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NECESSARY CONDITIONS FOR STABLE CONVERGENCE OF SEMIMARTINGALES

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Abstract. We prove an inverse to a theorem on stable convergence of semimartingales due to Feigin [*Stochastic Process. Appl.,* 19 (1985), pp. 125–134]. As a consequence, it can be stated (under some control in the jumps) that a sequence of martingales X^n converges stably to a continuous martingale X with conditionally independent increments if and only if the quadratic variations of X^n converge in probability to the quadratic variation of X for each $t \in \mathbf{R}^+$.

Key words. semimartingale, stable convergence, independent increments

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1. Introduction and results. The subject of this paper is the characterization of the behavior of a sequence of semimartingales when they converge stably to a continuous semimartingale with conditionally independent increments. In view of the results of [3] and more recently of [1], it remains only to prove some inverse theorems, and this is done based on the extension of existing results for weak convergence and a convenient characterization of stable convergence. The results presented here were announced in [4].

For general concepts about semimartingales and their characteristics we refer to [2, Chapters I, II].

Let us now define the sequence of semimartingales and the limit.

Let h denote a truncation function, and all the characteristics will be considered with respect to h.

(S) *Consider for each $n \in \mathbf{N}$ a d-dimensional semimartingale X^n defined on a stochastic basis $\mathcal{B}^n = (\Omega, \mathcal{F}, \mathbb{F}^n, \mathbf{P})$, with $\mathbb{F}^n = (\mathcal{F}_t^n \colon t \geqq 0)$. Denote by $(B^n, \widetilde{C}^n, \nu^n)$ the triplet of modified characteristics of X^n under $\mathbf{P}$ and by $[X^n, X^n]$ the $\mathbf{R}^d \otimes \mathbf{R}^d$-valued processes whose components are the (co-)quadratic variation $[X^{n,j}, X^{n,k}]$. Set for simplicity $X_0^n = 0$.*

As for the limit, consider a transition probability $\mathbf{Q}(\omega, d\gamma)$ from $(\Omega, \mathcal{F})$ into the Skorokhod space

$$(\mathbf{D}(\mathbf{R}^d),\ \mathcal{D}(\mathbf{R}^d),\ (\mathcal{D}_t \colon\ t \geqq 0)),$$

such that for each $\omega \in \Omega$ the canonical process on $\mathbf{D}(\mathbf{R}^d)$ defined by $X_t(\gamma) = \gamma_t$ is a semimartingale with independent increments and with characteristics $(B(\omega), C(\omega), 0)$.

Now we set

$$\overline{\Omega} = \Omega \times \mathbf{D}(\mathbf{R}^d), \qquad \overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{D}(\mathbf{R}^d),$$

$$\overline{\mathbf{P}}(d\omega, d\gamma) = \mathbf{P}(d\omega)\mathbf{Q}(\omega, d\gamma), \qquad \overline{\mathbb{F}} = \big(\cap_{s>t} (\mathcal{F} \otimes \mathcal{D}_s) \colon t \geqq 0 \big),$$

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and every process, variable or σ-field defined on $(\Omega, \mathcal{F})$ or $(\mathbf{D}(\mathbf{R}^d), \mathcal{D}(\mathbf{R}^d))$, is naturally extended to $(\overline{\Omega}, \overline{\mathcal{F}})$ with the same symbol, e.g., $X(\omega, \gamma) = X(\gamma)$, $X^n(\omega, \gamma) = X^n(\omega)$, etc.

(L) *Then let X be a process defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathbf{P}})$ with $(B, C, 0)$ as characteristics, that is, a process with $\mathcal{F}$-conditionally independent increments. Denote by $[X, X]$ the quadratic variation of X (defined as above) and observe that $\widetilde{C} = C$.*

(N) (Nesting condition.) *Assume there exists a sequence of real numbers (α_n) decreasing to 0, such that*

(1) $\mathcal{F}^n_{\alpha_n} \subset \mathcal{F}^{n+1}_{\alpha_{n+1}}$,
(2) $\mathcal{F} = \bigvee_n \mathcal{F}^n_{\alpha_n}$.

Then consider the conditions

$$[\gamma - \mathbf{R}^+]\colon \widetilde{C}^n_t \xrightarrow{\mathbf{P}} C_t \quad \forall\, t \in \mathbf{R},$$
$$[\delta - \mathbf{R}^+]\colon \mathbb{I}_{\{|x|>\varepsilon\}} * \nu^n_t \xrightarrow{\mathbf{P}} 0 \quad \forall\, t \in \mathbf{R}\ \forall\, \varepsilon > 0.$$

Denote $\operatorname{Var}(B^n)_t = \sum_{i=1}^d \operatorname{Var}(B^{n,i})_t$, where $\operatorname{Var}(A)_t$ stands for the total variation of a function A on $[0, t]$.

We will say that the sequence X^n converges $\mathcal{F}$-stably to X in $\mathbf{D}(\mathbf{R}^d)$ (denoted by $X^n \xrightarrow{\mathrm{d}} X$ ($\mathcal{F}$-stably)) if

$$\mathbf{E}\,(Y f(X^n)) \longrightarrow \mathbf{E}\,(Y f(X)) \quad \text{as } n \to +\infty$$

for all bounded and $\mathcal{F}$-measurable random variables Y and continuous and bounded functions $f\colon \mathbf{D}(\mathbf{R}^d) \to \mathbf{R}$.

THEOREM 1. *Assume conditions* **(S)** *and* **(L)** *are fulfilled. If*

(i) $X^n \xrightarrow{\mathrm{d}} X$ ($\mathcal{F}$-*stably*),
(ii) $(\operatorname{Var}(B^n)_t \mid \overline{\mathbf{P}})$ *is tight for all* $t \geqq 0$,

then

$$([X^n, X^n], X^n) \xrightarrow{\mathrm{d}} ([X, X], X) \quad (\mathcal{F}\text{-}stably)$$

and, in particular,

$$[X^n, X^n] \xrightarrow{\mathrm{d}} [X, X] \quad (\mathcal{F}\text{-}stably).$$

COROLLARY 1. *Under the assumptions of Theorem* 1

$$[X^n, X^n]_t \xrightarrow{\mathbf{P}} C_t \quad \forall\, t \geqq 0.$$

THEOREM 2. *Assume conditions* **(S)** *and* **(L)** *are fulfilled. If*

(i) $X^n \xrightarrow{\mathrm{d}} X$ ($\mathcal{F}$-*stably*),
(ii) $(\operatorname{Var}(B^n)_t \mid \overline{\mathbf{P}})$ *is tight,*

then

(1) $[\delta - \mathbf{R}^+]\colon \mathbb{I}_{\{|x|>\varepsilon\}} * \nu^n_t \xrightarrow{\mathbf{P}} 0 \quad \forall\, t \in \mathbf{R}^+\ \forall\, \varepsilon > 0$,
(2) $[\gamma - \mathbf{R}^+]\colon \widetilde{C}^n_t \xrightarrow{\mathbf{P}} C_t \quad \forall\, t \in \mathbf{R}^+$.

COROLLARY 2. *Assume conditions* **(S)**, **(L)**, *and* **(N)** *are fulfilled. Suppose that the X^n's are local martingales satisfying $|\Delta X^n_t| \leqq c$ for some constant c, and X is a*

continuous $\mathcal{F}$-conditionally Gaussian martingale (*for each fixed* ω, X *is a Gaussian martingale*).

Then the following statements are equivalent:

(1) $X^n \xrightarrow{\mathrm{d}} X$ (*$\mathcal{F}$-stably*),

(2) $[X^n, X^n]_t \xrightarrow{\mathbf{P}} [X, X]_t \quad \forall\, t \in \mathbf{R}^+$,

(3) $[\gamma - \mathbf{R}^+] + [\delta - \mathbf{R}^+]$.

2. Proofs. Let us first characterize stable convergence in a convenient way. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Denote by $\mathcal{P}(\mathcal{G})$ the set of probability measures $\widetilde{\mathbf{P}}$ on $(\Omega, \mathcal{F})$ such that $d\widetilde{\mathbf{P}} = Y d\mathbf{P}$ for any $\mathcal{G}$-measurable, positive, and bounded random variable Y. Given X a random variable, let $\sigma(X)$ denote the σ-field generated by X.

LEMMA 1. *Suppose that on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *a sequence of random elements* X^n *and* X *with values in a metric space* (E, δ) *is given. Let* $\mathcal{G} \subset \mathcal{F}$. *Then the following conditions* (1) *and* (2) *are equivalent*:

(1) $X^n \xrightarrow{\mathrm{d}} X$ (*$\mathcal{G}$-stably*),

(2) $X^n \xrightarrow{\mathcal{L}(\widetilde{\mathbf{P}})} X \quad \forall\, \widetilde{\mathbf{P}} \in \mathcal{P}(\mathcal{G})$.

If $E = \mathbf{R}^d$, δ *is the Euclidean metric in* $\mathbf{R}^d$, *and* $\sigma(X) \subset \mathcal{G}$, *then* (1) *and* (2) *are equivalent to* (3):

(3) $X^n \xrightarrow{\mathbf{P}} X$.

Proof. Equivalence (1) $\Leftrightarrow$ (2) follows from linearity and normalization.

(1) $\Rightarrow$ (3). We know that $\mathbf{E}\,(\exp\{i(\lambda X + \mu X^n)\}) \to \mathbf{E}\,(\exp\{i(\lambda X + \mu X)\})$ for the real and continuous functions $\cos(\lambda x)$, $\cos(\mu x)$, $\sin(\lambda x)$, $\sin(\mu x)$ and the bounded and $\mathcal{G}$-measurable random variables $\cos(\lambda X)$, $\cos(\mu X)$, $\sin(\lambda X)$, $\sin(\mu X)$. From this fact it follows that

$$(X, X^n) \xrightarrow{\mathcal{L}(\mathbf{P})} (X, X) \quad \text{and} \quad X^n - X \xrightarrow{\mathbf{P}} 0.$$

(3) $\Rightarrow$ (2). If $\widetilde{\mathbf{P}} \in \mathcal{P}$, then $\mathbf{P} \sim \widetilde{\mathbf{P}}$. So $X^n \xrightarrow{\widetilde{\mathbf{P}}} X$ and $X^n \xrightarrow{\mathcal{L}(\widetilde{\mathbf{P}})} X$.

Proof of Theorem 1. According to Lemma 1, it suffices to verify

$$([X^n, X^n], X^n) \xrightarrow{\mathcal{L}(\widetilde{\mathbf{P}})} ([X, X], X),$$

where $\widetilde{\mathbf{P}} \in \mathcal{P}$.

We know that the quadratic variation is invariant under an absolutely continuous change of measure, so it is natural to apply Theorem VI.6.1 of [2] under $\widetilde{\mathbf{P}}$. Let us verify its hypothesis. We know that

$$X^n \xrightarrow{\mathcal{L}(\widetilde{\mathbf{P}})} X$$

((1) $\Rightarrow$ (2) in Lemma 1). In order to prove that $(\mathrm{Var}\,(B^n)_t \,|\, \widetilde{\mathbf{P}})$ is tight, let us fix an $\varepsilon > 0$. As $\widetilde{\mathbf{P}} \sim \overline{\mathbf{P}}$, there exists a $\delta > 0$ such that

$$\overline{\mathbf{P}}(A) < \delta \Rightarrow \widetilde{\mathbf{P}}(A) < \varepsilon, \quad A \in \mathcal{F}.$$

Let K_ε be such that

$$\overline{\mathbf{P}}\{\mathrm{Var}\ (B^n)_t > K_\varepsilon\} < \delta \quad \forall\, n \in \mathbf{N};$$

then

$$\widetilde{\mathbf{P}}\{\text{Var } (B^n)_t > K_\varepsilon\} < \varepsilon \quad \forall\, n \in \mathbf{N}.$$

Now the application of Theorem VI.6.1 of [2] completes the proof:

$$([X^n, X^n], X^n) \xrightarrow{\mathcal{L}(\widetilde{\mathbf{P}})} ([X, X], X).$$

Proof of Corollary 1. In view of Theorem 1 and implication (1) ⇒ (3) in Lemma 1 the result follows from the fact that C_t is $\mathcal{F}$-measurable.

Proof of Theorem 2. (1) As X is continuous,

$$\sup_{s \leqq t} |\Delta X^n_s| \xrightarrow{\mathbf{P}} 0, \tag{2.1}$$

which is equivalent to $[\delta - \mathbf{R}^+]$ in virtue of the Lenglart–Rebolledo inequalities (see [2, section VI.4.22]).

(2) Let

$$X^n = M^n + B^n + \check{X}^n(h)$$

be the semimartingale decomposition of X^n:

- $\check{X}^n(h) = \sum_{s \leqq t} \Delta X^n_s - h(\Delta X^n_s)$,
- M^n is a local martingale with bounded jumps and $M^n_0 = 0$,
- B^n is a predictable process with finite variation in intervals.

By (2.1) we know

$$\mathbf{P}\big\{ \sup_{s \leqq t} |\check{X}^n(h)_s| > 0\big\} \longrightarrow 0.$$

Furthermore, on the set $\{\sup_{s \leqq t} |\Delta \check{X}^n_s| = 0\}$ we have

$$[X^n, X^n] = [M^n, M^n] + [B^n, B^n] + 2[M^n, B^n]$$

and

$$\sup_{s \leqq t} \big|[X^n, X^n] - [M^n, M^n]\big| \leqq \text{Var } (B^n)_t \sup_{s \leqq t} \big(|\Delta B^n_s| + 2|\Delta M^n_s|\big).$$

We know that Var $(B^n)_t$ is tight. Let us first check that

$$\sup_{s \leqq t} |\Delta B^n_s| \xrightarrow{\mathbf{P}} 0. \tag{2.2}$$

Observe that

$$|\Delta B^n_s| = h(x) * \nu^n(\{s\}, dx) \leqq \varepsilon + \Delta \mathbb{I}_{\{|x| > \varepsilon\}} * \nu^n_s \leqq \varepsilon + \mathbb{I}_{\{|x| > \varepsilon\}} * \nu^n_t$$

(assuming $h(x) = x$ on the set $\{|x| < \varepsilon\}$ and $|h(x)| \leqq 1$). So

$$\sup_{s \leqq t} |\Delta B^n_s| \leqq \varepsilon + \mathbb{I}_{\{|x| > \varepsilon\}} * \nu^n_t$$

which yields (2.2) in virtue of $[\delta - \mathbf{R}^+]$.

Let us then prove

$$\sup_{s\leqq t}|\Delta M_s^n| \xrightarrow{\mathbf{P}} 0. \tag{2.3}$$

We have

$$\Delta M_s^n = \Delta X_s^n - \Delta B_s^n - \Delta \check{X}(h)_s = h(\Delta X_s^n) - \Delta B_s^n,$$

so, as $|h(x)| \leqq |x|$,

$$|\Delta M_s^n| \leqq |\Delta X_s^n| + |\Delta B_s^n|$$

and (2.3) follows.

Thus, we have then proved that

$$\sup_{s\leqq t}\left|[X^n, X^n] - [M^n, M^n]\right| \xrightarrow{\mathbf{P}} 0$$

and, as Corollary 1 gives $[X^n, X^n]_t \xrightarrow{\mathbf{P}} C_t$, we obtain $[M^n, M^n]_t \xrightarrow{\mathbf{P}} C_t$. Now $[\gamma - \mathbf{R}^+]$ follows from a standard argument given in [5] establishing that under (2.1)

$$\sup_{s\leqq t}|\langle M^n, M^n\rangle - [M^n, M^n]| \xrightarrow{\mathbf{P}} 0. \tag{2.4}$$

Proof of Corollary 2. Let us first take a truncation function h that satisfies $h(x) = x$ for $|x| \leqq c$: we then have $B^n(h) = 0$.

(1) $\Rightarrow$ (2) is a particular case of Corollary 1.

(3) $\Rightarrow$ (1) is a particular case of Feigin's theorem [1]; see also [2, section VIII.5.42].

(2) $\Rightarrow$ (3). Since (2.1) $\forall\, t \in \mathbf{R}^+$ is equivalent to $[\delta - \mathbf{R}^+]$ and under (2.1) convergence (2.4) takes place, it is enough to prove that (2) $\Rightarrow$ (2.1).

Now it follows from (2) that

$$\sup_{s\leqq t}|\Delta[X^n, X^n]_s| \xrightarrow{\mathbf{P}} 0.$$

Since $|\Delta X^n|^2 = \sum_{i=1}^d \Delta[X^{n,i}, X^{n,i}]$, (2.1) holds.

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