Perpetual Options for Lévy Processes in the Bachelier Model

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Abstract

Solution to the optimal stopping problem

$$V(x) = \sup_{\tau} \mathbf{E} e^{-\delta\tau} g(x + X_\tau)$$

is given, where $X = \{X_t\}_{t\geq 0}$ is a Lévy process, τ is an arbitrary stopping time, $\delta \geq 0$ is a discount rate, and the reward function g takes the form $g_c(x) = (x-K)^+$ or $g_p(x) = (K-x)^+$ Results, interpreted as option prices of perpetual options in Bachelier's model are expressed in terms of the distribution of the overall supremum in case $g = g_c$ and overall infimum in case $g = g_p$ of the process X killed at rate δ. Closed form solutions are obtained under mixed exponentially distributed positive jumps with arbitrary negative jumps for g_c, and under arbitrary positive jumps and mixed exponentially distributed negative jumps for g_p. In case $g = g_c$ a prophet inequality comparing prices of perpetual look-back call options and perpetual call options is obtained.

Keywords and Phrases: Optimal stopping, Lévy process, mixtures of exponential distributions, perpetual American options, prophet inequalities

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1 Introduction and general results

1.1 Lévy processes

Let $X = \{X_t\}_{t\geq 0}$ be a real valued stochastic process defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ that satisfy the usual conditions. Assume that X

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is càdlàg, adapted, $X_0 = 0$, and for $0 \le s < t$ the random variable $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$ with a distribution that only depends on the difference $t - s$. X is a process with stationary independent increments (PIIS), or a Lévy process. We will also consider the process $\hat{X} = \{\hat{X}_t\}_{t \ge 0} = \{-X_t\}_{t \ge 0}$, called the *dual process*.

If $q \in \mathbf{R}$, Lévy-Khinchine formula states

$$(1) \qquad \mathbf{E}e^{iqX_t} = \exp\Big\{t\Big[ibq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbf{R}} (e^{iqy} - 1 - iqy\mathbf{1}_{\{|y|<1\}})\Pi(dy)\Big]\Big\},$$

where b and $\sigma \ge 0$ are real constants, and Π is a positive measure on $\mathbf{R} - \{0\}$ such that $\int (1 \wedge y^2)\Pi(dy) < +\infty$, called the Lévy measure. These parameters completely determine the law of the process. We always assume that the process does not degenerate, i.e. $\sigma \ne 0$ or $\Pi \ne 0$.

In order to use indistinctly Fourier or Laplace transforms, consider the set

$$(2) \qquad \mathbf{C_0} = \{c \in \mathbf{R}: \int_{\{|y|>1\}} e^{cy}\Pi(dy) < \infty\}.$$

$\mathbf{C_0}$ is a convex set that contains the origin, and consists of all $c \in \mathbf{R}$ such that $\mathbf{E}e^{cX_t} < \infty$ for some $t > 0$. Furthermore, if $z \in \mathbf{C}$ and $\Re(z) \in \mathbf{C_0}$ we can define the characteristic exponent

$$(3) \qquad \psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} (e^{zy} - 1 - izy\mathbf{1}_{\{|y|<1\}})\Pi(dy)$$

having $\mathbf{E}|e^{zX_t}| < \infty$ for all $t \ge 0$, and

$$\mathbf{E}e^{zX_t} = e^{t\psi(z)}.$$

Observe that if $z = iq$ the preceding formula gives (1). See Sato (1999) for details.

When considering X as a semimartingale, denote by $(bt, \sigma^2 t, \Pi(dy)dt)$, or for short (b, σ^2, Π) the triplet of predictable characteristics of X (see II.4.19 in Jacod and Shiryaev (1987)). If we denote by $(\hat{b}, \hat{\sigma^2}, \hat{\Pi})$ the triplet of $\hat{X}$, the dual process, we have $\hat{b} = -b$, $\hat{\sigma} = \sigma$, and $\hat{\Pi}(dx) = \Pi(-dx)$. Given X and $\delta \ge 0$ denote

$$(4) \qquad M = \sup_{0 \le t < \tau(\delta)} X_t \quad \text{and} \quad I = \inf_{0 \le t < \tau(\delta)} X_t,$$

where $\tau(\delta)$ is an exponential random variable with parameter $\delta > 0$, independent of X, and $\tau(0) = \infty$. M and I will be called the *supremum* and *infimum* of X in both cases $\delta = 0$ and $\delta > 0$. τ is a stopping time relative to $\mathbf{F}$ if

$$\tau: \Omega \to [0, +\infty] \text{ and } \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0.$$

Denote by $\mathcal{M}$ the class of all stopping times relative to $\mathbf{F}$.

For general reference on the subject see Jacod and Shiryaev (1987), Skorokhod (1991), Bertoin (1996), or Sato (1999).

1.2 General results on optimal stopping

Consider the following problem: given a Borel function $g\colon \mathbf{R} \to \mathbf{R}$, the reward function, a process X as above, and a discount rate $\delta \geq 0$, find a real function V and a stopping time τ^* such that

$$V(x) = \sup_{\tau \in \mathcal{M}} \mathbf{E} e^{-\delta\tau} g(x + X_\tau) = \mathbf{E} e^{-\delta\tau^*} g(x + X_{\tau^*}). \tag{5}$$

V is called the cost function, and τ^*, the stopping time that realizes the supremum, the optimal stopping time. We assume that $e^{-\delta\tau} g(x + X_\tau)\mathbf{1}_{\{\tau=\infty\}} = \limsup_{t\to\infty} e^{-\delta t} g(x + X_t)$.

In the present paper solution to the problem (5) is given, when considering reward functions

$$g_c(x) = (x - K)^+ \quad \text{and} \quad g_p(x) = (K - x)^+ \tag{6}$$

(with $x^+ = \max(x, 0)$) in terms of the distribution of the random variables M and I in (4). c and p stand for call and put options, and we interpret our results as the pricing of perpetual call and put options in the Bachelier (1900) model. In Section 2 closed form solutions are obtained for the call (respectively put) options, assuming that positive (resp. negative) jumps of X are distributed as a mixture of exponentials, and negative jumps (resp. positive) behave arbitrarily. In Section 3 we present a prophet inequality comparing the cost of a perpetual call and the expectation of the overall maximum of the process, i.e. the price of a perpetual look-back option. Section 4 contains the proofs, and Section 5 a conclusion. The presented results where partially announced in (Mordecki, (2000a)) were other examples are considered, and are complemented by the ones contained in the paper "Optimal Stopping and Perpetual Options for Lévy processes", (Mordecki, (2000b)), where similar results for reward functions $(e^x - K)^+$ and $(K - e^x)^+$ are obtained, and applications to finance are discussed in detail.

Theorem 1 *Let X be a Lévy process and $\delta \geq 0$. Denote M and I as in (4)*

(a) *If $\mathbf{E}M < \infty$, the solution to the problem (5) with reward $g_c(x) = (x-K)^+$ has cost function*

$$V_c(x) = \mathbf{E}(x + M - \mathbf{E}M - K)^+ \tag{7}$$

and optimal stopping time

$$\tau_c^* = \inf\{t \geq 0\colon x + X_t \geq K + \mathbf{E}M\}.$$

(b) *If $\mathbf{E}I > -\infty$, the solution to the problem (5) with reward $g_p(x) = (K - x)^+$ has cost function*

$$V_p(x) = \mathbf{E}(K + \mathbf{E}I - x - I)^+$$

and optimal stopping time

$$\tau_p^* = \inf\{t \geq 0\colon x + X_t \leq K + \mathbf{E}I\}.$$

Remark. As can bee seen in the proof, (b) follows easily from (a), considering the dual process. For this reason consider mainly the call case (a).

Theorem 1 essentially says, that although in general, the optimal stopping problem for a Markov process leds to a free boundary problem (Theorem III§8.15 in Shiryaev (1978)), in the case considered, this problem reduces to finding the distribution of M, a problem that has the advantage of having fixed boundary conditions. The conclusion is, that for a Lévy process, the optimal stopping problem with reward x^+ can be explicitly solved once the distribution of M is known. The case of a random walk with $\delta = 0$ was considered by Darling et al. (1972).

2 Closed solutions for mixed-exponential jumps

We specify now the Lévy measure in (1) in order to obtain closed solutions for the optimal stopping problem. Given $a = (a_1, \dots, a_n)$ with $\sum_{k=1}^n a_k = 1$, $a_k > 0$ for $k = 1, \dots, n$, and $\alpha = (\alpha_1, \dots, \alpha_n)$, with $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$, denote by

$$h_n(y; a, \alpha) = \sum_{k=1}^{n} a_k \alpha_k e^{-\alpha_k y}, \qquad y \geq 0,$$

the density of a mixture of n exponential random variables with parameters $\alpha_1, \dots, \alpha_n$ and mixture coefficients $a_1, \dots, a_n$.

Consider a Lévy process X with triplet (b, σ^2, Π), and Lévy measure given by

$$\Pi(y) = \begin{cases} \lambda h_n(y; a, \alpha) dy & y > 0, \\ \pi(dy) & y < 0, \end{cases} \tag{8}$$

where $\pi(dy)$ is an arbitrary Lévy measure with support on $(-\infty, 0)$, and a and α are as before. X has mixed-exponentially distributed positive jumps. Its dual process $\hat{X}$ has mixed exponentially distributed negative jumps and arbitrary positive jumps.

In this case, the set $\mathbf{C_0}$ in (2) contains the interval $[0, \alpha_1)$ and the characteristic exponent

$$\psi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{0} (e^{zy} - 1 - zy\mathbf{1}_{\{-1<y<0\}})\pi(dy)$$
$$+\lambda \int_{0}^{\infty} (e^{zy} - 1 - zy\mathbf{1}_{\{0<y<1\}}) h_n(y; a, \alpha) dy$$

has an analytical continuation to the set $\{z \in \mathbf{C} \colon \Re(z) \geq 0, z \neq \alpha_k, k = 1, \dots, n\}$ given by

$$\Psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{0} (e^{zy} - 1 - zy\mathbf{1}_{\{-1<y<0\}})\pi(dy) +$$

$$+\lambda \sum_{k=1}^{n} a_k \frac{z}{\alpha_k - z}, \tag{9}$$

with

$$a = b - \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k} \big[1 - (1+\alpha_k)e^{-\alpha_k}\big].$$

The distribution of the random variable M in (4) for (b, σ^2, Π) as above with $\sigma > 0$, under either the condition $\delta > 0$, or

$$\psi'(0+) = \lim_{p \to 0+} \frac{\psi(p)}{p} = a + \int_{-\infty}^{-1} y\pi(dy) + \lambda \sum_{k=1}^{n} \frac{a_k}{\alpha_k} < 0, \tag{10}$$

if $\delta = 0$, where the integral can take the value $-\infty$, is also a mixture of exponentials, and has density

$$f_M(y) = h_{n+1}(y; A, p) \tag{11}$$

where $p = (p_1, \ldots, p_{n+1})$, with

$$0 < p_1 < \alpha_1 < p_2 < \ldots < \alpha_n < p_{n+1} \tag{12}$$

are the real and positive roots of $\Psi(p) = \delta$ and $A = (A_1, \ldots, A_{n+1})$ are given by

$$A_j = \frac{\prod_{k=1}^{n} (\frac{p_j}{\alpha_k} - 1)}{\prod_{k=1, k \neq j}^{n+1} (\frac{p_j}{p_k} - 1)} \qquad j = 1, \ldots, n+1, \tag{13}$$

as follows form Theorem 1.1 in Mordecki (1999). This gives the following result.

Theorem 2 *Let X be a Lévy process with $\sigma > 0$, Lévy measure given by (8), and $\delta \geq 0$. Under the condition $\delta > 0$ or (10) when $\delta = 0$, we have*

$$\mathbf{E}M = \sum_{k=1}^{n+1} \frac{A_k}{p_k},$$

with p_k, $k = 1, \ldots, n+1$, the positive roots of $\Psi(p) = \delta$, that satisfy (12), and A_k given in (13). Furthermore, the optimal stopping problem for X with reward $g_c(x) = (x-K)^+$ has cost function

$$V_c(x) = \begin{cases} \sum_{k=1}^{n+1} \frac{A_k}{p_k} e^{-p_k(x^* - x)} & x \leq x^*, \\ \\ x - K & x > x^*, \end{cases}$$

with $x^* = K + \mathbf{E}M$, and optimal stopping time

$$\tau^* = \inf\{t \geq 0 \colon x + X_t \geq x^*\}.$$

Remarks. (1) In order to obtain closed solution for put options, we must specify the negative jumps of the Lévy process. Given X with Lévy measure in (8) the dual process $\hat{X}$ has negative mixed exponentially distributed negative jumps, and it is direct (see proof of (b) for Theorem 1), under the same assumptions, denoting $x_p^* = K + \mathbf{E}\hat{I}$

$$\sup_\tau \mathbf{E}(K - x - \hat{X}_\tau)^+ = \begin{cases} K - x & x \leq x_p^*, \\ \sum_{k=1}^{n+1} \frac{A_k}{p_k} e^{-p_k(x_p^* - x)} & x \leq x_p^*, \end{cases}$$

and the optimal stopping time is

$$\tau_p^* = \inf\{t \geq 0 \colon x + X_t \leq x_p^*\}.$$

(2) The results presented extend those in Taylor (1968) for the Wiener process, see also Shiryaev (1978), Mordecki (1998) for a compound Poisson process with exponential jumps, and Mordecki (1997) for a diffusion with exponential jumps. All these cases considered only $\delta = 0$.

3 A prophet inequality for Lévy processes

Our final result states a prophet inequality for the class of Lévy processes. Recall that if $K = 0$ and $g = g_g$ the cost function is given by

$$V(x) = \sup_\tau \mathbf{E} e^{-\delta\tau}(x + X_\tau)^+.$$

Theorem 3 *Let X be a Lévy process with $\mathbf{E}M < \infty$, and $x \geq 0$. Then, the following inequality holds:*

$$V(x) \leq \mathbf{E}(x + M) \leq eV(x).$$

This result generalizes the one obtained in Darling et al. (1972) for random walks, $x = 0$ and $\delta = 0$. It is called a prophet inequality in the sense that compares the expected reward of a gambler that has complete foresight, or a *prophet* that can choose the moment at which the maximum is attained, obtaining $x + M$, with the expectation of the best possible strategy in the class $\mathcal{M}$. For references on analogous results for sequences of random variables see Hill and Kertz (1990). For related results on prophet inequalities in finance see Kertz (1999).

4 Proofs

The proof of Theorem 1 is based on a result on random walks of independent interest, a slightly generalization of Darling et al. (1972).

Introduce a sequence $X, X_1, X_2, \dots$ of i.i.d. random variables. As usual, partial sums are denoted by

$$S_0 = 0, \quad S_n = \sum_{k=1}^{n} X_k, \quad n = 1, 2, \dots,$$

and τ denotes a stopping time with respect to the filtration generated by the random walk $S = \{S_n\}_{n \in \mathbf{N}}$. The optimal stopping problem for this random walk with reward g and discount rate $\alpha \in (0, 1]$ consists in finding a real function V and a stopping time τ^* such that

$$V(x) = \sup_{\tau} \mathbf{E}\alpha^{\tau} g(x + S_{\tau}) = \mathbf{E}\alpha^{\tau^*} g(x + S_{\tau^*}). \tag{14}$$

Let $\tau(\alpha)$ be a geometric random variable with $\mathbf{P}(\tau(\alpha) > k) = \alpha^k$, for $k = 1, 2, \dots$, independent of S with $\alpha \in (0, 1)$, and denote $\tau(1) = \infty$. The supremum of the random walk killed at rate α is

$$W = \inf_{0 \le n < \tau(\alpha)} S_n. \tag{15}$$

Proposition 1 *Consider a random walk S, $\tau(\alpha)$ and W as above. Assume $\mathbf{E}W < \infty$ and denote*

$$C(x) = \mathbf{E}(x + W - \mathbf{E}W)^+, \qquad \sigma^* = \inf\{n \ge 0 \colon x + S_n \ge \mathbf{E}W\}.$$

Then we have

- *(i)* $C(x) \ge x^+$,
- *(ii)* $C(x) \ge \alpha \mathbf{E} C(x + X)$,
- *(iii)* $C(x) = \mathbf{E}\alpha^{\sigma^*}(x + S_{\sigma^*})^+$.

and in consequence,

$$\sup_{\tau} \mathbf{E}\alpha^{\tau}(x + S_{\tau})^+ = \mathbf{E}(x + W - \mathbf{E}W)^+ = \mathbf{E}\alpha^{\sigma^*}(x + S_{\sigma^*})^+. \tag{16}$$

Remark. For $\alpha = 1$ this result is included in Darling et al. (1972).

Proof. (i) follows by Jensen's inequality. To see (ii) introduce the random variable J, independent of S and $\tau(\alpha)$, such that $\mathbf{P}(J = 0) = 1 - \alpha$, $\mathbf{P}(J = 1) = \alpha$. We claim

$$W =^d J(X + W)^+.$$

In fact,

$$\mathbf{E}e^{zJ(X+W)^+} = 1 - \alpha + \alpha \mathbf{E}e^{z(X+W)^+}$$

$$= \mathbf{E}e^{zW}\mathbf{1}_{\{\tau(\alpha)=1\}} + \sum_{k=1}^{\infty}\alpha\mathbf{E}(e^{z\sup_{0\le n<k} S_n}\mathbf{1}_{\{\tau(\alpha)=k\}}) = \mathbf{E}e^{zW}.$$

Now

$$C(x) = \mathbf{E}(x + J(X+W)^+ - \mathbf{E}W)^+ \ge \alpha\mathbf{E}(x + X + W - \mathbf{E}W)^+ = \alpha\mathbf{E}C(x+X).$$

In order to see (iii), observe that $\{\sigma^* < \tau(\alpha)\} = \{x + W \ge \mathbf{E}W\}$, and

$$\mathbf{E}(x+W)\mathbf{1}_{\{x+W\ge\mathbf{E}W\}} = \mathbf{E}(x+S_{\sigma^*})\mathbf{1}_{\{x+W\ge\mathbf{E}W\}} + \mathbf{E}(W - S_{\sigma^*})\mathbf{1}_{\{x+W\ge\mathbf{E}W\}} =$$

$$E\alpha^{\sigma^*}(x+S_{\sigma^*})^+ + \mathbf{E}M\mathbf{P}(\sigma^* < \tau(\alpha))$$

and from this

$$E\alpha^{\sigma^*}(x+S_{\sigma^*})^+ = \mathbf{E}(x+W-\mathbf{E}W)^+.$$

This proves (i), (ii) and (iii). The final conclusion (16) follows from Darling et al. (1972).

Proof of Theorem 1. First observe that (b) follows from (a).

$$\sup_\tau \mathbf{E}(K - x - X_\tau)^+ = \sup_\tau \mathbf{E}(K - x + \hat{X}_\tau)^+$$

$$= \mathbf{E}(K - x + \hat{M} - E\hat{M})^+ = \mathbf{E}(K + \mathbf{E}I - x - I)^+,$$

where $\hat{X}$ is the dual process, and $\hat{M} = \sup_{0\le t<\tau(\delta)}\hat{X}_t = -I$.

Let us now see (a). Without loss of generality we take $K=0$, so $g_c(x) = x^+$. First we verify that (7) holds. In order to see this, on one side we recall the fact that the cost function of the optimal stopping problem considered is the minimal δ-excessive majorant function of the reward g_c, and satisfies

$$V_c(x) = \lim_{n\to\infty}\lim_{N\to\infty} Q_n^N g_c(x) \tag{17}$$

where the operator Q_n is given by

$$Q_n g_c(x) = \max\{g_c(x), e^{-\delta 2^{-n}}\mathbf{E}g_c(x + X_{2^{-n}})\},$$

and Q_n^N is the N^{th} iteration of Q_n. (see Shiryaev (1978) and VIII.2.4 in Shiryaev (1999)). Fix now n. The limit $\lim_{N\to\infty} Q_n^N g_c(x)$ is the cost function for the optimal stopping problem for the random walk $X^n = \{X_{k2^{-n}} : k \in \mathbf{N}\}$ discounted at rate $\alpha_n = e^{-\delta 2^{-n}}$ with reward g_c, (Theorem II.23 in Shiryaev (1978)).

Take now $\delta \ge 0$ and $\tau(\delta)$ as in (4). For fixed n, and $\delta > 0$

$$\tau_n = \frac{[2^n\tau(\delta)] + 1}{2^n}$$

(with $[x]$ the integer part of x) is a geometric random variable taking values on the set $\{\frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{k}{2^n}, \dots\}$ with parameter $\alpha_n = e^{-\delta 2^{-n}}$, i.e. $\mathbf{P}(\tau_n > k2^{-n}) =$

$(e^{-\delta/2^n})^k$. If $\delta = 0$ put $\tau_n = \infty$. On the other side Proposition 1 give the solution to this discrete time optimal stopping problem. In conclusion

$$\lim_{N\to\infty} Q_n^N g_c(x) = \mathbf{E}(x + M^n - \mathbf{E}M^n)^+ \tag{18}$$

where the r.h.s. is the formula in Proposition 1, and

$$M^n = \sup\{X_{k2^{-n}} : 0 \le k < \tau_n\},$$

$M^n \le M$ and $\mathbf{E}M^n < \infty$. Now we make $n \to \infty$. First

$$M_n = \sup_{0\le k<\tau_n} X_{k2^{-n}} = \sup_{0\le k<\tau(\delta)} X_{k2^{-n}} \to M = \sup_{0\le t<\tau(\delta)} X_t \qquad \mathbf{P}\text{-a.s.} \tag{19}$$

because $\tau_n > \tau(\delta)$, $\tau_n \to \tau(\delta)$ and X is càdlàg. The result then follows taking limits in (18). The l.h.s. has limit $V_c(x)$ by (17) and

$$\mathbf{E}(x + M^n - \mathbf{E}M^n)^+ \to \mathbf{E}(x + M - \mathbf{E}M)^+$$

in view of (19) and dominated convergence since $\mathbf{E}M < \infty$. The fact that τ_c^* is the optimal stopping rule follows exactly as in the proof of (iii) in Proposition 1.

Proof of Theorem 2. We simply plug formula (11) in Theorem 1.

$$\mathbf{E}M = \int_0^\infty y h_{n+1}(y, A, p)dy = \sum_{k=1}^{n+1} \frac{A_k}{p_k}.$$

Furthermore, if $x \ge x^*$, $\mathbf{E}(x + M - \mathbf{E}M - K)^+ = x - K$. If $x < x^*$

$$V_c(x) = \mathbf{E}(x + M - \mathbf{E}M - K)^+ = \int_{x^*-x}^\infty (x + y - x^*)h_{n+1}(y, A, p)dy$$

$$= \sum_{k=1}^{n+1} \frac{A_k}{p_k} e^{-p_k(x^*-x)}.$$

concluding the proof.

Proof of Theorem 3. Let $\delta \ge 0$ and X be a Lévy process with $\mathbf{E}M$ finite. Assume $x \ge 0$. As for any $\tau \in \mathcal{M}$ inequality $(x+X_\tau)^+\mathbf{1}_{\{\tau<\tau(\delta)\}} \le x+M$, holds and $\mathbf{E}e^{-\delta\tau}(x + X_\tau)^+ = \mathbf{E}(x + X_\tau)^+\mathbf{1}_{\{\tau<\tau(\delta)\}}$, we have

$$V(x) = \sup_\tau \mathbf{E}e^{-\delta\tau}(x + X_\tau)^+ \le \mathbf{E}(x + M).$$

We have to prove now, that for $x \ge 0$,

$$\mathbf{E}(x + M) \le eV(x).$$

As $V(x) = \mathbf{E}(x + M - \mathbf{E}M)^+$, this amounts to say, that for $x \geq 0$

$$q(x) = \frac{\mathbf{E}(x + M)}{\mathbf{E}(x + M - \mathbf{E}M)^+} \leq e. \tag{20}$$

If $x \geq \mathbf{E}M$, $\mathbf{E}(x + M - \mathbf{E}M)^+ = x$, and $q(x) \leq 2 < e$.

Denoting $B(x) = \mathbf{P}(M \geq x)$, we claim

$$B(x + y) \geq B(x)B(y). \tag{21}$$

To see this consider, for $z \geq 0$

$$\tau(z) = \inf\{t \geq 0 \colon X_t \geq z\}.$$

Based on $X_{\tau(x)} \geq x$ on the set $\tau(x) < \tau(\delta)$, and the lack of memory of $\tau(\delta)$

$$\begin{aligned} B(x+y) = \mathbf{P}(\sup_{0 \leq t < \tau(\delta)} X_t \geq x + y) &= \mathbf{P}(\sup_{0 \leq t < \tau(\delta)} X_t \geq x + y \mid \tau(x) < \tau(\delta))B(x) \\ &\leq \mathbf{P}(\sup_{0 \leq t < \tau(\delta)} X_t \geq X_{\tau(x)} + y \mid \tau(x) < \tau(\delta))B(x) \\ = \mathbf{P}(\sup_{\tau(x) \leq t < \tau(\delta)} X_t - X_{\tau(x)} &\geq y \mid \tau(x) < \tau(\delta))B(x) = B(y)B(x), \end{aligned}$$

and (21) is proved.

Consider now the auxiliary function $f = f(x)$, given by

$$f(x) = e^{\frac{x}{x^*}} \int_x^{+\infty} B(y)dy, \qquad x \geq 0,$$

where $x^* = \mathbf{E}M = \int_0^{+\infty} B(y)dy$. Differentiation gives

$$f'(x) = e^{\frac{x}{x^*}} \Big(\frac{1}{x^*} \int_x^{+\infty} B(x)dx - B(x) \Big)$$

In order to conclude that $f'(x) \geq 0$ for all $x \geq 0$, we integrate (21)

$$\int_x^{+\infty} B(y)dy = \int_0^{+\infty} B(x + y)dy \geq B(x) \int_0^{+\infty} B(y)dy = B(x)x^*$$

and deduce that $f(x)$ is non decreasing. As $f(0) = x^*$ we conclude

$$f(x) = e^{\frac{x}{x^*}} \int_x^{+\infty} B(y)dy \geq x^*. \tag{22}$$

Now, applying (22) with $0 \leq x \leq x^*$

$$q(x) = \frac{x + x^*}{\int_{x^*-x}^{+\infty} B(y)dy} \leq \frac{x + x^*}{x^*} \exp\Big(\frac{x^* - x}{x^*}\Big) = c(x).$$

It is direct to see that the function $c(x)$ is decreasing, and that $c(0) = e$, concluding the proof.

5 Conclusion

The presented paper shows how to reduce a discounted optimal stopping problem for a Lévy process X, with rewards $(x-K)^+$ and $(K-x)^+$ to the computation of the distribution of the supremum M and infimum I of the underlying process X killed at the discount rate. It is interesting to note, that in general, the optimal stopping problem of a Markov process, leds to a free boundary problem, and the computation of the supremum of a process has fixed boundary conditions. Second, recent results on exact distributions of M (resp. I) when positive (resp. negative) jumps of X are distributed according a mixture of exponentials and negative (resp. positive) are arbitrary are used, in order to obtain closed solutions to the optimal stopping problems. Third, a prophet type inequality is obtained, that can be seen as a comparison between prices of "perpetual look-back call options" and perpetual call options. The presented results complement the ones contained in the paper "Optimal Stopping and Perpetual Options for Lévy processes", (Mordecki, (2000b)), where reward functions $(e^x-K)^+$ and $(K-e^x)^+$ are considered, and applications to finance are discussed in detail.

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