P_k(\frac{x-k}{d}) - (k+1),$$

and from (8) follows that $P_k'(x) = -P_{k-1}(x)$. Then

$$du'(x) - u(x) = e^{\frac{x-k}{d}} P_k'(\frac{x-k}{d}) + k + 1$$

$$= 1 - e^{\frac{x-k}{d}} P_{k-1}(\frac{x-k}{d}) + k = 1 - u(x-1) \tag{37}$$

In this way (36) is proved for $x \in (k, k+1)$. It is clear from (35), that in order to verify (36) when $x = 1, 2, \ldots$ it is enough to see that the functions $u = u(x)$ and $u' = u'(x)$ are continuous in these points.

Let us examine the continity of u. In view of (7)

$$u(k) = P_k(0) - (k+1) = d_k - (k+1)$$

and by (9)

$$\lim_{x \uparrow k} u(x) = e^{1/d} P_{k-1}(1/d) - k = d_k - (k+1),$$

and the continuity of u follows. For the function u' in view of (37)

$$\lim_{x \downarrow k} u'(x) = \frac{1}{d} \lim_{x \downarrow k} [1 + u(x-1) - u(x)]$$

$$= \frac{1}{d}[1 + u(k-1) - u(k)],$$

and

$$\lim_{x \uparrow k} u'(x) = \frac{1}{d} \lim_{x \uparrow k} [1 + u(x - x \wedge 1) - u(x)]$$

$$= \frac{1}{d}[1 + u(k-1) - u(k)],$$

concluding the proof. $\square$

2.2. In the following Lemma we study the behaviour of $u = u(x)$ and its derivative.

Lemma 2 *The functions $u = u(x)$ and $u' = u'(x)$ satisfy*

a) $u(0) = 0, \;\; u'(0) = \frac{1}{d}$.

b) u and u' are strictly increasing.

c) $\lim_{x\to+\infty} u(x) = +\infty$. $\lim_{x\to+\infty} u'(x) = \begin{cases} \frac{1}{d-1}, & d > 1 \\ +\infty, & 0 < d \leq 1. \end{cases}$

d) If $d = 1$, for all $x \geq 0$ we have

$$x^2 \leq u(x) \leq x^2 + x.$$

e) If $d = 1$, then

$$\lim_{x\to+\infty} \frac{xu'(x) - u(x)}{x^2} = 1.$$

Proof. a) As $u(x) = e^{x/d} - 1$ when $x \in [0, 1)$ we obtain

$$u(0) = 0 \quad \text{and} \quad u'(0) = \frac{1}{d}.$$

b) In order to prove that u and u' are strictly increasing, as they are both absolutely continuous, we will see that

(38) $\qquad u''(x)$ is positive and continuous for all $x \neq 1, \quad x \geq 0,$

As $u(x) = e^{x/d} - 1$ if $x \in (0, 1)$, by differentiation $u''(x) = \frac{1}{d^2}e^{x/d}$ is continuous and positive if $x \in [0, 1)$. Take now $x > 1$. From (36), for $x > 1$ we obtain

(39)
$$du''(x) = \int_{x-1}^{x} u''(y)dy.$$

As u'' is bounded on compact intervals, the continuity if $x \neq 1$ follows. In view of (39) follows that

$$u''(x) > 0, \qquad x > 1.$$

In fact, let $x_0 = \inf\{x \geq 0 \colon u''(x) = 0\}$. We have $u''(x_0) = 0$ and $u''(x) > 0$ when $x < x_0$ contradicting (39).

So $u = u(x)$ and $u' = u'(x)$ are both strictly increasing functions.

c) Let us now compute the limits at infinite. If $x > 1$, then from (36) follows that

$$du'(x) = u(x) - u(x-1) + 1 = 1 + \int_{x-1}^{x} u'(y)dy.$$

From this, taking into account the monotonicity of the function $u' = u'(x)$ we obtain

$$1 + u'(x-1) < du'(x) < 1 + u'(x)$$

and taking limits we obtain the second limit in c). Finally, this ensures the convergence of $u(x) \to +\infty$ when $x \to +\infty$, proving c).

d) Let $d = 1$. Denote $\Delta(x) = u(x) - x^2$. We want to establish

$$\Delta(x) \geq 0 \quad \text{for all} \quad x \geq 0. \tag{40}$$

For this function

$$\mathcal{K}\Delta(x) = \mathcal{K}u - \mathcal{K}(x^2) = 0.$$

This means

$$\Delta'(x) = \Delta(x) - \Delta(x-1) = \int_{x-1}^{x} \Delta'(t)dt.$$

If $x \in (0,1)$, we know $\Delta(x) = e^x - 1 - x^2$, and in consequence $\Delta'(x) > 0$ on this interval. So, with the same argument as in the proof of (38) we obtain that $\Delta'(x) > 0$ for all $x > 0$. As $\Delta(0) = 0$, (40) is proved.

Denote now

$$\delta(x) = x^2 + x - u(x).$$

In a similar way, as was done for Δ, we obtain that

$$\delta'(x) = 2x + 1 - e^x > 0 \quad \text{for} \quad 0 < x < 1,$$

and for $x > 1$

$$\delta'(x) = \int_{x-1}^{x} \delta'(t)dt.$$

concluding that $\delta(x) \geq 0$ for all $x \geq 0$ and the proof of d).

e) From d) we obtain

$$\lim_{x\to+\infty} \frac{u(x)}{x^2} = 1 \tag{41}$$

Now, denoting $w(x) = xu'(x) - u(x)$, we have

$$w'(x) = xu''(x).$$

So, by L'Hôpital rule

$$\lim_{x\to+\infty} \frac{w(x)}{x^2} = \lim_{x\to+\infty} \frac{xu''(x)}{2x} = \lim_{x\to+\infty} \frac{u''(x)}{2}.$$

But, in view of (41), and L'Hôpital rule again

$$\lim_{x\to+\infty} \frac{u''(x)}{2} = \lim_{x\to+\infty} \frac{u'(x)}{2x} = \lim_{x\to+\infty} \frac{u(x)}{x^2} = 1.$$

proving d). $\square$

3. We will also need the following

Lemma 3 *Let $d > 1$. Then the function $w = w(x)$ defined by*

$$w(x) = x - (d-1)u(x), \qquad x \geq 0,$$

is strictly increasing and

$$\lim_{x\to+\infty} w(x) = \frac{1}{2(d-1)}. \tag{42}$$

Proof. By part c) of Lemma 2

$$w'(x) = 1 - (d-1)u'(x) > 0 \quad \text{for all} \quad x \geq 0$$

so the funtion w is stricltly incresing. In consequence the limit in (42) exists.

Now, by (36)

$$\mathcal{K}w(x) = dw'(x) - w(x) = 1 - x, \qquad x < 1,$$

and

$$\mathcal{K}w(x) = dw'(x) - w(x) + w(x-1) = 0, \qquad x \geq 1. \tag{43}$$

Integrating (43) over $[1, x]$, follows

$$d[w(x) - w(1)] = \int_1^x [w(y) - w(y-1)]dy$$

$$= \int_{x-1}^x w(y)dy + \int_0^1 w(y)dy$$

Taking into account (36)

$$dw(x) = \int_{x-1}^x w(y)dy + \frac{1}{2}.$$

Now, the monotonicity of $w(x)$ gives

$$w(x-1) + \frac{1}{2} < dw(x) < w(x) + \frac{1}{2},$$

and taking limits as $x \to +\infty$ we conclude the proof. $\square$

3 Proofs of the Theorems

3.1. The process $X = (X_t)_{t\geq 0}$, defined in (13), is an homogeneous Markov process with stochastic differential

$$dX_t = adt - (X_{t-} \wedge 1)dN_t.$$

If $w = w(x)$, $x \geq 0$, is a continously differentiable function, then, Itô's formula for pure jump processes gives (see [6] ch. 3 §6)

$$w(X_t) = w(x) + \lambda \int_0^t \mathcal{K}w(X_{r-})dr + M_t, \quad t \geq 0, \tag{44}$$

with $\mathcal{K} = \mathcal{K}(\frac{a}{\lambda})$ in (35) and the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = \int_0^t [w(X_{r-} - X_{r-} \wedge 1) - w(X_{r-})]d(N_r - \lambda r)$$

is a local martingale. From (44) and Lemma 1 follows, that the process $(u(X_t) - \lambda t)_{t\geq 0}$ is a local martingale, with the function $u = u(x; \frac{a}{\lambda})$ defined in (7) and $d = \frac{a}{\lambda}$.

Proof of Theorem 1.

(i) Assume $a + c \leq \lambda$. As $E(N_t - (a + c)t) = (\lambda - a - c)t \geq 0$ and the process $\{N_t - (a + c)t\}$ has stationary independent increments, it follows that $P(\sup_t(N_t - (a + c)t) = +\infty) = 1$ (see [7]). Thi s means, that for the stopping time

$$\tau_H = \inf\{t \geq 0 \colon N_t - (a + c)t \geq H\}.$$

we have $P(\tau_H < +\infty) = 1$. Then

$$s(x) \geq s(0) \geq E(\sup_{0\leq r\leq \tau_H} (N_r - ar) - c\tau_H)$$

$$\geq E((N_{\tau_H} - (a + c)\tau_H) \geq H.$$

As H is arbitrary, the proof of (i) is concluded.

(ii) Let $c = 0$ and $a > \lambda$. For $t > 0$ denote

$$s_t(x) = E(\max[x, \sup_{0\leq r\leq t} (N_r - ar)]).$$

By monotnonous convergence

$$s_t(x) \uparrow s(x), \qquad t \to +\infty.$$

Taking into account, that $E(N_t) = \lambda t,\ t \geq 0$, we obtain that

$$s_t(x) = E(X_t - (a - \lambda)t), \qquad X_0 = x,$$

with the process $X = (X_t)_{t \geq 0}$ defined in (13).

Denote $w(x) = x - \frac{a-\lambda}{\lambda} u(x)$. By Lemma 3, we have

$$\lim_{x \to +\infty} w(x) = \frac{\lambda}{2(a - \lambda)}.$$

Furthermore, as $X_t \geq at - N_t$, and $a > \lambda$, then, $X_t \to +\infty$ with probability one as $t \to +\infty$. As the process $\{\frac{a-\lambda}{\lambda} u(X_t) - (a - \lambda)t\}$ is a martingale, we obtain

$$s(x) - \frac{a - \lambda}{\lambda} u(x) = \lim_{t \to +\infty} E(s_t(x)) - E(\frac{a - \lambda}{\lambda} u(X_t) - (a - \lambda)t)$$

$$= \lim_{t \to +\infty} E(X_t - \frac{a - \lambda}{\lambda} u(X_t)) = \lim_{t \to +\infty} E(w(X_t)) = \frac{\lambda}{2(a - \lambda)}$$

by bounded convergence, concluding the proof of (ii).

(iii) First we observe, that when $c > 0$ and $a + c > \lambda$ the supremum in (1) can be taken over $\mathcal{M}_0$. To see this, take an arbitrary stopping time $\tau \in \mathcal{M}$ with $E(\tau) = +\infty$. Consider δ such that $\lambda - a < \delta < c$.

$$E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau) \leq x + E[\sup_{0 \leq t < +\tau} (N_t - at) - c\tau]$$

$$\leq x + E[\sup_{0 \leq t < +\tau} (N_t - (a + \delta)t) - (c - \delta)\tau] = -\infty$$

because, in accordance with (10)

$$E(\sup_{0 \leq t < +\infty} (N_t - (a + \delta)t)) = \frac{\lambda}{a + \delta - \lambda} < +\infty.$$

Now, if $c \geq \lambda$, $a > 0$ and $E(\tau) < +\infty$, then $E(N_\tau) = \lambda E(\tau)$ and

$$E(\max[x, \sup_{0 \leq t \leq \tau} (N_t - at)] - c\tau)) \leq x + E(\sup_{0 \leq t \leq \tau} (N_t - at) - c\tau)$$

$$\leq x + E(N_\tau - c\tau) \leq x - (c - \lambda)E(\tau) \leq x.$$

On the other hand, taking $\tau_s^* = 0$

$$E(\max[x, \sup_{0\le t\le \tau_s^*}(N_t - at)] - c\tau_s^*) = x,$$

and follows that $\tau_s^* = 0$ is the optimal stopping time, and $s(x) = x,\ x \ge 0$.

(iv) Let $c < \lambda < a + c$. In view of Lemma 2 (14) has an unique solution x_s^* such that $0 < x_s^* < +\infty$.

Let us first prove, that for τ_s^* defined by (12), identity (15) holds.

As $u(x) \le u(x_s^*)$ if $x \le x_s^*$, then

$$-\lambda t \le u(X_{\tau_s^*\wedge t}) - \lambda(\tau_s^* \wedge t) \le u(x_s^*)$$

and, as a consequence, the local martingale $\{u(X_{\tau_s^*\wedge t}) - \lambda(\tau_s^* \wedge t)\}_{t\ge 0}$ is in fact a martingale. Therefore

$$\lambda E(\tau_s^* \wedge t) = E(u(X_{\tau_s^*\wedge t})) - u(x) \tag{45}$$

From this, we deduce that $E(\tau_s^*) < +\infty$, so $P(\tau_s^* < +\infty) = 1$, concluding that

$$u(X_{\tau_s^*\wedge t}) \to u(x_s^*) \quad \text{as} \quad t \to +\infty.$$

Now, taking limits in (45) as $t \to +\infty$ we obtain (15).

We know by (iii) that we can take the supremum over $\mathcal{M}_0$. If $\tau \in \mathcal{M}_0$, $E(\tau) < +\infty$, then $E(N_\tau) = \lambda E(\tau)$. In consequence

$$E(\max[x, \sup_{0\le t\le \tau}(N_t - at)] - c\tau) = E(X_\tau - (a + c - \lambda)\tau).$$

Denote by $\tilde{s} = \tilde{s}(x)$ by

$$s(x) = \begin{cases} x, & \text{if } x \ge x^* \\ x_s^* + \frac{a+c-\lambda}{\lambda}[u(x, \frac{a}{\lambda}) - u(x_s^*; \frac{a}{\lambda})], & \text{if } 0 \le x < x^* \end{cases}$$

We want to prove $\tilde{s} = s$ with s in (1). The following relation takes place

(A)$s(x) = \max(x, x_s^*) - (a + c - \lambda)E(\tau_s^*) = E(X_{\tau_s^*} - (a + c - \lambda)\tau_s^*)$,

that is direct if $x \ge x_s^*$ and follows from (15) when $x < x_s^*$. In consequence, in order to complete the proof of the Theorem, it remains to see that for any stopping time $\tau \in \mathcal{M}_0$

(B)$\tilde{s}(x) \ge E(X_\tau - (a + c - \lambda)\tau)$.

In view of Lemma 2 the function $u = u(x)$ is convex, so $s(x) \geq x$. To prove (B) it is enough to see that the process $Z = (Z_t)_{t\geq 0}$ defined by

$$Z_t = \tilde{s}(X_{\tau\wedge t}) - (a + c - \lambda)(\tau \wedge t)$$

is a supermartingale. From (44) follows that the process Z is a local supermartingale, if

$$\lambda\mathcal{K}\tilde{s}(x) - (a + c - \lambda) \leq 0, \qquad x \geq 0, \tag{46}$$

with $\mathcal{K} = \mathcal{K}(\frac{a}{\lambda})$ defined in (35).

For $x \leq x_s^*$ (46) takes places in view of Lemma 1 (in fact we have an identity). If $x > x_s^*$, then $\tilde{s}(x) = x$ and

$$\lambda\mathcal{K}\tilde{s}(x) - (a + c - \lambda) = a + \lambda\tilde{s}(x - x \wedge 1) - \lambda x - (a + c - \lambda)$$

$$\leq a + \lambda\tilde{s}(x_s^* - x_s^* \wedge 1) - \lambda x_s^* - (a + c - \lambda) = \lambda\mathcal{K}\tilde{s}(x_s^*) - (a + c - \lambda) = 0,$$

where we used the convexity of the function $\tilde{s}$. In this way the process $Z = (Z_t)_{t\geq 0}$ is a local supermartingale, and as

$$Z_t \geq -(a + c - \lambda)t, \qquad t \geq 0,$$

and $E(\tau) < +\infty$, we deduce that Z is a supermartingale and the proof is concluded. □

3.2. The process $Y = (Y_t)_{t\geq 0}$ defined in (19) is an homogeneous Markov process with stochastic differential

$$dY_t = dN_t - \mathbf{I}_{\{Y_{t-}>0\}} b dt. \tag{47}$$

If $w = w(x)$, $x \geq 0$, is a continously differentiable function, Itô's formula gives (see [6])

$$w(Y_t) = w(x) + \lambda\int_0^t \mathcal{L}w(Y_{r-})dr + M_t, \tag{48}$$

where

$$\mathcal{L}w(x) = -\frac{b}{\lambda}\mathbf{I}_{\{x>0\}}w'(x) + w(x+1) - w(x), \tag{49}$$

and the process $M = (M_t)_{t\geq 0}$, given by

$$M_t = \int_0^t [w(Y_{r-} + 1) - w(Y_{r-})]d(N_r - \lambda r)$$

is a local martingale.

Proof of Theorem 2.

(i) Let $\lambda + c \le b$. Take $H > 0$ and define

$$\tau_H = \inf\{t \ge 0 \colon (b-c)t - N_t \ge H\}.$$

As on the proof of (i) in Theorem 1 we conclude that $P(\tau_H < +\infty) = 1$. We have

$$\begin{aligned} v(x) &\ge E(\max[x, \sup_{0\le t\le \tau_H}(bt - N_t)] - c\tau_H) \\ &\ge E((b-c)\tau_H - N_{\tau_H}) \ge H. \end{aligned}$$

As H is arbitrary, the proof of (i) is concluded.

(ii) Let $c = 0$ and $b < \lambda$. For $t > 0$ denote

$$v_t(y) = E(\max[x, \sup_{0\le r\le t}(br - N_r)]).$$

By monotonous convergence

$$v_t(y) \uparrow v(y) \quad \text{as} \quad t \to +\infty.$$

Taking into account the definition of Y in (19) and the fact that $E(N_t) = \lambda t$, we obtain that

$$v_t(x) = E[Y_t - (\lambda - b)t], \qquad Y_0 = x.$$

Consider the function $\tilde v = \tilde v(x)$ defined by

$$\tilde v(x) = x + \frac{1}{\alpha^*} e^{-\alpha^* x},$$

with α^* the positive root of the equation (17) that, as $b < \lambda$ has an unique positive solution. A direct computation shows that the function $\tilde v = \tilde v(x)$ satisfies the equation

$$\lambda \mathcal{L} \tilde v(x) - (\lambda - b)t = 0.$$

In view of (46) the process $K = (K_t)_{t\ge 0}$ defined by

$$K_t = \tilde v(Y_t) - (\lambda - b)t$$

is a local martingale.

From the definition of $\tilde{v} = \tilde{v}(x)$ follows that

$$x \leq \tilde{v}(x) \leq x + \frac{1}{\alpha^*}, \qquad x \geq 0, \tag{50}$$

and

$$\lim_{x\to+\infty} |\tilde{v}(x) - x| = 0.$$

Notice now, that

$$\max(0, N_t - bt) \leq Y_t \leq x + N_t \tag{51}$$

so the local martingale $K = (K_t)_{t\geq 0}$ is uniformly integrable and in consequence a martingale. Finally, from (51) and $\lambda > b$ follows that

$$Y_t \to +\infty, \quad t \to +\infty \quad P\text{-a.s.} \tag{52}$$

From this fact, and (50), we obtain

$$\lim_{t\to+\infty} |\tilde{v}(Y_t) - Y_t| = 0.$$

So, we have

$$\begin{aligned} v(x) &= \lim_{t\to+\infty} v_t(x) = \lim_{t\to+\infty} E(Y_t - (b-\lambda)t) \\ &= \lim_{t\to+\infty} E(\tilde{v}(Y_t) - (b-\lambda)t) = \tilde{v}(x), \end{aligned}$$

and the proof of (ii) is concluded.

(iii) Let $c \geq b$. Then, for any $\tau \in \mathcal{M}$

$$E(\max[x, \sup_{0\leq t\leq\tau}(bt - N_t)] - c\tau) \leq y + (b-c)E(\tau) \leq y,$$

and, if we take $\tau_s^* = 0$

$$E(\max[x, \sup_{0\leq t\leq\tau_s^*}(bt - N_t) - c\tau_s^*)] = x,$$

proving (iii).

(iv) Assume $c < b < c + \lambda$. Let us see, that in (2) it is enough to take the supremum over $\mathcal{M}_0$. Take δ positive, such that $b - \lambda < \delta < c$. Consider an arbitrary stopping time τ with $E(\tau) = +\infty$. As by (16)

$$E(\sup_{0\leq t<+\infty}((b-\delta)t - N_t) < \infty,$$

we obtain

$$E(\max[x, \sup_{0\le t\le\tau}(bt - N_t)] - c\tau)) \le x+$$

$$+E(\sup_{0\le t<+\infty}((b-\delta)t - N_t) - (c-\delta)E(\tau) = -\infty,$$

See now, that in view of Lemma 2 equation (20) has a unique positive solution x_v^*. Define now the function $\tilde{v} = \tilde{v}(x)$ by

$$\tilde{v}(x) = \begin{cases} x, & x \ge x_v^* \\ x + \frac{c}{\lambda}u(x_v^* - x; \frac{b}{\lambda}), & 0 \le x < x_v^*. \end{cases}$$

In order to prove (iv) we verify $\tilde{v} = v$, with v in (2). For this, it is enough to verify the following two assertions:

(A) If $\tau_v^* = \inf\{t \ge 0 : Y_t \ge x_v^*\}$, then

$$\tilde{v}(y) = E(\max(y, \sup_{0\le t\le\tau_v^*}(bt - N_t)) - c\tau_v^*).$$

(B) For all $\tau \in \mathcal{M}_0$

$$\tilde{v}(x) \ge E(\max[x, \sup_{0\le t\le\tau}(bt - N_t)] - c\tau),$$

We begin by (B). If $\tau \in \mathcal{M}_0$ then $E(N_\tau) = \lambda E(\tau)$ and in consequence

$$E(\max[x, \sup_{0\le t\le\tau}(bt - N_t)] - c\tau) = E(Y_\tau - (c+\lambda-b)\tau)$$

$$\le E(\tilde{v}(Y_\tau) - (c+\lambda-b)\tau),$$

because $\tilde{v}(x) \ge x$, as the function u is convex (Lemma 2).

Then, in order to conclude the proof of (B) it is enough to see that the process $V = (V_t)_{t\ge0}$ given by

$$V_t = \tilde{v}(Y_{t\wedge\tau}) - (c+\lambda-b)(t\wedge\tau)$$

is a supermartingale. As $V_t \ge -(c+\lambda-b)t$, by (46) it is enuogh to verify

$$\lambda\mathcal{L}\tilde{v}(x) - (c+\lambda-b) \le 0 \quad \text{for} \quad x \ge 0. \tag{53}$$

with $\mathcal{L}$ defined in (48). Extend the definition of the function u in a continuous way, as $u(x) = 0$ if $x < 0$. Now, using Lemma 1 we obtain, for $0 \leq x < x_v^*$

$$\lambda \mathcal{L}\tilde{v}(x) = -b\mathbf{I}_{\{y>0\}}[1 - \frac{c}{\lambda}u'(x_v^* - x)] + \lambda + c[u(x_v^* - x - 1) - u(x_v^* - x)]$$

$$= -b + \lambda + c\mathcal{K}u(y)|_{y=x_v^*-x} = c + \lambda - b \leq 0.$$

When $x \geq x_v^*$, then $\tilde{v}(x) = x$ and

$$\lambda \mathcal{L}\tilde{v}(x) - (c + \lambda - b) = -c < 0,$$

and in this way the inequality (51) holds and (B) is proved.

Let us see (A). Define $V^* = (V_t^*)_{t\geq 0}$ by

$$V_t^* = \tilde{v}(Y_{t\wedge\tau_v^*}) - (c + \lambda - b)(t \wedge \tau_v^*).$$

In view of (46) and (52) V^* is a local martingale. Furthermore, this process is bounded on each interval of the form $[0, T]$, $T < +\infty$, and in consequence is a martingale. So

$$\tilde{v}(x) = E(V_t^*) = E(\tilde{v}(Y_{t\wedge\tau_v^*})) - (c + \lambda - b)E(t \wedge \tau_v^*). \tag{54}$$

As $0 \leq \tilde{v}(Y_{t\wedge\tau_v^*}) \leq \max(x, x_v^*)$, we can take limits as $t \to +\infty$ in (54) obtaining

$$\tilde{v}(x) = E(\tilde{v}(Y_{\tau_v^*} - (c + \lambda - b)\tau_v^*) = E(Y_{\tau_v^*} - (c + \lambda - b)\tau_v^*)$$

$$= E(\max[x, \sup_{0\leq t\tau_v^*}(bt - N_t) - c\tau_v^*).$$

concluding (A). From (54), we also deduce (21)

$$E(\tau_v^*) = \frac{1}{c + \lambda - b}(\tilde{v}(x_v^*) - \tilde{v}(x)) = \frac{1}{c + \lambda - b}(x_v^* - x - \frac{c}{\lambda}u(x_v^* - x)).$$

concluding the proof of Theorem 2. □

3.3. *Proof of Theorem 3.*

(i) For $T > 0$ equation (24) has an unique positive root $x_s^*(T)$ by Lemma 2. The constant $c(T)$ defined by

$$\frac{a + c - \lambda}{\lambda}u'(x_s^*(T); \frac{a}{\lambda}) = 1,$$

satisfies

$$\lambda - a < c(T) < \lambda.$$

So, by (iv) in Theorem 1 the stopping time

$$\tau_s^*(T) = \inf\{t \geq 0 \colon X_t \geq x_s^*(T)\},$$

with $X = (X_t)_{t\geq 0}$ defined by (11) is optimal for the problem (1). By the election of $c = c(T)$, and (15)

$$E(\tau_s^*(T)) = T.$$

Also, in view of (1), (11) and (24)

$$E(\sup_{0\leq t\leq \tau_s^*(T)} (N_t - at)) = s(0) + c(T)T = x_s^*(T) - (\lambda - a)T,$$

and for any $\tau \in \mathcal{M}_T$

$$E(\sup_{0\leq t\leq \tau} (N_t - at)) \leq s(0) + c(T)E(\tau) \leq x_s^*(T) - (\lambda - a)T.$$

proving (25). Finally, (26) and (27) are direct consequences of c) in Lemma 2, concluding (i).

(ii) For a positive T, define $x_v^*(T)$ as the root of the equation

$$\frac{xu'(x) - u(x)}{1 + (1 - \frac{b}{\lambda})u'(x)} = \lambda T, \tag{55}$$

where $u = u(x) = u(x; \frac{b}{\lambda})$. In order to see that the equation (55) has an unique root, we note, that for $b > \lambda$ the function

$$f(x) = \frac{xu'(x) - u(x)}{1 + (1 - \frac{b}{\lambda})u'(x)}$$

is increasing (because is the quotient of an increasing function over a decreasing one), $f(0) = 0$ and $f(x) \to +\infty$ if $u'(x) \to (\frac{b}{\lambda} - 1)^{-1}$.

If $b \leq \lambda$, L'Hôpital rule gives $\lim_{x\to+\infty} f(x) = +\infty$, and, as $f(0) = 0$, we confirm the existence of only one root, because

$$f'(x) = \frac{u''(x)[x + u(x)(1 - \frac{b}{\lambda})]}{[1 + (1 - \frac{b}{\lambda})u'(x)]^2} \geq 0.$$

Let us see that the constant $c(T)$ defined by the relation

$$\frac{c(T)}{\lambda}u'(x_v^*(T);\frac{b}{\lambda})=1 \tag{56}$$

satisfies $b-\lambda<c(T)<b$. As, by Lemma 2 $u'(x_v^*(T))>u'(0)=\frac{\lambda}{b}$,

$$c=\frac{\lambda}{u'(x_v^*(T))}<b.$$

On the other side, if $b\leq\lambda$ the second inequality is inmediate. If $\lambda<b$, then by b) in Lemma 2,

$$u'(x_v^*(T))<\lim_{x\to+\infty}u'(x)=\frac{\lambda}{b-\lambda}$$

and the second inequality follows.

Then, we are in case (iv) of Theorem 2. The stopping time

$$\tau_v^*(T)=\inf\{t\geq 0\colon Y_t\geq x_v^*(T)\}$$

is optimal for the problem (2) with $c=c(T)$, $x=0$. Then,

$$E(\tau_v^*(T))=\frac{1}{c+\lambda-b}(x_v^*(T)-\frac{c}{\lambda}u(x_v^*))=T.$$

Furthermore, (2), (18) and (56) gives

$$E(\sup_{0\leq t\leq\tau_v^*(T)}(bt-N_t))=v(0)+c(T)E(\tau_v^*(T))=\frac{c(T)}{\lambda}u(x_v^*(T))+c(T)T$$

$$=\frac{u(x_v^*)(T)+\lambda T}{u'(x_v^*(T))}=x_v^*(T)-(\lambda-b)T,$$

concluding the proof of (31).

To see (30), take $b=\lambda$, and denote $w(x)=xu'(x)-u(x)$. We have $x_v^*(T)=w^{-1}(T)=\psi(T)$. We know,

$$\lim_{x\to+\infty}\frac{w(x)}{x^2}=1,$$

this means

$$\frac{T}{(w^{-1}(T))^2} \to 1,$$

and taking square roots, we obtain (30) concluding the proof of Theorem 3. $\square$

3.4. *Proof of Theorem 4.*

(i) Assume $\lambda > a$. For $H > 0$, denote

$$\tau_H = \{t \geq 0 : x + N_t - at \geq H\}.$$

As in the proof of (i) in Theorem 1, we obtain $P(\tau_H < +\infty = 1)$. Then

$$c(x) = \sup_{\tau \geq 0} E(x + N_\tau - a\tau)^+ \geq E(x + N_{\tau_H} - a\tau_H)^+ \geq H.$$

As H is arbitrary, the proof of (i) is concluded.

(ii) Take $a > \lambda$. Denote $U_t = x + N_t - at$. The process $U = (U_t)_{t \geq 0}$ has an infinitesimal operator of the form

$$\mathcal{U} f(x) = \lambda(f(x+1) - f(x)) - af'(x).$$

Itô's formula in this case reads, with c in (31)

$$c(U_t) = c(x) + \int_0^t \mathcal{U} c(U_{r-}) dr + M_t, \qquad t \geq 0, \tag{57}$$

with the process $M = (M_t)_{t \geq 0}$ given by

$$M_t = \int_0^t [c(U_{r-} + 1) - c(U_{r-}))] d(N_r - \lambda r)$$

a local martingale. Let us now verify that for the function $c = c(x)$ defined in (31)

$$\mathcal{U} c(x) = 0 \quad \text{if} \quad x < x_c^*. \tag{58}$$

$$\mathcal{U} c(x) < 0 \quad \text{if} \quad x \geq x_c^*. \tag{59}$$

In fact, (59) is inmediate. In order to see (58), taking into account (36)

$$\mathcal{U} c(x) = \lambda(c(x+1) - c(x)) - ac'(x)$$

$$= \lambda - a + (a-\lambda)(u(x_c^* - 1) - u(x_c^* - x) + \frac{a}{\lambda} u'(x_c^* - x)) = 0.$$

Now, from (57) we obtain that the stopped local martingale $M^* = \{M_{t \wedge \tau_c^*}\}_{t \geq 0}$ with τ_c^* in (32) is uniformly bounded:

$$-c(x) \leq M_{t \wedge \tau_c^*} \leq c(x_c^*) + 1,$$

so, taking expected values and limits, we obtain $E(M_{\tau_c^*}) = 0$ that means, by (31). As on the set $\{\tau_c^* = +\infty\}$ we have

$$c(x + N_{\tau_c^*} - a\tau_c^*) = (x + N_{\tau_c^*} - a\tau_c^*)^+ = 0.$$

we deduce

(A) $c(x) = E(c(x + N_{\tau_c^*} - a\tau_c^*)) = E(x + N_{\tau_c^*} - a\tau_c^*)^+$.

To complete the proof, we will see

(B) For any $\tau \in \bar{\mathcal{M}}$

$$c(x) \geq E(x + N_\tau - a\tau)^+.$$

For the process M defined by (57), as $-\mathcal{U}c(x) \geq 0$ we have

$$M_t \geq -c(x).$$

This fact, and Fatou's Lemma gives, that the process M is a supermartingale. Now, using this fact, and $c(x) \geq x^+$ we obtain

$$c(x) \geq E(c(x + N_\tau - a\tau)) \geq E(x + N_\tau - a\tau)^+,$$

concluding the proof of Theorem 4. □

3.5. *Proof of Theorem 5.*

(i) Is analogous to the proof of (i) in Theorem 4.

(ii) Take $b < \lambda$. Denote $D_t = x + bt - N_t$. The process $D = (D_t)_{t \geq 0}$ has in infinitesimal operator of the form

$$\mathcal{D}f(x) = bf'(x) - \lambda(f(x-1) - f(x)).$$

It is direct to see that

$$\begin{cases} \mathcal{D}p(x) = 0, & \text{if} \quad x < x_p^*. \\ \mathcal{D}p(x) = b - \lambda, & \text{if} \quad x > x_p^*. \end{cases} \tag{60}$$

Itô's formula in this case is

$$(61) \qquad p(D_t) = p(x) + \int_0^t \mathcal{D}(D_{r-})dr + M_t, \quad t \geq 0,$$

with the process $M = (M_t)_{t\geq 0}$ given by

$$M_t = \int_0^t [p(D_{r-} - 1) - p(D_{r-}))]d(N_r - \lambda r)$$

a local martingale. Now, from (60) we obtain that the stopped local martingale $M^* = \{M_{t\wedge\tau_p^*}\}_{t\geq 0}$ with τ_p^* defined in (34) is uniformly bounded:

$$-p(x) \leq M_{t\wedge\tau_p^*} \leq p(x_p^*) = x_p^*,$$

so, taking expected values and limits, we obtain $E(M_{\tau_p^*}) = 0$. As on the set $\{\tau_p^* = +\infty\}$ we have $= p(D_{\tau_p^*}) = (D_{\tau_c^*})^+ = 0$, (61) gives

(A) $p(x) = E(p(x + b\tau_p^* - N_{\tau_p^*}) = E(x + b\tau_p^* - N_{\tau_p^*})^+$.

To complete the proof, we will see

(B) For any $\tau \in \bar{\mathcal{M}}$

$$p(x) \geq E(x + b\tau - N_\tau)^+.$$

For the process M defined by (61), as $-\mathcal{D}p(x) \geq 0$ we have

$$M_t \geq -p(x).$$

This fact, and Fatou's Lemma gives, that the process M is a supermartingale. Now, using this fact, and $p(x) \geq x^+$ we obtain

$$p(x) \geq E(p(x + b\tau - N_\tau)) \geq E(x + b\tau - N_\tau)^+,$$

concluding the proof of the Theorem $\square$

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