Smoothing of paths and weak approximation of the occupation measure of Lévy processes

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Abstract

Consider a real-valued Lévy process with non-zero Brownian component and jumps with locally finite variation. We obtain an invariance principle theorem for the speed of approximation of its occupation measure by means of functionals defined on regularizations of the paths.

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1 Lévy processes

Let $X = \{X_t : t \geq 0\}$ be a real-valued Lévy process, defined on a probability space $(\Omega, \mathcal{F}, P)$, that we represent by

$$X_t = \sigma W_t + S_t + mt. \tag{1}$$

Here $W = \{W_t : t \geq 0\}$ is a standard Wiener process, $S = \{S_t : t \geq 0\}$ a pure jump process with càdlàg paths, m and σ are real constants, and we assume that the Gaussian part does not vanish, i.e. $\sigma > 0$. Denote by $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ the minimal filtration generated by X, that satisfy the usual assumptions (see Jacod and Shiryaev (1987)).

Furthermore, assume that

(FV) the jump part of the process has locally finite variation, i.e. for each positive t, $\sum_{0<r\leq t} |\Delta S_r|$ is almost surely finite,

where, as usual, we denote $f(r-)$ the left limit of a càdlàg function f on the point r, and $\Delta f_r = f(r) - f(r-)$ is the magnitude of its jump at this point.

In view of (FV), the random variable S_t satisfies

$$S_t = \sum_{0<s\leq t} \Delta X_s.$$

Given a positive constant a, it will be useful to define the processes $S^a = \{S^a_t : t \geq 0\}$ and $X^a = \{X^a_t : t \geq 0\}$ by

$$S^a_t = \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\{|\Delta X_s|\geq a\}}, \tag{2}$$

that is the ($a.s.$ finite) sum of jumps of the process greater or equal than a, and

$$X^a_t = mt + \sigma W_t + S^a_t, \tag{3}$$

respectively.

The characteristic function of the random variable X_t has the standard form $E\left(e^{zX_t}\right) = e^{t\kappa(z)}$, where the function $\kappa(z)$ (defined for the complex values of z such that this expectation is finite) has the form

$$\kappa(z) = mz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1)\Pi(dy). \tag{4}$$

Here $\Pi(dy)$, the Lévy-Khinchine measure of the process, is a non-negative measure defined on $\mathbb{R} \setminus \{0\}$ that, in accordance with condition (FV) above, satisfies $\int (1 \wedge |y|)\Pi(dy) < \infty$. We denote by $\nu_t(dy)$ the Poisson jump measure of the process on the interval $[0, t]$. Note that for each $t > 0$, we have $a.s.$ $\nu_t\big(\{|x| \geq \delta\}\big) < \infty$ for every $\delta > 0$. For general references on Lévy processes see Skorokhod (1991), Bertoin (1996) or Sato (1999).

2 Regularized Lévy processes

We now describe the regularization of the trajectories, that, in our context, is interpreted as a partial observation of the process through a physical device. Let $\psi\colon \mathbb{R} \to \mathbb{R}^+$ be a C^1 function with compact support, say $\mathrm{supp}(\psi) \subset [-1, 1]$, such that $\int_{-1}^{1} \psi(t)dt = 1$ and, for $\varepsilon > 0$, define the approximation of unity

$$\psi_\varepsilon(t) = \frac{1}{\varepsilon}\psi\Big(\frac{t}{\varepsilon}\Big).$$

We denote by $\|\psi\| = \big(\int_{-1}^{1} \psi^2(t)dt\big)^{1/2}$ the norm of ψ in $L^2(\mathbb{R}, dt)$. The regularization $X^\varepsilon = \{X^\varepsilon_t : t \geq 0\}$ of the process is obtained by convolution with ψ_ε in the following way:

$$X^\varepsilon_t = \big(\psi_\varepsilon * X\big)_t = \int_{\mathbb{R}} \psi_\varepsilon(t-s)X_s ds = \int_{-1}^{1} \psi(-w)X_{t+w\varepsilon}dw, \tag{5}$$

where we set $X_s = W_s = S_s = 0$ if $s < 0$. In the same way, we define $W^\varepsilon = \{W_t^\varepsilon : t \geq 0\}$ and $S^\varepsilon = \{S_t^\varepsilon : t \geq 0\}$, and obtain that $X_t^\varepsilon = mt - \varepsilon m\alpha + \sigma W_t^\varepsilon + S_t^\varepsilon$ where $\alpha = \int_{\mathbb{R}} w\psi(w)dw$.

Observe that the regularized processes inherits the regularity properties of ψ, so that X^ε has C^1 paths. For further reference, we compute the time-derivative (denoted with a dot) of the regularized process, that can be written as a stochastic integral:

$$\begin{aligned} \dot{X}_t^\varepsilon &= \int_{\mathbb{R}} \frac{\partial}{\partial t}\big(\psi_\varepsilon(t-s)\big) X_s ds = \frac{1}{\varepsilon}\int_{-1}^{1} \dot{\psi}(-w)\left(X_{t+\varepsilon w} - X_{t-\varepsilon}\right) dw \\ &= \int_{\mathbb{R}} \psi_\varepsilon(t-s) dX_s = \frac{1}{\varepsilon}\int_{-1}^{1} \psi(-w) d^w\big(X_{t+\varepsilon w}\big). \end{aligned} \tag{6}$$

Similar formulae hold for W^ε, S^ε and $S^{a,\varepsilon}$. In particular,

$$\dot{W}_t^\varepsilon = \frac{1}{\varepsilon}\int_{-1}^{1} \dot{\psi}(-w)(W_{t+\varepsilon w} - W_{t-\varepsilon})dw,$$

which implies, for $t \in [0,T]$ and $0 < \varepsilon < 1$:

$$|\varepsilon \dot{W}_t^\varepsilon| \leq 2\|\dot{\psi}\|_\infty \sup_{|h|<\varepsilon, t\in[0,T]} |W_{t+h} - W_{t-\varepsilon}| \leq C_\eta(\omega)\varepsilon^{1/2-\eta}, \tag{7}$$

with $\eta \in (0, 1/2)$ arbitrary, and $C_\eta(\omega)$ a random constant independent of ε; we also have that $\sqrt{\varepsilon}\dot{W}_t^\varepsilon$ has centered Gaussian distribution, with variance $\|\psi\|^2$.

If $F\colon \mathbb{R}^+ \to \mathbb{R}$ is a C^1 function, we denote the *number of crossings* of the level u by the function F on an interval $I = [s,t]$, by

$$N_u^F[s,t] = \sharp\{r\colon F_r = u, r \in I\}, \tag{8}$$

that is, the number of roots belonging to I of the equation $F_t = u$. It is easy to verify, that, for a given continuous function $f\colon \mathbb{R} \to \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} f(u) N_u^F[0,T]du = \int_0^T f\big(F_t\big)|\dot{F}_t|dt. \tag{9}$$

3 Main Result

The aim of Theorem 1 below is to approximate the occupation measure of the process X on the interval $[0,T]$ by a re-normalization of the number of crossings of the process $X^\varepsilon = \{X_t^\varepsilon : t \geq 0\}$ with horizontal levels on the same time interval.

Theorem 1 *Consider a Lévy process $X = \{X_t\colon t \geq 0\}$ with characteristic exponent given in* (4)*, $\sigma > 0$, finite variation jump component, and the regularization $X^\varepsilon = \{X_t^\varepsilon : t \geq 0\}$ defined in* (5)*. Then, for each C^2-function $f\colon \mathbb{R} \to \mathbb{R}$*

with bounded second derivative, we have

$$\frac{1}{\sqrt{\varepsilon}}\left[\int_{\mathbb{R}} f(u) C_\varepsilon \sqrt{\varepsilon} N_u^{X^\varepsilon}[0,t]du - \sigma\int_0^t f(X_s)ds\right] - C_\varepsilon \int_0^t f(X_s^\varepsilon)|\dot{S}_s^\varepsilon|ds$$
$$\Rightarrow D\int_0^t f(X_s)dB_s \quad (10)$$

as $\varepsilon \to 0$, *where:*

- $B = \{B_t : t \geq 0\}$ *is a Wiener process independent of* X*;*
- *The first constant is*

$$C_\varepsilon = \frac{\sigma}{E(\sqrt{\varepsilon}|\sigma \dot{W}_1^\varepsilon + m|)} \to \frac{1}{||\psi||}\sqrt{\frac{\pi}{2}} = C_0 \quad (\varepsilon \to 0). \tag{11}$$

- *The second constant is*

$$D^2 = 2\sigma^2 \int_0^2 \big(r(t)\operatorname{Arsin} r(t) + \sqrt{1-r^2(t)} - 1\big)dt, \tag{12}$$

  where $r(t)$ *is a covariance function defined by*

$$r(t) = \frac{1}{||\psi||^2}\int \psi(t-u)\psi(-u)du.$$

- $\Rightarrow$ *denotes weak convergence in the space* $\mathcal{C} = \mathcal{C}([0,+\infty),\mathbb{R})$ *of continous functions.*

Before proving the Theorem we make some remarks on the statement.

Remarks.

1.- A simple consequence of Theorem 1 is that for each $t > 0$, one has

$$\int_{\mathbb{R}} f(u) C_\varepsilon \sqrt{\varepsilon} N_u^{X^\varepsilon}[0,t]du \to \sigma \int_0^t f(X_s)ds \text{ in probability} \tag{13}$$

as $\varepsilon \to 0$. This result can be used to estimate σ from the observation of the smoothed path X^ε. Results of type (13) are well-known for semimartingales having continuous paths (Azaïs & Wschebor, 1997) and also other classes of processes (Azaïs & Wschebor, 1996), where almost sure convergence is proved.

2.- Theorem 1 contains the speed of convergence in (13). This allows to make inference on σ from the observation of X^ε.

Analogous results for processes with continuous paths are in Berzin & León (1994) for Brownian motion and in Perera & Wschebor (1998, 2002) for certain classes of continuous semi-martingales having Itô-integrals as martingale part. Even if X is a Brownian motion, the proof below seems to be simpler and more direct than previously published ones.

There exist also some related results for Brownian motion and general diffusions, where the approximation X^ε of the actual path X is replaced by polygonal approximation and the smooth function f by a Dirac-delta function, or considering functionals defined on random walks. See for example, Dacunha-Castelle and Florens (1986), Florens (1993), Génon-Catalot and Jacod (1993), Borodin and Ibragimov (1994), and Jacod (1998, 2000). In this context, if ε is the size of the discretization in time, then the speed of convergence turns out to be of the order $\varepsilon^{1/4}$.

3.- We shall prove (see Proposition 3 in next section) that for each $t > 0$ the bias term

$$\mathcal{L}_\varepsilon(f,t) = C_\varepsilon \int_0^t f(X_s^\varepsilon)\big|\dot{S}_s^\varepsilon\big|ds \tag{14}$$

in (10), almost surely converges, as $\varepsilon \to 0$, to

$$\mathcal{L}_0(f,t) = C_0 \sum_{0<s\leq t} L(f,s)\big|\Delta X_s\big| \tag{15}$$

where

$$L(f,t) = \int_{-1}^1 \psi(z) f\Big(X_{t-}\int_z^1 \psi(w)dw + X_t \int_{-1}^z \psi(w)dw\Big)dz. \tag{16}$$

It follows that one can replace Theorem 1 by the statement

$$\frac{1}{\sqrt{\varepsilon}}\left[\int_{\mathbb{R}} f(u) C_\varepsilon \sqrt{\varepsilon} N_u^{X^\varepsilon}[0,t]du - \sigma \int_0^t f(X_s)ds\right] - \mathcal{L}_0(f,t) \tag{17}$$

converges cilindrically to the law of $D\int_0^t f(X_s)dB_s$.

With this statement, the bias term does not depend on ε, but excepting the case of trivial f, we lose weak convergence when the jump part of X does not vanish.

4 Proofs

Proof of Theorem 1

In order to prove the Theorem we first observe, in view of (9), that

$$\int_{\mathbb{R}} f(u) N_u^{X^\varepsilon}[0,t]du = \int_0^t f(X_s^\varepsilon)|\dot{X}_s^\varepsilon|ds, \qquad a.s.$$

Write our expression as the sum of three terms:

$$
\begin{aligned}
&\frac{1}{\sqrt{\varepsilon}}\int_0^t f(X_s^\varepsilon)C_\varepsilon\sqrt{\varepsilon}|\dot{X}_s^\varepsilon|ds-\frac{\sigma}{\sqrt{\varepsilon}}\int_0^t f(X_s)ds-C_\varepsilon\int_0^t f(X_s^\varepsilon)|\dot{S}_s^\varepsilon|ds\\
&\quad=C_\varepsilon\int_0^t f(X_s^\varepsilon)\big(|\dot{X}_s^\varepsilon|-|\sigma\dot{W}_s^\varepsilon+m|-|\dot{S}_s^\varepsilon|\big)ds\\
&\qquad+C_\varepsilon\int_0^t f(X_s^\varepsilon)|\sigma\dot{W}_s^\varepsilon+m|ds-C_\varepsilon\int_0^t f(X_s)|\sigma\dot{W}_{s-\varepsilon}^\varepsilon+m|ds\\
&\qquad+\frac{1}{\sqrt{\varepsilon}}\int_0^t\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_{s-\varepsilon}^\varepsilon+m|-\sigma\big)f(X_s)ds.
\end{aligned}
$$

We now introduce the following simplification, that will be useful for the proof. Given an arbitrary $\delta\in(0,1)$ there exists $b>0$ such that there is no jump with absolute value greater than b, with probability greater that $1-\delta$. The given Lévy process can be written as $X_t=(X_t-S_t^b)+S_t^b$, where the random processes $\{S_t^b:t\geq 0\}$ (defined in (2)) and $\{X_t-S_t^b:t\geq 0\}$ are independent. A standard argument shows that it is enough to prove the result for the process $\{X_t-S_t^b:0\leq t\leq T\}$, so, in what follows, we assume that the support of N is contained in the interval $[-b,b]$. Under this additional hypothesis, it is easy to see that for each $t\geq 0$ the random variable X_t has finite moments of all orders.

In what follows, the parameter of the various processes we will consider vary in a fixed interval $[0,T]$.

We divide the proof into three steps:

1. Proof of
$$Z_t^{1,\varepsilon}=\int_0^t f(X_s^\varepsilon)\big(|\dot{X}_s^\varepsilon|-|\sigma\dot{W}_s^\varepsilon+m|-|\dot{S}_s^\varepsilon|\big)ds\Rightarrow 0. \tag{18}$$

2. Proof of
$$Z_t^{2,\varepsilon}=\int_0^t f(X_s^\varepsilon)|\sigma\dot{W}_s^\varepsilon+m|ds-\int_0^t f(X_s)|\sigma\dot{W}_{s-\varepsilon}^\varepsilon+m|ds\Rightarrow 0 \tag{19}$$

3. Proof of
$$Z_t^{3,\varepsilon}=\frac{1}{\sqrt{\varepsilon}}\int_0^t\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_{s-\varepsilon}^\varepsilon+m|-\sigma\big)f(X_s)ds\Rightarrow D\int_0^t f(X_t)dB_s. \tag{20}$$

Proof of Step 1. We prove

$$\sup_{0\leq t\leq T}\left|\int_0^t f(X_s^\varepsilon)\big(|\dot{X}_s^\varepsilon|-|\sigma\dot{W}_s^\varepsilon+m|-|\dot{S}_s^\varepsilon|\big)ds\right|\to 0\quad a.s.\quad(\varepsilon\to 0).$$

Since there exists an almost surely finite random variable $M(\omega)$ such that

$$\sup_{0\leq s\leq T}|f(X_s^\varepsilon)|\leq M(\omega),\qquad \sup_{0\leq s\leq T}|f(X_s)|\leq M(\omega), \tag{21}$$

it is enough to prove that

$$\int_0^T \Big||\dot X_s^\varepsilon| - |\sigma \dot W_s^\varepsilon + m| - |\dot S_s^\varepsilon|\Big| ds \to 0 \quad a.s. \quad (\varepsilon \to 0).$$

We have

$$\begin{aligned}\Big||\dot X_s^\varepsilon| - |\sigma \dot W_s^\varepsilon + m| - |\dot S_s^\varepsilon|\Big| &\le 2\min\big(|\sigma \dot W_s^\varepsilon + m|, |\dot S_s^\varepsilon|\big)\\ &\le 2\min\big(|\sigma \dot W_s^\varepsilon + m|, |\dot S_s^{a,\varepsilon}|\big) + 2|\dot S_s^{a,\varepsilon} - \dot S_s^\varepsilon|.\end{aligned}$$

We claim that

$$\sup_{0<\varepsilon\le 1} \int_0^T |\dot S_s^\varepsilon - \dot S_s^{a,\varepsilon}| ds \to 0 \quad a.s. \quad (a \to 0). \tag{22}$$

In fact, denoting $g(x) = |x|\mathbf{1}_{\{|x|<a\}}$, in view of (6), we have

$$\begin{aligned}|\dot S_t^\varepsilon - \dot S_t^{a,\varepsilon}| &\le \Big|\frac{1}{\varepsilon}\int_{-1}^1 |\dot\psi(-w)| \sum_{t-\varepsilon<v\le t+\varepsilon w} g(\Delta X_v) dw\Big|\\ &\le \frac{2}{\varepsilon}\|\dot\psi\|_\infty \sum_{t-\varepsilon<v\le t+\varepsilon} g(\Delta X_v).\end{aligned}$$

Furthermore, if $G(t) = \sum_{0<v\le t} g(\Delta X_v)$ and $0<\varepsilon\le 1$, we obtain

$$\begin{aligned}\int_0^T |\dot S_s^\varepsilon - \dot S_s^{a,\varepsilon}| ds &\le \frac{2}{\varepsilon}\|\dot\psi\|_\infty \int_0^T \big(G(t+\varepsilon) - G(t-\varepsilon)\big) dt\\ &= \frac{2}{\varepsilon}\|\dot\psi\|_\infty \int_0^T \Big(\int_{t-\varepsilon}^{t+\varepsilon} G(dv)\Big) dt\\ &\le \frac{2}{\varepsilon}\|\dot\psi\|_\infty \int_0^{T+1} \Big(\int_{v-\varepsilon}^{v+\varepsilon} dt\Big) G(dv)\\ &= 4\|\dot\psi\|_\infty G(T+1)\\ &= 4\|\dot\psi\|_\infty \sum_{0<t\le T+1} |\Delta X_t| \mathbf{1}_{\{|\Delta X_t|<a\}},\end{aligned}$$

and (22) follows.

Consider now a fixed. Put $\tau_0 = 0$, and denote the succesive epochs of jump with absolute value not smaller that a by

$$\tau_n = \inf\{t > \tau_{n-1} : |\Delta X_t| \ge a\} \quad (n = 1, 2, \dots).$$

Denote also by $N_t = \max\{n \colon \tau_n \le t\}$ $(t \ge 0)$ the number of these jumps up to time t. Fix $\omega \in \Omega$, and choose $\varepsilon > 0$ such that the intervals $(\tau_n - \varepsilon, \tau_n + \varepsilon)$ $(n = 1, \dots, N_T)$ are disjoint. Taking into account that $\dot S^{a,\varepsilon} = 0$ outside these intervals, and applying (7) with $\eta = 1/4$, we obtain for $s \in [0,T]$:

$$\min\big(|\sigma \dot W_s^\varepsilon + m|, |\dot S_s^{a,\varepsilon}|\big) \le \hat C_{1/4}(\omega)\varepsilon^{-3/4} \sum_{n=1}^{N_T} \mathbf{1}_{(\tau_n-\varepsilon,\tau_n+\varepsilon)}(s).$$

where $\hat{C}_{1/4}$ is a new constant depending on $\omega \in \Omega$, and on the parameters m, σ. We then obtain

$$\int_0^T \min\left(|\sigma \dot{W}_s^\varepsilon + m|, |\dot{S}_s^{a,\varepsilon}|\right) ds \leq \sum_{n=1}^{N_T} \int_{\tau_n - \varepsilon}^{\tau_n + \varepsilon} \hat{C}_{1/4}(\omega) \varepsilon^{-3/4} ds$$
$$= 2\hat{C}_{1/4}(\omega) N_T(\omega) \varepsilon^{1/4}.$$

From this inequality and (22) the statement of Step 1 follows.

Proof of Step 2. First observe that

$$\int_0^t f(X_s)|\sigma \dot{W}_{s-\varepsilon}^\varepsilon + m| ds = \int_{-\varepsilon}^{t-\varepsilon} f(X_{s+\varepsilon})|\sigma \dot{W}_s^\varepsilon + m| ds$$

which implies

$$Z_t^{2,\varepsilon} = \int_0^t \left(f(X_s^\varepsilon) - f(X_{s+\varepsilon})\right)|\sigma \dot{W}_s^\varepsilon + m| ds$$
$$- \int_{-\varepsilon}^0 f(X_{s+\varepsilon})|\sigma \dot{W}_s^\varepsilon + m| ds + \int_{t-\varepsilon}^t f(X_{s+\varepsilon})|\sigma \dot{W}_s^\varepsilon + m| ds.$$

For the second integral, we have

$$\left|\int_{-\varepsilon}^0 f(X_{s+\varepsilon})|\sigma \dot{W}_s^\varepsilon + m| ds\right| \leq M(\omega)\hat{C}_{1/4}(\omega)\varepsilon^{1/4},$$

where $M(\omega)$ is given in (21). (Remember that $W_s = 0$ if $s < 0$, but W_s^ε does not necessarily vanishes for $s < 0$.) A similar bound holds for the third integral.

So, in order to obtain (19), we must prove that

$$\hat{Z}_t^{2,\varepsilon} = \int_0^t \left(f(X_s^\varepsilon) - f(X_{s+\varepsilon})\right)|\sigma \dot{W}_s^\varepsilon + m| ds \Rightarrow 0 \tag{23}$$

Denote

$$f_t^\varepsilon = f(X_t^\varepsilon) - f(X_{t+\varepsilon}), \qquad g_t^\varepsilon = |\sigma \dot{W}_t^\varepsilon + m|.$$

Given $S < T$ we compute the second moment:

$$E\left(\hat{Z}_T^{2,\varepsilon} - \hat{Z}_S^{2,\varepsilon}\right)^2 = 2\iint_{S \leq s+2\varepsilon \leq t \leq T} E(f_s^\varepsilon f_t^\varepsilon g_s^\varepsilon g_t^\varepsilon) ds dt \tag{24}$$
$$+ 2\iint_{S \leq s \leq t \leq s+2\varepsilon \leq T} E(f_s^\varepsilon f_t^\varepsilon g_s^\varepsilon g_t^\varepsilon) ds dt = 2(L_1 + L_2).$$

We denote by Δ_t^ε the increment

$$\Delta_t^\varepsilon = X_t^\varepsilon - X_{t+\varepsilon} = \int_{-1}^1 \psi(-w)(X_{t+w\varepsilon} - X_{t+\varepsilon}) dw$$
$$= \sigma \Delta_t^{\varepsilon,W} + \Delta_t^{\varepsilon,S} - \varepsilon m(\alpha + 1)$$

(with the obvious notation). Furthermore

$$E\big(\Delta_t^{\varepsilon,W}\big)^2 = \iint \psi(-u)\psi(-v)E\big((W_{t+\varepsilon u} - W_{t+\varepsilon})(W_{t+\varepsilon v} - W_{t+\varepsilon})\big)dudv$$
$$= \varepsilon \iint \psi(-u)\psi(-v)(1 - u \vee v)dudv.$$

Now we apply Taylor's expansion:

$$f_t^\varepsilon = f(X_{t+\varepsilon} + \Delta_t^\varepsilon) - f(X_{t+\varepsilon}) = f'(X_{t+\varepsilon})\Delta_t^\varepsilon + \frac{1}{2}f''(X_{t+\varepsilon} + \theta\Delta_t^\varepsilon)(\Delta_t^\varepsilon)^2/2$$
$$= f'(X_{t-\varepsilon})\Delta_t^\varepsilon + \big(f'(X_{t+\varepsilon}) - f'(X_{t-\varepsilon})\big)\Delta_t^\varepsilon \tag{25}$$
$$+ \frac{1}{2}f''(X_{t+\varepsilon} + \theta\Delta_t^\varepsilon)(\Delta_t^\varepsilon)^2/2 \tag{26}$$

where $0 < \theta < 1$.

Take now conditional expectations in the integrand corresponding to L_1 in (28):

$$E(f_s^\varepsilon f_t^\varepsilon g_s^\varepsilon g_t^\varepsilon) = E\big(f_s^\varepsilon g_s^\varepsilon E(f_t^\varepsilon g_t^\varepsilon / \mathcal{F}_{t-\varepsilon})\big).$$

Plug the Taylor expansion for f_t^ε into the last expectation and consider each term. First, as $\Delta_t^\varepsilon g_t^\varepsilon$ is independent of $\mathcal{F}_{t-\varepsilon}$,

$$E\big(f'(X_{t-\varepsilon})\Delta_t^\varepsilon g_t^\varepsilon / \mathcal{F}_{t-\varepsilon}\big) = f'(X_{t-\varepsilon})E(\Delta_t^\varepsilon g_t^\varepsilon)$$
$$= f'(X_{t-\varepsilon})\Big[\sigma E(\Delta_t^{\varepsilon,W} g_t^\varepsilon) + E(\Delta_t^{\varepsilon,S})E(g_t^\varepsilon) - \varepsilon m(\alpha+1)E(g_t^\varepsilon)\Big]$$

since S and W are independent processes.

For the first term in brackets, substracting $E\big(\Delta_t^{\varepsilon,W}|\sigma\dot{W}_t^\varepsilon|\big) = 0$, we have

$$\big|E(\Delta_t^{\varepsilon,W} g_t^\varepsilon)\big| = \Big|E\big(\Delta_t^{\varepsilon,W}(|\sigma\dot{W}_t^\varepsilon + m| - |\sigma\dot{W}_t^\varepsilon|)\big)\Big|$$
$$\leq |m|E\big|\Delta_t^{\varepsilon,W}\big| = (const)\varepsilon^{1/2}.$$

In what concerns the second term in brackets,

$$E\big|\Delta_t^{\varepsilon,S}\big| \leq \|\psi\|_\infty E\Big(\sum_{t-\varepsilon < s \leq t+\varepsilon} |\Delta X_s|\Big) = \|\psi\|_\infty 2\varepsilon \int |x|\Pi(dx),$$

so that

$$\big|E(\Delta_t^{\varepsilon,S})E(g_t^\varepsilon)\big| \leq (const)\varepsilon^{1/2},$$

and we obtain:

$$\big|E\big(f'(X_{t-\varepsilon})\Delta_t^\varepsilon g_t^\varepsilon \mid \mathcal{F}_{t-\varepsilon}\big)\big| \leq |f'(X_{t-\varepsilon})| \ \ (const)\ \varepsilon^{1/2}$$
$$\leq \big(\|f''\|_\infty |X_{t-\varepsilon}| + 1\big)(const)\varepsilon^{1/2}.$$

Furthermore

$$
\begin{aligned}
&\Big|E\Big(\big(f'(X_{t+\varepsilon})-f'(X_{t-\varepsilon})\big)\Delta_t^\varepsilon g_t^\varepsilon \mid \mathcal{F}_{t-\varepsilon}\Big)\Big| \\
&\qquad = \Big|E\Big(\big(f''(X_{t-\varepsilon}+\theta'(X_{t+\varepsilon}-X_{t-\varepsilon})\big)(X_{t+\varepsilon}-X_{t-\varepsilon})\Delta_t^\varepsilon g_t^\varepsilon \mid \mathcal{F}_{t-\varepsilon}\Big)\Big| \\
&\qquad\qquad \le \|f''\|_\infty E\Big(\big|\sigma(W_{t+\varepsilon}-W_{t-\varepsilon})+S_{t+\varepsilon}-S_{t-\varepsilon}+2m\varepsilon\big| \times |\Delta_t^\varepsilon| g_t^\varepsilon\Big),
\end{aligned}
$$

where $0<\theta'<1$. A standard computation with normal distributions shows that:

$$E\big(|\Delta_t^{\varepsilon,W}|^2\,[g_t^\varepsilon]^2\,\big) \le (const)$$

So, by Cauchy-Schwarz's inequality we obtain

$$E\Big(\big|\sigma(W_{t+\varepsilon}-W_{t-\varepsilon})+S_{t+\varepsilon}-S_{t-\varepsilon}+2m\varepsilon\big| \times |\Delta_t^{\varepsilon,W}| g_t^\varepsilon\Big) \le (const)\varepsilon^{1/2}.$$

Also

$$E\Big(\big|W_{t+\varepsilon}-W_{t-\varepsilon}\big| \times |\Delta_t^{\varepsilon,S}| g_t^\varepsilon\Big) = E\Big(\big|W_{t+\varepsilon}-W_{t-\varepsilon}\big| g_t^\varepsilon\Big) E|\Delta_t^{\varepsilon,S}| \le (const)\varepsilon.$$

As for the other term

$$
\begin{aligned}
&E\Big(\big|S_{t+\varepsilon}-S_{t-\varepsilon}\big| \times |\Delta_t^{\varepsilon,S}| g_t^\varepsilon\Big) \\
&\qquad\qquad = E\Big(\big|(S_{t+\varepsilon}-S_{t-\varepsilon})\Delta_t^{\varepsilon,S}\big|\Big) E|g_t^\varepsilon| \le (const)\varepsilon^{1/2},
\end{aligned}
$$

because $E|g_t^\varepsilon| \le (const)\varepsilon^{-1/2}$ and

$$
\begin{aligned}
&E\Big(\big|(S_{t+\varepsilon}-S_{t-\varepsilon})\Delta_t^{\varepsilon,S}\big|\Big) \\
&\qquad\qquad = E\Big|\int_{-1}^{1}\psi(-w)(S_{t+\varepsilon}-S_{t-\varepsilon})(S_{t+w\varepsilon}-S_{t+\varepsilon})dw\Big| \le (const)\varepsilon.
\end{aligned}
$$

Let us now consider the result of plugging the last term of (26) into the conditional expectation. We have:

$$
\begin{aligned}
&\Big|E\big(f''(X_{t+\varepsilon}+\theta\Delta_t^\varepsilon)(\Delta_t^\varepsilon)^2 g_t^\varepsilon/\mathcal{F}_{t-\varepsilon}\big)\Big| \le \\
&\qquad (const)\|f''\|_\infty E\big(\sigma^2(\Delta_t^{\varepsilon,W})^2 g_t^\varepsilon + (\Delta_t^{\varepsilon,S})^2 g_t^\varepsilon + \varepsilon^2 g_t^\varepsilon\big) \le (const)\varepsilon^{1/2}, \qquad (27)
\end{aligned}
$$

based on similar computations. Summing up, we obtain (in the integral L_1):

$$E(f_s^\varepsilon g_s^\varepsilon/\mathcal{F}_{t-\varepsilon}) \le (const)\varepsilon^{1/2}.$$

This also shows that

$$E(f_s^\varepsilon g_s^\varepsilon) \le (const)\varepsilon^{1/2},$$

so that

$$L_1 \leq (const)(T-S)^2\varepsilon. \tag{28}$$

On the other hand, let us show that for $s,t \in [0,T]$ and $0 < \varepsilon \leq 1$ the expectation $E\big(f_s^\varepsilon f_t^\varepsilon g_s^\varepsilon g_t^\varepsilon\big)$ is bounded.

Applying Cauchy-Schwarz's inequality, it suffices to prove the boundedness of

$$E\left\{(f_t^\varepsilon g_t^\varepsilon)^2\right\}$$

for $t \in [0,T]$ and $0 < \varepsilon < 1$. Check that

$$\begin{aligned}&E\left\{(f_t^\varepsilon g_t^\varepsilon)^2\right\} \leq (const)\left(\|f''\|_\infty + 1\right)^2 \times \\ &\Big[E\Big\{X_{t-\varepsilon}^2 \left(\Delta_t^\varepsilon g_t^\varepsilon\right)^2 + E\Big\{\left(\Delta_t^\varepsilon\right)^4 \left(g_t^\varepsilon\right)^2\Big\}\Big\} + E\left\{(X_{t+\varepsilon} - X_{t-\varepsilon})^2(\Delta_t^\varepsilon)^2(g_t^\varepsilon)^2\right\}\Big],\end{aligned}$$

and the proof of the boundedness of this expression follows in much a similar way as the one of L_1.

This implies, first, that

$$E\big((\hat{Z}_T^{2,\varepsilon} - \hat{Z}_S^{2,\varepsilon})^2\big) \leq (const)(T-S)^2$$

for $0 \leq S,T \leq T_0$, hence that $\{\hat{Z}_T^{2,\varepsilon} : 0 \leq T \leq T_0\}$ is tight in $\mathcal{C}([0,T_0],\mathbb{R})$ and, second, that

$$E\big((\hat{Z}_T^{2,\varepsilon})^2\big) \leq (const)T^2\varepsilon,$$

so that, for fixed T, $\hat{Z}_T^{2,\varepsilon} \to 0$ $(\varepsilon \to 0)$ in L^2. This proves (23).

Proof of Step 3. Introduce the processes $y^\varepsilon = \{y_t^\varepsilon : t \geq 0\}$ and $Y^\varepsilon = \{Y_t^\varepsilon : t \geq 0\}$ defined by

$$y_t^\varepsilon = C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_{t-\varepsilon}^\varepsilon + m| - \sigma, \qquad Y_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}}\int_0^t y_s^\varepsilon ds, \qquad t \geq 0.$$

Let us prove that

$$Y^\varepsilon \Rightarrow DB, \tag{29}$$

where $B = \{B_t : t \geq 0\}$ is a Wiener process independent of X, and D the constant in (12).

In order to see this, first observe that, since y_t^ε depends on the increments of the process W on the interval $[t-\varepsilon, t+\varepsilon]$, the process y^ε is 2ε-dependent, and as a consequence, the process Y^ε has asymptotically independent increments as $\varepsilon \to 0$. This means that any cluster point in the weak topology for the family of processes $\{Y^\varepsilon : t \geq 0\}$ as $\varepsilon \to 0$ is a process with independent increments. Analogous arguments give that any cluster point has stationary increments. In order to complete the proof of (29), we prove the tightness in the space of continuous functions.

$$
\begin{aligned}
E(Y_t^\varepsilon - Y_s^\varepsilon)^4 &= \frac{1}{\varepsilon^2} E\Big(\int_s^t y_u^\varepsilon du\Big)^4 \\
&= \frac{4!}{\varepsilon^2}\int_s^t du_1 \int_{u_1}^{u_1+2\varepsilon} du_2 \int_{u_2}^t du_3 \int_{u_3}^{u_3+2\varepsilon} du_4 E(y_{u_1}^\varepsilon y_{u_2}^\varepsilon y_{u_3}^\varepsilon y_{u_4}^\varepsilon) \\
&\qquad\qquad \leq 4\times 4!(s-t)^2 E\big(y_u^\varepsilon\big)^4 \leq (const)(t-s)^2,
\end{aligned}
$$

where we have used (i) the 2ε-dependence of $\{\dot W_t^\varepsilon : t\geq 0\}$, (ii) the fact that due to the choice of C_ε we have $E\big(y_t^\varepsilon\big) = 0$, and (iii) the fact that $E\big(y_u^\varepsilon\big)^4$ converges to a finite limit, as $\varepsilon\to 0$. This proves the tightness property (see 12.51 in Billingsley (1968)). As Y^ε is a centered process, in order to conclude (29) it remains to compute the constant D. This constant can be obtained as

$$D^2 = \lim_{\varepsilon\to 0} E\big(Y_1^\varepsilon\big)^2.$$

Now,

$$
\begin{aligned}
E\big(Y_1^\varepsilon\big)^2 &= \frac{1}{\varepsilon}\int_0^1\int_0^1 E\big(y_s^\varepsilon y_t^\varepsilon\big) ds dt = \frac{2}{\varepsilon}\int_0^1 dt \int_t^{(t+\varepsilon)\wedge 1} E\big(y_s^\varepsilon y_t^\varepsilon\big) \\
\sim \frac{2}{\varepsilon}\int_0^{2\varepsilon} E\big(y_1^\varepsilon y_{1+t}^\varepsilon\big) dt &= 2\int_0^2 E\big(y_1^\varepsilon y_{1+\varepsilon u}^\varepsilon\big) dt \to 2\sigma^2 \int_0^2 E\big(g(U_0)g(U_u)\big) du.
\end{aligned}
$$

with U defined in (30), and $g(x)$ defined in (32). The rest of the computation of the constant D is presented in the following result.

Given $\varepsilon > 0$ define the process $U^\varepsilon = \{U_t^\varepsilon : t\geq 0\}$ by

$$U_t^\varepsilon = \frac{\sqrt{\varepsilon}}{\|\psi\|}\dot W_{\varepsilon t-\varepsilon}^\varepsilon \tag{30}$$

For $t\geq 2$, U^ε is a centered Gaussian stationary process with covariance function

$$
\begin{aligned}
r(t) = E(U_2^\varepsilon U_{2+t}^\varepsilon) &= \frac{\varepsilon}{\|\psi\|^2} E(\dot W_\varepsilon^\varepsilon \dot W_{\varepsilon(1+t)}^\varepsilon) \\
&= \frac{1}{\|\psi\|^2}\int \psi(t-u)\psi(-u) du, \qquad (31)
\end{aligned}
$$

(where we used (6)). We conclude that the distribution of U^ε does not depend on ε (excluding the interval $[0,2]$), and introduce the process U as a centered Gaussian stationary process with covariance given by (31), that can be put in place of U^ε for our purposes. Observe that $E(U_t)^2 = 1$.

Lemma 2 *Define*

$$g(x) = \sqrt{\frac{\pi}{2}}|x| - 1, \ (x\in\mathbb{R}). \tag{32}$$

Then

(1) $E\big(g(U_t)\big) = 0$.

(2) $E\big(g(U_0)g(U_t)\big) = r(t)\operatorname{Arsin} r(t) + \sqrt{1-r^2(t)} - 1$.

Proof. As U_0 is a standard Gaussian random variable (1) is direct. In order to see (2), denote by

$$p(x,y,r) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\Big[x^2+y^2-2rxy\Big]\right\},$$

the density of the Gaussian bidimensional vector (U_0, U_t) with $r = r(t)$. If we denote $f(r) = E\big(g(U_0)g(U_t)\big)$, it is not difficult to verify the following formal calculations:

$$\begin{aligned} f''(r) &= \frac{\partial}{\partial r}\iint_{\mathbb{R}^2} g(x)g(y)\frac{\partial}{\partial r}p(x,y,r)dxdy \\ &= \frac{\partial}{\partial r}\iint_{\mathbb{R}^2} g(x)g(y)\frac{\partial^2}{\partial x\partial y}p(x,y,r)dxdy \\ &= \frac{\partial}{\partial r}\iint_{\mathbb{R}^2} g'(x)g'(y)p(x,y,r)dxdy = \frac{\partial}{\partial r}f'(r) \\ &= \iint_{\mathbb{R}^2} g''(x)g''(y)p(x,y,r)dxdy = 2\pi p(0,0,r) = \frac{1}{\sqrt{1-r^2}}. \end{aligned} \tag{33}$$

Here $g''(x) = 2\sqrt{\pi/2}\,\delta_0$ where δ_0 denotes a Dirac delta function at the origin, we twice use $\frac{\partial}{\partial r}p(x,y,r) = \frac{\partial^2}{\partial x\partial y}p(x,y,r)$, and twice integrate by parts. When $r = 0$ the random variables U_0 and U_t are independent. This gives $f(0) = E\big(g(U_0)g(U_t)\big) = E\big(g(U_0)\big)^2 = 0$, by (1); and, by the intermediate step (33) we also have $f'(0) = E\big(g'(U_0)g'(U_t)\big) = E\big(g'(U_0)\big)^2 = 0$. Finally, integrating twice we get

$$E\big(g(U_0)g(U_t)\big) = r(t)\operatorname{Arsin} r(t) + \sqrt{1-r^2(t)} - 1,$$

concluding the proof of the Lemma.

We now claim that

$$(Y^\varepsilon, W) \Rightarrow (DB, W) \tag{34}$$

where (B, W) is a pair of standard independent Wiener processes.

For this, see first that

$$\begin{aligned} &E\Big(\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}^\varepsilon_{t-\varepsilon} + m| - \sigma\big)W_s\Big) \\ &\qquad\qquad = E\Big(\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}^\varepsilon_{t-\varepsilon} + m| - \sigma\big)\big(W_{t\wedge s} - W_{(t-\varepsilon)\wedge s}\big)\Big). \end{aligned}$$

Now

$$\begin{aligned}
\left|E(Y_t^\varepsilon W_s)\right| &= \Big|\frac{1}{D\sqrt{\varepsilon}}\int_0^t E\Big(\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_{r-\varepsilon}^\varepsilon+m|-\sigma\big)W_s\Big)dr\Big| \\
&= \Big|\frac{1}{D\sqrt{\varepsilon}}\int_{(s-\varepsilon)\wedge t}^{s\wedge t} E\Big(\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_{r}^\varepsilon+m|-\sigma\big)W_s\Big)dr\Big| \\
&\leq \frac{\varepsilon}{D\sqrt{\varepsilon}}\Big(E\big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_{t-\varepsilon}^\varepsilon+m|-\sigma\big)^2 E(W_t-W_{t-\varepsilon})^2\Big)^{1/2} \\
&= (const)\varepsilon.
\end{aligned}$$

This means that $E(Y_t^\varepsilon W_s) \to 0$ $(\varepsilon \to 0)$, and, as it is direct to obtain that $\{Y_t^\varepsilon W_s\}_{\varepsilon>0}$ is uniformly integrable, we obtain (34). As a consequence, since the jump part is independent from the continous part in our Lévy process, we have the weak convergence

$$(Y^\varepsilon, W, S) \Rightarrow (DB, W, S)$$

where B is independent of X.

Let us finally see (20). Observe that for each $\varepsilon > 0$, *a.s.* the process Y^ε has locally finite variation. Applying Ito's formula:

$$\begin{aligned}
\frac{1}{\sqrt{\varepsilon}}\int_0^T \big(C_\varepsilon\sqrt{\varepsilon}|\sigma\dot{W}_t^\varepsilon+m|-\sigma\big)f(X_t)dt &= \int_0^T f(X_t)dY_t^\varepsilon \\
&= f(X_T)Y_T^\varepsilon - \int_0^T Y_t^\varepsilon df(X_t), \qquad (35)
\end{aligned}$$

where using the hypothesis that f is C^2 it follows that $\{f(X_t)\}$ is a semimartingale. The process (Y^ε, X) is adapted, and weakly converges to (B, X).

As the integrator in the right hand member of (35) is fixed one can verify that the hypotheses of Theorem 2.2 in Kurtz and Protter (1991, see Remark 2.5) hold true, thus obtaining

$$f(X_T)Y_T^\varepsilon - \int_0^T Y_t^\varepsilon df(X_t) \Rightarrow f(X_T)B_T - \int_0^T B_t df(X_t)$$

Now, we apply Ito's formula, taking into account that the quadratic covariation $[X, B] = 0$ and we get

$$f(X_T)B_T - \int_0^T B_t df(X_t) = \int_0^T f(X_t)dB_t,$$

completing the proof of (20).

To finish, we state and prove the proposition announced in Remark 3 after the statement of Theorem 1.

Proposition 3 *Assume that $X = \{X_t : t \geq 0\}$ and f satisfy the hypothesis of Theorem 1. Then, for the processes defined in* (15) *and* (14), *for each $t \geq 0$, almost surely*

$$\mathcal{L}_\varepsilon(f,t) \to \mathcal{L}_0(f,t),$$

as $\varepsilon \to 0$.

Proof. On account of (22) it suffices to show that for fixed $a > 0$, almost surely

$$C_0 \int_0^t f(X_s^{a,\varepsilon}) |\dot{S}_s^{a,\varepsilon}| ds \to \mathcal{L}_0^a(f,t), \tag{36}$$

as $\varepsilon \to 0$, where $\mathcal{L}_0^a(f,t)$ is obtained from (15) when the process X is replaced by X^a.

Using the same notations as in the last part of Proof of Step 1 in Theorem 1, we can write $a.s.$, for ε sufficiently small

$$\int_0^t f(X_s^{a,\varepsilon}) |\dot{S}_s^{a,\varepsilon}| ds = \sum_{n=1}^{N_t} \int_{\tau_n - \varepsilon}^{\tau_n + \varepsilon} f(X_s^{a,\varepsilon}) |\dot{S}_s^{a,\varepsilon}| ds. \tag{37}$$

Observe that for $\tau_n - \varepsilon < s < \tau_n + \varepsilon$ one has

$$\dot{S}_s^{a,\varepsilon} = \frac{1}{\varepsilon} \psi\Big(\frac{s - \tau_n}{\varepsilon}\Big) \Delta X_{\tau_n}$$

so that

$$\int_{\tau_n - \varepsilon}^{\tau_n + \varepsilon} f(X_s^{a,\varepsilon}) |\dot{S}_s^{a,\varepsilon}| ds = \frac{1}{\varepsilon} \int_{\tau_n - \varepsilon}^{\tau_n + \varepsilon} f(X_s^{a,\varepsilon}) \psi\Big(\frac{s - \tau_n}{\varepsilon}\Big) |\Delta X_{\tau_n}| \ ds.$$

Making the change of variables $z = (s - \tau_n)/\varepsilon$ in each integral, we obtain

$$\int_0^t f(X_s^{a,\varepsilon}) |\dot{S}_s^{a,\varepsilon}| ds = \sum_{n=1}^{N_t} |\Delta X_{\tau_n}| \int_{-1}^{1} f\big(X_{\tau_n + \varepsilon z}^{a,\varepsilon}\big) \psi(z) dz.$$

To compute the limit as $\varepsilon \to 0$ in the right hand member of the last equality, use that

$$X_{\tau_n + \varepsilon z}^{a,\varepsilon} \to X_{\tau_n}^a \int_{-1}^{z} \psi(w) dw + X_{\tau_n -}^a \int_z^1 \psi(w) dw.$$

This proves the statement.

References

[1] Azaïs, J-M. and Wschebor, M. Almost sure oscillation of certain random processes, *Bernoulli* **2** (3), 1996, 257–270.

[2] Azaïs, J-M. and Wschebor, M. Oscillation presque sûre de martingales continues, *Séminaire de Probabilités XXXI, Lecture Notes in Mathematics 1655,* Springer, 1997, 69–76.

[3] Bertoin, J. *Lévy Processes.* Cambridge University Press, Cambridge, 1966.

[4] Berzin, C. and León, J.R. Weak convergence of the integrated number of level crossings to the local time of the Wiener process. *Comptes R. Acad. Sc. Paris*, Sér. I **319**, 1994, 1311–1316.

[5] Billingsley, P. *Convergence of probability measures.* Wiley, New York, London, Sydney, 1968.

[6] Borodin, A.N. and Ibragimov, I.A. Limit theorems for functionals of random walks. (Predel'nye teoremy dlya funktsionalov ot sluchajnykh bluzhdanij.) (Russian) Trudy Matematicheskogo Instituta Imeni V. A. Steklova. 195. Nauka, Sankt-Peterburg, 1994. (See also: Borodin, A.N. and Ibragimov, I.A. Limit theorems for functionals of random walks. (English) Proceedings of the Steklov Institute of Mathematics. 195. American Mathematical Society (AMS), Providence, RI, 1995.)

[7] Dacunha-Castelle, D. and Florens-Zmirou, D. Estimation of the coefficient of a diffusion from discrete observations, *Stochastics* **19**, 1986, 263–284.

[8] Florens-Zmirou, D. On estimating the difussion coefficient from discrete observations, *J. Appl. Prob.* **30**, 1993, 790–804.

[9] Génon-Catalot, V. and Jacod, J. On the estimation of the diffusion coefficient for multidimensional diffusion processes, *Ann.Inst. H.Poincaré, Probab. Stat.* **29**, 1993, 119–151.

[10] Jacod, J. Rates of convergence to the local time of a diffusion. *Ann.Inst. H.Poincaré, Probab. Stat.* **34**, 1998, 505–544.

[11] Jacod, J. Non-parametric kernel estimation of the diffusion coefficient of a diffusion. *Scand. J. Statist.* **27** (1), 2000, 83–96.

[12] Jacod, J. and Shiryaev, A.N. *Limit Theorems for Stochastic Processes.* Springer, Berlin, Heidelberg, 1987.

[13] Kurtz, T. G. and Protter, P. Weak limit theorems for stochastic integrals and stochastic differential equations. *Ann. Probab.* **19** (3), 1991, 1035–1070.

[14] Perera, G. and Wschebor, M. Crossings and occupation measures for a class of semimartingales, *Ann. Probab.* **26** (1), 1998, 253–266.

[15] Perera, G. and Wschebor, M. Inference on the Variance and Smoothing of the Paths of Diffusions, *Ann.Inst. H.Poincaré, Probab. Stat.* **38** (6), 2002, 1009–1022.

[16] Sato, Ken-iti. *Lévy processes and infinitely divisible distributions.* Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999.

[17] Skorokhod, A. V. *Random processes with independent increments.* Kluwer Academic Publishers, Dordrecht, 1991.