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DYNKIN GAMES FOR LÉVY PROCESSES*

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Abstract. We obtain a verification theorem for solving a Dynkin game driven by a Lévy process. The result requires finding two averaging functions that, composed, respectively, with the supremum and the infimum of the process, summed, and taking the expectation, provide the value function of the game. The optimal stopping rules are the respective hitting times of the support sets of the averaging functions. We also provide some hints to understand in which type of problems the verification result can be used. The proof relies on fluctuation identities of the underlying Lévy process. We illustrate our result with examples, three of them with affine payoffs for different processes and a fourth consisting of a mathematical finance application. The smooth pasting property of the solutions is not always present.

Key words. Dynkin game, Lévy processes, Wiener–Hopf factorization

MSC codes. 60G40, 60G51, 91A15

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1. Introduction.

By a Dynkin game (DG) we understand an optimal stopping game involving two players, called the *max* and *min* players. The max player chooses the moment to stop a stochastic process and receives the value of the stopped process from the min player; the min player chooses the moment to stop another stochastic process and has to pay the value of this second stopped process to the max player. The game ends once the first of these two moments has been chosen. In the present paper, these moments are modeled by stopping times under a common filtration, and the payoff processes are compositions with different functions G_1 and G_2 of a single stochastic processes $X = \{X_t : t \geq 0\}$, an underlying Lévy process. As a two player game, when a minimax theorem holds, we say that the game has a value, and if there exists a pair of strategies that realizes this value, this pair of stopping times is called a Nash equilibrium.

A variant of this game was first formulated by Dynkin in 1969 (see [13]), as an extension of the classical optimal stopping problem for sequences of random variables, proving the existence of a value for the game, and finding a pair of approximate optimal stopping times. The methodology to prove these results is nowadays called the *martingale approach,* initiated by Snell [39] for the optimal stopping problem (see the presentation of the martingale and Markovian approaches to optimal stopping in [35]). In the very same year, two related publications appeared. First, a game formulated for Markov chains by Frid in [17], and second, one formulated for the Wiener process by Gusein-Zade in [19]. In these last two papers, each player has a set where he can stop the process, these two sets being disjoint.

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The existence of the value of these games is further considered in different frameworks. To mention two different approaches, Alario-Nazaret, Lepeltier, and Marchal establish, for optional processes, the existence of a saddle point under the so-called "Mokobodzki's assumption" [1, Thm. 2.3], and Cvitanić and Karatzas [11] construct the value as the solution of a doubly reflected backward stochastic differential equation, in this case the underlying process being a diffusion.

Our formulation, initiated by Neveu [34, Chap. VI-6], is the stochastic game proposed by Ekström and Peskir [14] (see also Peskir [35]). More precisely, Ekström and Peskir [14] prove, for a strong Markov process, with right continuous and quasi-left-continuous trajectories, under the order condition $G_1 \leq G_2$ and classical integrability conditions, that the game has a value and also a Nash equilibrium, where the stopping times are the hitting times of the sets where the value coincides with the respective payoff functions. This fine result is achieved in several steps, based first on the martingale approach, using the Wald–Bellman equations and afterward the Markovian approach. The Nash equilibrium strategies for each player are stopping times, respectively, obtained as hitting times of certain Borel sets. Our purpose is then, in this framework, to find explicit solutions when the underlying Markov process is a general Lévy process.

For games when the order condition does not hold, see, for instance, Touzi and Vieille [40] and more recently Christensen and Schultz [10]. Games where the integrability condition does not hold, in the framework of positive diffusions, are considered by Ekström and Villeneuve in [15], where interesting examples are provided.

As in the case of optimal stopping problems, one of the main motivations comes from mathematical finance. In this setting, the max player represents the holder of an American type contract (like in the optimal stopping situation), whereas the min player is the issuer of the contract, who can choose the exercise time of the contract by paying a penalty to the holder. This contract receives the name of *callable* American contracts. In this direction, we can mention the work of Kifer [22], who studies the nonarbitrage pricing of such a contract, for both the Cox–Ross–Rubinstein binomial model and the Black–Scholes model; Kyprianou in [27], where explicit solutions for put callable perpetual options are found; and Emmerling in [16], where explicit solutions of callable call options are provided. These last two works consider the Black–Scholes model.

When considering particular classes of processes, a situation in which one expects to obtain more explicit solutions to particular problems, there are important contributions when the underlying process is a diffusion. In addition to the aforementioned results of Kyprianou [27] and Emmerling [16], Alvarez [3] relies on the fundamental solutions of the second-order differential equation associated with the generator of the diffusion process in order to find the value of a game and characterize equilibrium points in some particular situations. In this respect, it is well known that in order to find explicit solutions to *boundary* problems for a diffusion process, such as computing hitting barrier probabilities, finding a stationary distribution of a reflected diffusion within an interval, solving an optimal stopping problem, or solving a stopping game as above, the fundamental solutions often contain all the necessary information to solve the aforementioned problems.

However, the generalization of these ideas or procedures to Lévy processes encounters significant difficulties. The key problem is the overshoot: boundary conditions in Lévy processes are not limited to one or two points (the extremes of the interval where the process evolves) but extend over the entire line, as the process, when stopped, can potentially reach any point. In analytic terms, many boundary problems for

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diffusions can be approached by solving an ordinary differential equation on an interval or a half-line, whereas the same type of problem for a Lévy process requires to solve an integro-differential equation, which, in general, does not have explicit solutions.

At least two strategies have been proposed to face this difficulty. An approach is to consider Lévy processes with one-sided jumps, called *spectrally* (say, negative) Lévy processes. This fact simplifies the boundary conditions when the process does not jump over a boundary. Even in the case when the process does overshoot, the exponential nature of the distribution of one of the Wiener–Hopf factors (the maximum in the case of negative jumps) can help to obtain useful information regarding the solution of the boundary problem, as the optimal threshold for the American put option for spectrally negative Lévy process found by Chan [8]. In many of these cases, the scale function—which is computable for large classes of these processes (see Hubalek and Kyprianou [20] and Kuznetsov [26])—provides valuable information for solving some problems. Examples of this situation are the DGs for spectrally Lévy processes considered by Bardoux and Kyprianou in [5] (the McKean game) and [6] (the Shepp–Shiryaev game).

A second approach leverages the memoryless property of exponential random variables: when the jump is exponential, the overshoot is independent of the departure point, potentially allowing for closed-form solutions to various boundary problems. Based on this principle, it is possible to consider boundary problems for Lévy processes with jump distributions to include phase-type distributions [4], or those with rational transforms [29], or even when the density of the Lévy measure is given by an infinite series of exponential functions with positive coefficients, as considered in [24]. Examples of closed solutions for optimal stopping of Lévy processes with exponential-like jumps can be found, for instance, in [4], [23], or [30]. Closed solutions for DGs in the context of mathematical finance (i.e., the pricing of certain callable perpetual contracts) for processes with positive exponential jumps have been found in Gapeev and Kühn [18].

At the same time, there have been efforts to solve optimal stopping problems for general Lévy processes using the distributions of the supremum and infimum of the process. The first work addressing one-sided optimal stopping problems through the distribution of the supremum was performed for general random walks by Darling, Liggett, and Taylor [12] and generalized to Lévy processes by Mordecki [30] (see also [2]). Posterior developments in optimal stopping and the supremum can be found in the monograph by Kyprianou [28]. For two-sided problems, both the infimum and the supremum are expected to participate in the solution. A verification theorem for Lévy processes (as a particular case of Hunt processes) is presented in [33]. Another verification theorem for the two-sided case was presented in [9]. A third verification theorem for Lévy processes, where also examples with closed solutions are included, was provided by Mordecki and Oliú [32]. In a parallelism with diffusions, one can think that the distributions of the infimum and the supremum of a Lévy process play the role of the fundamental solutions of a diffusion, i.e., they contain all the information necessary to solve boundary problems and, if known, allow for an explicit solution of the problem.

As we mentioned, in the present paper we analyze a DG in the formulation of Ekström and Peskir [14], with the particularity that the underlying process is a Lévy process. Although the two-sided optimal stopping (with only one max player) is an optimization problem, and a DG with two players requires finding an equilibrium, i.e., it is a minimax-problem, we found that some ideas of [32] are applicable, and a (somewhat similar) verification theorem is obtained. More precisely, the verification

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theorem proposes to construct the value function in terms of the infimum and supremum of the Lévy process, with the help of two *averaging* functions, which should be found. The optimal stopping rules are the support of the respective averaging functions. Naturally, as the distribution of the infimum and the supremum of the process are involved, the Wiener–Hopf factorization obtained by Rogozin [37] for Lévy processes (see also [7] or [28]) appears as a natural tool to find explicit solutions for DGs under this approach.

We afterward apply this verification result in two situations. We first consider three examples with symmetric affine payoff functions and Lévy processes with known Wiener–Hopf factors, finding explicit solutions. The processes considered are the Brownian motion with drift, the Cramér–Lundberg process with exponential jumps, and the compound Poisson process with double-sided exponential jumps. These three examples share analytical properties, and an explicit solution to a symmetric game is presented (symmetric games have not been considered in the literature). Second, we present a financial application, by the consideration of a perpetual callable futures contract. It is a classical perpetual contract, where not only the holder can select the execution time, but the issuer can also choose it, giving rise to a DG. The stock in this financial problem is modeled through a Kou's process [23].

The rest of the paper is organized as follows. Section 2 includes some preliminaries, necessary to introduce in section 3 the main verification theorem. This section also includes the proof of this main result and the computation of the lateral derivatives of the value function at the contact points, in terms of the averaging functions and the possible mass of the supremum (or infimum) of the process at the origin. Section 4 presents an application. The DG is driven by a process with known Wiener–Hopf factors, and the payoff functions are affine. For three processes (Brownian motion with drift, Cramér–Lundberg, and compound Poisson) we obtain the explicit solutions of the considered DG and illustrate each situation with numerical examples. The smooth pasting property is analyzed in each example, as in some cases it does not hold. Section 5 presents the computation of optimal strategies and value function in a callable perpetual futures contract, with an underlying Kou's process [23]. In section 6 some conclusions are presented.

2. Preliminaries. Let $X = \{X_t \colon t \geq 0\}$ be a Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}_x)$ departing from $X_0 = x$. Assume that the filtration $\mathbf{F}$ satisfies the usual conditions (see [21]). The corresponding expectation is denoted by $\mathbf{E}_x$, and for short we denote $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{P} = \mathbf{P}_0$. The Lévy–Khintchine formula characterizes the law of the process, stating, for $z \in i\mathbb{R}$, that $\mathbf{E}\, e^{zX_t} = e^{t\Psi(z)}$ with

$$\Psi(z) = cz + \frac{\sigma^2}{2} z^2 + \int_{\mathbb{R}} \left(e^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}} \right) \Pi(dy),$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dy)$ is a nonnegative measure (the jump measure) that satisfies the integrability condition $\int_{\mathbb{R}} (1 \wedge y^2)\Pi(dy) < \infty$. For general references on Lévy processes see [7] or [28]. Given the stochastic basis $\mathcal{B}$ the set of stopping times is the set of random variables

$$\mathcal{M} = \{\tau \colon \Omega \to [0, \infty] \text{ such that } \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Observe that we allow the possibility $\tau = \infty$, as several stopping rules that participate in the solution of the considered problems are within this class.

The Wiener–Hopf factorization relates the characteristic functions of the overall infimum and supremum of a Lévy process, stopped at an independent exponential

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time, with the characteristic exponent given by the Lévy–Khintchine formula [7, Chap. VI]. These two random variables play a key role in the solution of two-sided problems and are defined, respectively, by

$$I = \inf\{X_t : 0 \le t \le e_r\} \qquad \text{and} \qquad S = \sup\{X_t : 0 \le t \le e_r\}, \tag{2.1}$$

where e_r is an exponential random variable of parameter $r > 0$, independent of X. Observe that as $r > 0$, both random variables S and I are proper. It is also worth saying that the parameter r plays the role of a discount factor in Definition 2.1 of the DG and remains the same constant throughout the paper. Then, Wiener–Hopf factorization (see [37] or [7, Thm. VII.5]) states

$$\frac{r}{r - \Psi(z)} = \mathbf{E}(e^{zS})\,\mathbf{E}(e^{zI}). \tag{2.2}$$

This equation allows one, in some cases, to compute the distribution of the supremum and/or the infimum of a Lévy process (see, for instance, [25] or [29]).

Our definition of a DG requires two technical conditions, which we present below. Consider $G_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$, a Lévy process X, and a positive discount factor r.

The first condition (see [14, eq. (2.1)]) is the integrability condition

$$\mathbf{E}_x\left(\sup_{t\ge 0}\left|e^{-rt}G_i(X_t)\right|\right) < \infty \quad \text{for } i = 1, 2. \tag{2.3}$$

The second condition involves the behavior at infinity, requiring that

$$\lim_{t\to\infty} e^{-rt}G_i(X_t) = 0, \quad i = 1, 2. \tag{2.4}$$

Then, for an arbitrary stopping time $\tau \in \mathcal{M}$, we define

$$e^{-r\tau}G_i(X_\tau)\mathbf{1}_{\{\tau=\infty\}} = 0, \quad i = 1, 2,$$

which is consistent in the sense that

$$\lim_{t\to\infty} \mathbf{E}_x\left(e^{-r(\tau\wedge t)}G_i(X_{\tau\wedge t})\right) = \mathbf{E}_x\left(e^{-r\tau}G_i(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right), \quad i = 1, 2, \tag{2.5}$$

where the limit is obtained by dominated convergence using (2.3) and (2.4).

Definition 2.1 (Dynkin game). *Let $G_1(x) \le G_2(x)$ be two real continuous payoff functions. Consider a Lévy process X and a discount factor $r > 0$, such that conditions* (2.3) *and* (2.4) *above hold. Given two stopping times σ and τ, define the expected payoff*

$$\mathbf{M}_x(\sigma, \tau) = \mathbf{E}_x\left(e^{-r\tau}G_1(X_\tau)\mathbf{1}_{\{\tau\le\sigma\}} + e^{-r\sigma}G_2(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}}\right).$$

The DG problem consists in finding the value function $V(x)$ and two optimal stopping rules σ^* and τ^* such that

$$V(x) = \inf_\sigma \sup_\tau \mathbf{M}_x(\sigma, \tau) = \sup_\tau \inf_\sigma \mathbf{M}_x(\sigma, \tau) = \mathbf{M}_x(\sigma^*, \tau^*). \tag{2.6}$$

It is important to mention that, in terms of game theory, there are two players: the first one is the max player, who chooses a strategy τ; the second one is the min player and chooses a strategy σ. As a result of these elections, the min player pays

$$e^{-r\tau}G_1(X_\tau)\mathbf{1}_{\{\tau\le\sigma\}} + e^{-r\sigma}G_2(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}}$$

to the max player. In this context the pair of optimal strategies (σ^*, τ^*) is called a Nash equilibrium.

The following simple result presents inequalities useful for solving a DG (see [14, eq. (1.4)]).

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PROPOSITION 2.2. *Consider a pair* $(\tilde{\sigma},\tilde{\tau})$ *of stopping times for the problem in Definition* 2.1 *and define the real valued function* $\widetilde{V}(x)$ *by*

$$\widetilde{V}(x)=\mathbf{M}_x(\tilde{\sigma},\tilde{\tau}). \tag{MG}$$

Assume that the two following conditions hold:

$$\widetilde{V}(x)\geq \mathbf{M}_x(\tilde{\sigma},\tau) \quad \text{for all } \tau\in\mathcal{M}, \tag{SPMG}$$

$$\widetilde{V}(x)\leq \mathbf{M}_x(\sigma,\tilde{\tau}) \quad \text{for all } \sigma\in\mathcal{M}. \tag{SBMG}$$

Then, the DG in Definition 2.1 *has value function* $\widetilde{V}(x)$*, and* $(\tilde{\sigma},\tilde{\tau})$ *is a pair of optimal stopping times, meaning that* (2.6) *holds.*

3. Main result. In this section we present a verification theorem that constructs the solution of a DG in terms of the supremum and infimum of the process, combined with two (unknown) averaging functions.

THEOREM 3.1. *Consider a Lévy process* X*, a discount rate* $r>0$*, and two continuous reward functions* $G_1\leq G_2\colon \mathbb{R}\to\mathbb{R}$*. Assume that there exist two points* $x_I<x_S$ *and two continuous nondecreasing functions:* Q_I *s.t.* $Q_I(x)=0$ *for* $x_I\leq x$ *and* Q_S *s.t.* $Q_S(x)=0$ *for* $x\leq x_S$ *(named* averaging functions*), and s.t.*

$$\mathbf{E}_x\, Q_I(I)+\mathbf{E}_x\, Q_S(S)=\begin{cases} G_1(x) & \text{when } x_S\leq x,\\ \\ G_2(x) & \text{when } x\leq x_I, \end{cases} \tag{3.1}$$

where I *and* S *are defined in* (2.1)*. Define the function*

$$V(x)=\mathbf{E}_x\, Q_I(I)+\mathbf{E}_x\, Q_S(S) \text{ for all } x\in\mathbb{R}. \tag{3.2}$$

Then, if the condition

$$G_1(x)\leq V(x)\leq G_2(x) \tag{3.3}$$

holds for all $x\in[x_I,x_S]$*, the DG of Definition* 2.1 *has value function* $V(x)$ *in* (3.2) *and optimal stopping rules given by*

$$\sigma^*=\inf\{t\geq 0\colon X_t\leq x_I\} \qquad \text{and} \qquad \tau^*=\inf\{t\geq 0\colon X_t\geq x_S\}. \tag{3.4}$$

Remark 3.2. Observe that, taking into account the asymptotic behavior of a Lévy process (see Theorem VI.12 in [7]) the stopping times in (3.4) satisfy $\mathbf{P}(\sigma^*\wedge\tau^*<\infty)=1$. Furthermore, in view of (2.5) and condition (3.3), for an arbitrary stopping time in $\mathcal{M}$, we have

$$\mathbf{E}_x\left(e^{-r\tau}V(X_\tau)\mathbf{1}_{\{\tau=\infty\}}\right)=0. \tag{3.5}$$

Despite the fact that we do not require specific conditions on G_1 and G_2 some simple arguments show necessary conditions in order to be able to apply the theorem above.

COROLLARY 3.3. *If Theorem* 3.1 *holds, we obtain the following necessary conditions on* G_1 *and* G_2*:*

(i) $G_1(x)$ *is increasing for* $x\geq x_S$*,*
(ii) $G_2(x)$ *is increasing for* $x\leq x_I$*,*
(iii) $\lim_{x\to-\infty}G_2(x)<0$*,*
(iv) $\lim_{x\to+\infty}G_1(x)>0$*,*
(v) $G_1(x)<G_2(x)$ *for* $x\in\mathbb{R}$*.*

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Proof. From the representation (3.2), by dominated convergence, taking into account the signs and monotonicity properties of Q_I and Q_S, conditions (i)–(iv) follow. Condition (v) is obtained as optimal stopping sets are disjoint. ∎

3.1. Proof of Theorem 3.1. The proof of the main result is carried out in the following two lemmas.

LEMMA 3.4 (MG). *Consider a DG as in Definition* 2.1. *Assume that there exist points x_S, x_I and averaging functions Q_S, Q_I that satisfy the hypothesis of Theorem* 3.1. *Then, for V and (τ^*, σ^*) defined by* (3.2) *and* (3.4), *the equality* (MG) *holds.*

Proof. Denote $U = \mathbb{R} \setminus (x_I, x_S)$. Define $G\colon U \to \mathbb{R}$ by

$$G(x) = \begin{cases} G_1(x) & \text{when } x_S \le x, \\ \\ G_2(x) & \text{when } x \le x_I. \end{cases}$$

Denote $\rho^* = \sigma^* \wedge \tau^*$. Observe that as X is a Lévy process, $\mathbf{P}_x(\rho^* < \infty) = 1$. As $X_{\rho^*} \in U$ and $V = G$ on U, we have

$$\mathbf{E}_x(e^{-r\rho^*} G(X_{\rho^*})) = \mathbf{E}_x(e^{-r\rho^*} V(X_{\rho^*})). \tag{3.6}$$

On the other side, by the monotonicity of the functions Q_S and Q_I,

$$\begin{aligned} V(x) &= \mathbf{E}_x\, Q_I(I) + \mathbf{E}_x\, Q_S(S) = \mathbf{E}_x \left(Q_I \left(\inf_{0 \le t \le e_r} X_t \right) + Q_S \left(\sup_{0 \le t \le e_r} X_t \right) \right) \\ &= \mathbf{E}_x \left(\inf_{0 \le t \le e_r} Q_I(X_t) + \sup_{0 \le t \le e_r} Q_S(X_t) \right). \end{aligned}$$

Observe that if $e_r < \rho^*$, we have $x_I < I \le S < x_S$ (i.e., the process still lives within $[x_I, x_S]$), and then

$$\inf_{0 \le t \le e_r} Q_I(X_t) = \sup_{0 \le t \le e_r} Q_S(X_t) = 0,$$

because $Q_I(x) = Q_S(x) = 0$ in $[x_I, x_S]$. Also note that

$$\inf_{0 \le t < \rho^*} Q_I(X_t) = \sup_{0 \le t < \rho^*} Q_S(X_t) = 0.$$

In conclusion,

$$\begin{aligned} \mathbf{E}_x\, Q_I(I) &= \mathbf{E}_x \left(\inf_{0 \le t \le e_r} Q_I(X_t) \right) = \mathbf{E}_x \left(\inf_{\rho^* \le t \le e_r} Q_I(X_t) \right) =: q_I(x), \\ \mathbf{E}_x\, Q_S(S) &= \mathbf{E}_x \left(\sup_{0 \le t \le e_r} Q_S(X_t) \right) = \mathbf{E}_x \left(\sup_{\rho^* \le t \le e_r} Q_S(X_t) \right) =: q_S(x), \end{aligned}$$

and we have

$$V(x) = q_I(x) + q_S(x).$$

Consider now $\tilde{X} = \{\tilde{X}_s = X_{\rho^*+s} - X_{\rho^*} : s \ge 0\}$ that, by the strong Markov property, is independent of $\mathcal{F}_{\rho^*}$ and has the same distribution as X, and denote by $\tilde{\mathbf{E}}_x$ the expectation w.r.t. $\tilde{X}$. Based on these considerations, we have

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$$
\begin{aligned}
q_I(x) &= \mathbf{E}_x\, Q_I(I) = \mathbf{E}_x \left(\sup_{\rho^* \le t \le e_r} Q_I(X_t) \right) \\
&= \mathbf{E}_x \left(\int_{\rho^*}^{\infty} \sup_{\rho^* \le t \le u} Q_I(X_t) r e^{-ru} du \right) \qquad (3.7) \\
&= \mathbf{E}_x \left(e^{-r\rho^*} \int_0^{\infty} \sup_{\rho^* \le t \le \rho^* + v} Q_I(X_t) r e^{-rv} dv \right) \qquad (3.8) \\
&= \mathbf{E}_x \left(e^{-r\rho^*} \int_0^{\infty} \sup_{\rho^* \le t \le \rho^* + v} Q_I(X_{\rho^*} + X_t - X_{\rho^*}) r e^{-rv} dv \right) \qquad (3.9) \\
&= \mathbf{E}_x \left(e^{-r\rho^*} \int_0^{\infty} \sup_{0 \le s \le v} Q_I(X_{\rho^*} + X_{\rho^*+s} - X_{\rho^*}) r e^{-rv} dv \right) \qquad (3.10) \\
&= \mathbf{E}_x \left(e^{-r\rho^*} \int_0^{\infty} \sup_{0 \le s \le v} Q_I(X_{\rho^*} + \tilde{X}_s) r e^{-rv} dv \right) \\
&= \mathbf{E}_x \left(e^{-r\rho^*} \tilde{\mathbf{E}}_{X_{\rho^*}} \left[\int_0^{\infty} \sup_{0 \le s \le v} Q_I(\tilde{X}_s) r e^{-rv} dv \right] \right) \\
&= \mathbf{E}_x \left(e^{-r\rho^*} \tilde{\mathbf{E}}_{X_{\rho^*}} \left[\sup_{0 \le s \le e_r} Q_I(\tilde{X}_s) \right] \right) = \mathbf{E}_x \left(e^{-r\rho^*} q_I(X_{\rho^*}) \right),
\end{aligned}
$$

where in (3.7) we take expectation w.r.t. the independent variable e_r, we change variables according to $v = u - \rho^*$ to pass from (3.7) to (3.8), and denote $s = t - \rho^*$ to pass from (3.9) to (3.10). The same relation holds with q_S and Q_S, i.e.,

$$
q_S(x) = \mathbf{E}_x\, Q_S(S) = \mathbf{E}_x \left(e^{-r\rho^*} q_S(X_{\rho^*}) \right).
$$

Summing up these two relations, in view of (3.6), we obtain

$$
\begin{aligned}
V(x) &= \mathbf{E}_x \left(e^{-r\rho^*} V(X_{\rho^*}) \right) = \mathbf{E}_x \left(e^{-r\rho^*} G(X_{\rho^*}) \right) \\
&= \mathbf{E}_x \left(e^{-r\tau^*} G_1(X_{\tau^*}) \mathbf{1}_{\{\tau^* \le \sigma^*\}} + e^{-r\sigma^*} G_2(X_{\sigma^*}) \mathbf{1}_{\{\sigma^* < \tau^*\}} \right),
\end{aligned}
$$

concluding the proof of the lemma. ∎

LEMMA 3.5 (SPMG-SBMG). *Consider the DG of Definition* 2.1. *Assume that there exist points x_I, x_S and averaging functions Q_I, Q_S that satisfy the hypothesis of Theorem* 3.1. *Then, for V defined by* (3.2),

(i) *the supermartingale inequality* (SPMG) *holds,*
(ii) *the submartingale inequality* (SBMG) *holds.*

Proof. Let us see (i). As $Q_I(x) = 0$ for $x \ge x_S$, in view of the definition (3.4) of σ^*, for an arbitrary stopping time $\tau \in \mathcal{M}$, we have the following a.s. identity:

$$
\inf_{0 \le t \le e_r} Q_I(X_t) = \inf_{\tau \wedge \sigma^* \le t \le e_r} Q_I(X_t) \mathbf{1}_{\{\tau \wedge \sigma^* \le e_r\}}. \qquad (3.11)
$$

On the other side, as $Q_S(x) \ge 0$, we have

$$
\sup_{0 \le t \le e_r} Q_S(X_t) \ge \sup_{\tau \wedge \sigma^* \le t \le e_r} Q_I(X_t) \mathbf{1}_{\{\tau \wedge \sigma^* \le e_r\}}. \qquad (3.12)
$$

With these relations in view, we obtain

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$$
\begin{aligned}
V(x) &= \mathbf{E}_x\, Q_I(I) + \mathbf{E}_x\, Q_S(S) \\
&= \mathbf{E}_x \left(\inf_{0\le t\le e_r} Q_I(X_t) + \sup_{0\le t\le e_r} Q_S(X_t) \right) && (3.13)\\
&\ge \mathbf{E}_x \left[\left(\inf_{\tau\wedge\sigma^*\le t\le e_r} Q_I(X_t) + \sup_{\tau\wedge\sigma^*\le t\le e_r} Q_S(X_t) \right) \mathbf{1}_{\{\tau\wedge\sigma^*<e_r\}} \right] && (3.14)\\
&= \mathbf{E}_x \left[\int_{\tau\wedge\sigma^*}^{\infty} \left(\inf_{\tau\wedge\sigma^*\le t\le v} Q_I(X_t) + \sup_{\tau\wedge\sigma^*\le t\le v} Q_S(X_t) \right) re^{-rv}\, dv \right] && (3.15)\\
&= \mathbf{E}_x \left[e^{-r(\tau\wedge\sigma^*)} V(X_{\tau\wedge\sigma^*}) \mathbf{1}_{\{\tau<\infty\}} \right] = \mathbf{E}_x \left[e^{-r(\tau\wedge\sigma^*)} V(X_{\tau\wedge\sigma^*}) \right], && (3.16)
\end{aligned}
$$

where we pass from (3.13) to (3.14) in view of (3.11) and (3.12), from line (3.14) to (3.15) integrating w.r.t. the exponential density of the independent random variable e_r, and the passage from (3.15) to (3.16) is the same procedures as in (3.7) to (3.10); and, the last equality uses (3.5). Finally, taking into account our hypothesis (3.3), and the fact that $V = G_2$ for $x \le x_I$, we obtain

$$
e^{-r(\tau\wedge\sigma^*)} V(X_{\tau\wedge\sigma^*}) \ge e^{-r\tau} G_1(X_\tau) \mathbf{1}_{\{\tau\le\sigma^*\}} + e^{-r\sigma^*} G_2(X_{\sigma^*}) \mathbf{1}_{\{\sigma^*<\tau\}}.
$$

Taking expectations, the inequality (SPMG) is proved, concluding the proof of (i). The proof of (ii) is analogous. ∎

Proof of Theorem 3.1. The proof is an application of Proposition 2.2, where Lemmas 3.4 gives (MG), and 3.5 gives the conditions (SPMG) and (SBMG). ∎

3.2. Smooth pasting. One of the interesting features of Theorem 3.1 is that, although it is a verification result in the sense that it requires a conjecture to identify a solution (a common characteristic in many optimal stopping problems), it does not rely on the smooth pasting condition. This condition is typically used in stopping problems within the diffusion framework [36]. This distinction is significant because, as illustrated in our examples and well documented in the context of Lévy processes, the smooth pasting condition does not consistently hold (see, for instance, the discussion in [2] and the examples in [28]). Due to its similarity, we adopt the approach of [32], where the following result is obtained.

PROPOSITION 3.6 (Theorem 3.1 in [32]). *Consider a function of the form*

$$
W(x) = \mathbf{E}_x\, Q(S). \tag{3.17}
$$

Assume that the three following conditions hold:

(i) $Q(x) = 0$ *for* $x \le x_0$ *and* $Q(x)$ *is nondecreasing and in class* $C^2[x_0, \infty)$,
(ii) *there exist* $A > 0$ *and* $\alpha > 0$ *s.t.* $|Q''(x)| \le Ae^{\alpha x}$,
(iii) *for* $\alpha > 0$ *above, it holds* $\mathbf{E}\, e^{\alpha X_1} < e^r$.

Then

$$
W'(x_0+) - W'(x_0-) = Q'(x_0+)\, \mathbf{P}(S = 0), \tag{3.18}
$$

$$
W'(x+) - W'(x-) = 0 \quad \text{for } x > x_0. \tag{3.19}
$$

To apply this result in our situation, observe that the function $V(x)$ is the sum of two functions of the form (3.17), because $\mathbf{E}_x\, Q_I(I) = \mathbf{E}_{-x}\, Q_I(-\hat{S})$, where $\hat{S}$ is the supremum of the dual process $\hat{X} = -X$. In conclusion, if the averaging functions Q_I, Q_S meet conditions (i) to (iii), and the payoffs G_1, G_2 are differentiable, we will have smooth pasting at x_I if and only if the condition $\mathbf{P}(I = 0) = 0$ holds, and smooth

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pasting at x_S if and only if the condition $\mathbf{P}(S=0)=0$ holds. This will be discussed more in detail in the examples of the next section.

3.3. Integrability. To end the section, we provide the following result in order to check conditions (2.3) and (2.4). Observe that, unless the functions G_i are bounded, condition $r>0$ is strictly necessary for (2.3) to hold true.

PROPOSITION 3.7. *Consider a Lévy process X, a discount rate $r>0$, and a function $G\colon \mathbb{R}\to\mathbb{R}$, and assume that there exist positive constants A, B, α such that*

$$|G(x)| \le A + B\cosh(\alpha x) \tag{3.20}$$

and

$$\mathbf{E}\, e^{\alpha X_1} < e^r \qquad \text{and} \qquad \mathbf{E}\, e^{-\alpha X_1} < e^r. \tag{3.21}$$

Then, the function G satisfies conditions (2.3) *and* (2.4).

Proof. In view of (3.21), the characteristic exponent $\Psi(z)$ has an analytic extension to the strip $\Re(z)\in[-\alpha,\alpha]$ in the complex plane (see [38, Thm. 25.17]). From this follows that $\mathbf{E}\,|X_1|<\infty$ and that $\mathbf{E}\,X_1=\Psi'(0)$. Consider the Lévy process $\{\bar{X}_t=\alpha X_t - rt\colon t\ge 0\}$. Its characteristic exponent $\bar{\Psi}(z)$ satisfies $\bar{\Psi}(1)=\Psi(\alpha)-r<0$ (the first statement in (3.21)). From this follows $\mathbf{E}\,\bar{X}_1<0$, meaning that $\lim_{t\to\infty}(\alpha X_t - rt)=-\infty$. A similar reasoning based on the second statement in (3.21) implies $\lim_{t\to\infty}(-\alpha X_t - rt)=-\infty$. In view of (3.20), condition (2.4) holds.

In order to see now condition (2.3) we use [30, Lem. 1], which can be stated for the Lévy process $\{\alpha X_t - rt\colon t\ge 0\}$ without discount, as the equivalence of the following two statements:

$$\mathbf{E}(e^{\sup_{t\ge 0}(\alpha X_t - rt)})<\infty, \tag{3.22}$$

$$\mathbf{E}(e^{\alpha X_1 - r})<1. \tag{3.23}$$

Furthermore

$$\sup_{t\ge 0} e^{-rt}|G(X_t)| \le A + \frac{B}{2}e^{\sup_{t\ge 0}(\alpha X_t - rt)} + \frac{B}{2}e^{\sup_{t\ge 0}(-\alpha X_t - rt)}. \tag{3.24}$$

As condition (3.23) is equivalent to the first statement in (3.21) (the same for $-\alpha$), in view of condition (3.22) and (3.24), the statement (2.3) follows. ∎

4. Three processes with affine payoff. In this section we consider three examples that are similar from an analytic point of view: Brownian motion with drift, a Cramér–Lundberg process with exponential claims, and a compound Poisson process with double-sided exponential jumps. These examples share the structure of characteristic function for the infimum and the supremum.

More precisely, we assume that the infimum I and the supremum S in (2.1) have characteristic functions given, respectively, by

$$\mathbf{E}\, e^{zI} = \pi_I + (1-\pi_I)\frac{r_I}{r_I+z}, \tag{4.1}$$

$$\mathbf{E}\, e^{zS} = \pi_S + (1-\pi_S)\frac{r_S}{r_S-z}, \tag{4.2}$$

where $0\le \pi_I, \pi_S \le 1$ are probabilities, and $-r_I<0<r_S$ are the two roots of the equation

$$\Psi(z)=r.$$

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It is important to note that the Brownian motion with drift is obtained when $\pi_I = \pi_S = 0$; the Cramér–Lundberg process is obtained when $\pi_I > 0$ and $\pi_S = 0$, and the compound Poisson process is obtained when $\pi_I > 0$ and $\pi_S > 0$. Moreover, a similar methodology is expected to apply to other processes with rational-transform distributed jumps, albeit at the cost or more involved computations (see [29]).

In view of (4.1) and (4.2), the random variables $-I$ and S have (possibly) defective exponential distributions with respective parameters r_I and r_S and atoms of size π_I, π_S. Then, with a slight abuse of notation, the respective densities can be denoted by

$$f_I(x) = \pi_I \delta_0(x) + (1-\pi_I) r_I e^{r_I x}, \quad x \le 0, \tag{4.3}$$

$$f_S(x) = \pi_S \delta_0(x) + (1-\pi_S) r_S e^{-r_S x}, \quad x \ge 0, \tag{4.4}$$

where $\delta_0(x)$ denotes the Dirac mass measure at $x = 0$.

For simplicity to obtain tractable solutions, we assume

$$G_1(x) = x - \delta \qquad \text{and} \qquad G_2(x) = x + \delta \tag{4.5}$$

for some $\delta > 0$.

Observe now that the three process considered above (Brownian motion with drift, a Cramér–Lundberg process with exponential claims, and a compound Poisson process with double-sided exponential jumps) have a characteristic exponent that is analytic in a certain strip $-\alpha \le \Re(z) \le \alpha$, and for that $\alpha > 0$ there exist positive constants A, B such that $|G_i(x)| \le A + B\cosh(\alpha x)$, $(i = 1, 2)$. This allows one to apply Proposition 3.7 to verify the conditions of Theorem 3.1 in the considered examples.

For simplicity in notation, introduce the constants $E_I, F_I, G_I, E_S, F_S, G_S$ that depend only on the parameters of the problem

$$E_I = -\mathbf{E}\, I = \frac{1-\pi_I}{r_I} > 0, \qquad E_S = \mathbf{E}\, S = \frac{1-\pi_S}{r_S} > 0, \tag{4.6}$$

$$F_I = \frac{1}{\mathbf{E}\, e^{-r_I S}} = \frac{r_I + r_S}{r_I + \pi_I r_S} > 1, \qquad F_S = \frac{1}{\mathbf{E}\, e^{r_S I}} = \frac{r_I + r_S}{\pi_S r_I + r_S} > 1, \tag{4.7}$$

$$G_I = F_I - 1 = \frac{(1-\pi_I) r_S}{r_I + \pi_I r_S} > 0, \qquad G_S = F_S - 1 = \frac{(1-\pi_S) r_I}{\pi_S r_I + r_S} > 0. \tag{4.8}$$

THEOREM 4.1. *Under the assumptions* (4.3) *and* (4.4) *for the densities of the respective infimum and supremum of the process X, and for the linear payoffs $G_1(x) = x - \delta$, $G_2(x) = x + \delta$, for some $\delta > 0$, the DG of Definition* 2.1 *has optimal thresholds given by*

$$x_I = -\delta - E_I + F_I \frac{E_S e^{r_I u} - E_I G_S}{e^{(r_I + r_S)u} - G_I G_S}, \tag{4.9}$$

$$x_S = \delta + E_S + F_S \frac{-E_I e^{r_S u} + E_S G_I}{e^{(r_I + r_S)u} - G_I G_S}, \tag{4.10}$$

where u is the unique solution of

$$u - 2\delta - E_I - E_S = \frac{-E_I F_S e^{r_S u} - E_S F_I e^{r_I u} + 2 E_S F_S G_I}{e^{(r_I + r_S)u} - G_I G_S}. \tag{4.11}$$

Moreover, the value function is given by

$$V(x) = \begin{cases} x + \delta & \text{if } x \le x_I, \\ A_I e^{-r_I(x - x_S)} + A_S e^{r_S(x - x_I)} & \text{if } x_I \le x \le x_S, \\ x - \delta & \text{if } x \ge x_S \end{cases} \tag{4.12}$$

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with

$$A_I = \frac{-E_I e^{r_S u} + E_S G_I}{e^{(r_I+r_S)u} - G_I G_S}, \quad A_S = \frac{E_S e^{r_I u} - E_I G_S}{e^{(r_I+r_S)u} - G_I G_S}. \tag{4.13}$$

Proof. In order to find functions $Q_I(x)$ and $Q_S(x)$ s.t. (3.1) holds, we can compute $\mathbf{E}_x\, Q_I(I)$ for $x \le x_I$ and $\mathbf{E}_x\, Q_S(S)$ for $x \ge x_S$ (using (4.3) and (4.4))

$$\begin{aligned} q_I(x) = \mathbf{E}_x\, Q_I(I) &= \int_{(-\infty,0]} Q_I(x+y) f_I(y) dy \\ &= \int_{(-\infty,0]} (\pi_I \delta_0(y) + (1-\pi_I) r_I e^{r_I y})\, Q_I(x+y) dy \\ &= \int_{-\infty}^{x \wedge x_I} \left(\pi_I \delta_0(z-x) + (1-\pi_I) r_I e^{r_I (z-x)}\right) Q_I(z) dz \\ &= \pi_I Q_I(x) + (1-\pi_I)\, e^{-r_I x} \int_{-\infty}^{x \wedge x_I} r_I e^{r_I z} Q_I(z) dz. \end{aligned} \tag{4.14}$$

Analogously

$$q_S(x) = \mathbf{E}_x\, Q_S(S) = \pi_S Q_S(x) + (1-\pi_S)\, e^{r_S x} \int_{x \vee x_S}^{+\infty} r_S e^{-r_S z} Q_S(z) dz. \tag{4.15}$$

For convenience, we introduce the notation

$$A_I = A_I(x_I, x_S) = (1-\pi_I) e^{-r_I x_S} \int_{-\infty}^{x_I} Q_I(z) r_I e^{r_I z} dz, \tag{4.16}$$

$$A_S = A_S(x_I, x_S) = (1-\pi_S) e^{r_S x_I} \int_{x_S}^{+\infty} Q_S(z) r_S e^{-r_S z} dz. \tag{4.17}$$

For the solution, we impose the condition (3.1) with the payoff functions of the problem. We first impose the continuity conditions at the optimal thresholds x_I and x_S. We obtain the system

$$\begin{cases} e^{r_I u} A_I + A_S = x_I + \delta, \\ A_I + e^{r_S u} A_S = x_S - \delta, \end{cases} \tag{4.18}$$

where we denoted

$$u = x_S - x_I.$$

As we have jumps, the boundary conditions necessarily comprise the whole half-lines $x \le x_I$ and $x \ge x_S$, instead of only the boundary point. This corresponds to the fact that the process can jump outside the interval $[x_I, x_S]$. Accordingly we impose (3.1) for $x \le x_I$, which states

$$\pi_I Q_I(x) + (1-\pi_I)\, e^{-r_I x} \int_{-\infty}^{x} r_I e^{r_I z} Q_I(z) dz + e^{r_S (x - x_I)} A_S = x + \delta. \tag{4.19}$$

The relationship (4.19) is an ordinary differential equation to find Q_I. If we multiply by $e^{r_I x}$ both sides, assume enough regularity, and take derivatives, we have

$$\begin{aligned} &\pi_I \left(r_I e^{r_I x} Q_I(x) + e^{r_I x} Q_I'(x)\right) + (1-\pi_I)\, r_I e^{r_I x} Q_I(x) + (r_I + r_S) e^{r_I x} e^{r_S (x - x_I)} A_S \\ &\quad = r_I (x+\delta) e^{r_I x} + e^{r_I x}. \end{aligned}$$

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Therefore, the function $Q_I(x)$ satisfies the differential equation

$$\frac{\pi_I}{r_I}Q_I'(x)+Q_I(x)=x+\delta+\frac{1}{r_I}-\left(1+\frac{r_S}{r_I}\right)A_S e^{r_S(x-x_I)}.$$

In the case that $\pi_I=0$ we directly obtain the candidate Q_I to use in the verification Theorem 3.1. When $\pi_I>0$, we assume that $Q_I(x)\sim x$ when $x\to-\infty$, in order to find a candidate. Solving the differential equation according to this condition, with the introduced above notation, we find the function

$$Q_I(x)=x+\delta+E_I-F_IA_Se^{r_S(x-x_I)},\quad x\le x_I. \tag{4.20}$$

Similarly, for $x\ge x_S$, we obtain the differential equation

$$\pi_SQ_S(x)+(1-\pi_S)\,e^{r_Sx}\int_x^{+\infty}r_Se^{-r_Sz}Q_S(z)dz+e^{-r_I(x-x_S)}A_I=x-\delta$$

with solution

$$Q_S(x)=x-\delta-E_S-F_SA_Ie^{-r_I(x-x_I)},\quad x\ge x_S. \tag{4.21}$$

Using now the functions (4.20) and (4.21), we impose conditions $Q_I(x_I)=0$ and $Q_S(x_S)=0$ required in Theorem 3.1 to get

$$\begin{cases}x_I+\delta+E_I-F_IA_S=0,\\ x_S-\delta-E_S-F_SA_I=0.\end{cases} \tag{4.22}$$

To solve the system above, we substitute $x_I-\delta$ and $x_S+\delta$ from (4.18) obtaining the linear system

$$\begin{cases}e^{r_Iu}A_I-G_IA_S=-E_I,\\ -G_SA_I+e^{r_Su}A_S=E_S,\end{cases}$$

which has solutions A_I and A_S given by (4.13).

We replace A_I and A_S in the system (4.22), we subtract both equations, and we obtain (4.11), where we used $E_SF_SG_I=E_IF_IG_S$.

The left-hand side in (4.11) is a linear increasing function in u, with value at $u=0$ strictly less than $-(E_I+E_S)$, as $\delta>0$. The numerator of the right-hand side (using the constants (4.6), (4.7), and (4.8)) results in

$$-E_IF_Se^{r_Su}-F_IE_Se^{r_Iu}+2E_SF_SG_I\le0,$$

which is a decreasing function in u and the denominator of the right-hand side is an increasing function in u. These properties imply that the right-hand side of (4.11) is a negative increasing function converging zero, and that its value at $u=0$ is $-(E_I+E_S)$. Therefore, we conclude that (4.11) has a unique positive solution for all $\delta>0$.

Finally, with this value of the root u, from (4.22) we find that the optimal strategies x_I and X_S are (4.9) and (4.10).

In order to compute $V(x)$ for $x\in[x_I,x_S]$, we refer to (3.2). Observe that $Q_I(x)=Q_S(x)=0$ within this interval, so, taking into account (4.14)–(4.15) and (4.16)–(4.17), we obtain (4.12). ∎

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4.1. Brownian motion with drift. We consider that $X = \{X_t : t \geq 0\}$ is a Brownian motion with drift,

$$X_t = x + ct + \sigma W_t, \quad t \geq 0, \tag{4.23}$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion. The characteristic exponent of the process is

$$\Psi(z) = \frac{\sigma^2}{2} z^2 + cz,$$

and therefore the denominator in the Wiener–Hopf factorization is zero when $\Psi(z) = r$ that has real roots $-r_I < 0 < r_S$, with

$$r_I = \frac{\sqrt{c^2 + 2r\sigma^2} + c}{\sigma^2} \qquad \text{and} \qquad r_S = \frac{\sqrt{c^2 + 2r\sigma^2} - c}{\sigma^2}. \tag{4.24}$$

As $r_I r_S = 2r/\sigma^2$, Wiener–Hopf factorization reads

$$\frac{r}{r - \Psi(z)} = \frac{2r}{\sigma^2 (r_I + z)(r_S - z)} = \frac{r_I}{r_I + z} \frac{r_S}{r_S - z},$$

giving densities

$$f_I(x) = r_I e^{r_I x}, \quad x \leq 0,$$
$$f_S(x) = r_S e^{-r_S x}, \quad x \geq 0,$$

i.e., the formulas in (4.3) and (4.4) with $\pi_I = \pi_S = 0$. In conclusion, applying the results of Theorem 4.1, we obtain the following results.

COROLLARY 4.2. *The DG of Definition* 2.1 *for the Brownian motion with drift in* (4.23) *and payoff functions* $G_1(x) = x - \delta$, $G_2(x) = x + \delta$ *has optimum stopping rules given by hitting times of the levels*

$$x_I = -\delta - \frac{1}{r_I} - \frac{\sqrt{c^2 + 2r\sigma^2}}{r} \frac{e^{r_I u} - 1}{e^{(r_I + r_S)u} - 1},$$
$$x_S = \delta + \frac{1}{r_S} + \frac{\sqrt{c^2 + 2r\sigma^2}}{r} \frac{e^{r_S u} - 1}{e^{(r_I + r_S)u} - 1},$$

where r_I, r_S *are the roots in* (4.24)*, and* u *is the unique positive solution of the equation*

$$u - 2\delta = \frac{\sqrt{c^2 + 2r\sigma^2}}{r} \frac{(e^{r_S u} - 1)(e^{r_I u} - 1)}{e^{(r_I + r_S)u} - 1}$$

and value function (4.12) *with*

$$A_I = -\frac{1}{r_I} \frac{e^{r_S u} - 1}{e^{(r_I + r_S)u} - 1}, \quad A_S = \frac{1}{r_S} \frac{e^{r_I u} - 1}{e^{(r_I + r_S)u} - 1}.$$

Moreover, as a consequence of the discussion in subsection 3.2*, in view of the fact that the infimum and supremum of the process have no atoms, the value function is everywhere differentiable, i.e., we have smooth pasting.*

Remark 4.3. The symmetric case is obtained when $c = 0$, with roots $-r_I = r_S = \sqrt{2r}/\sigma$. Furthermore, in this symmetric case, $A_I = -A_S$, $-x_I = x_S$, and the function $V(x)$ in (4.12) presents central symmetry around the origin.

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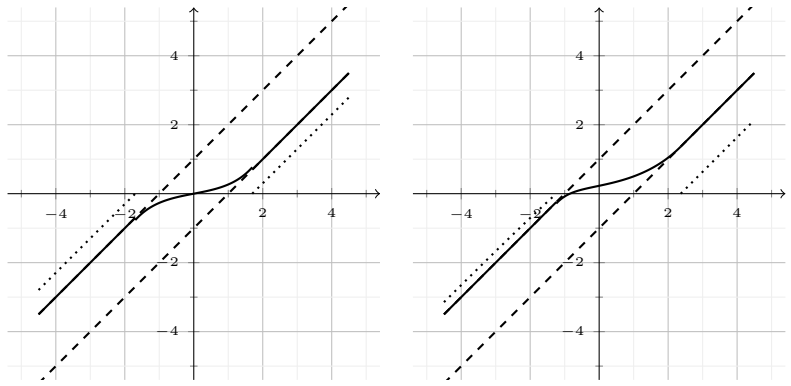


FIG. 1. *Brownian motion. Value function $V(x)$ (solid), $G_1(x)$, $G_2(x)$ (dashed), and $Q_I(x)$, $Q_S(x)$ (dotted). Left: symmetric case. Right: asymmetric case.*

Numerical examples. To illustrate our results in this case we consider two examples. In the first one we choose the standard Brownian motion, with $c = 0$ and $\sigma = 1$, and the second example is the Brownian motion with drift the parameters $c = 1$ and $\sigma = 1$. We consider $r = 1$ and $\delta = 1$. The results obtained are $x_I = -x_S = -1.6955$ and $x_I = -1.2426$ and $x_S = 2.3659$, respectively. In Figure 1, we show the corresponding value functions.

4.2. Cramér–Lundberg process. We consider the Cramér–Lundberg process $X = \{X_t : t \geq 0\}$ with exponential jumps, given by

$$X_t = x + ct - \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)}, \tag{4.25}$$

where $N^{(1)} = \{N_t^{(1)} : t \geq 0\}$ is a Poisson process with intensity λ_1 and $Y^{(1)} = \{Y_i^{(1)} : i \geq 1\}$ is a sequence of independent identically distributed exponential random variables with parameter α_1. The two processes $N^{(1)}$, and $Y^{(1)}$ are independent.

The characteristic exponent of the process is

$$\Psi(z) = cz - \lambda_1 \frac{z}{\alpha_1 + z},$$

and therefore the denominator in the Wiener–Hopf factorization is zero when $\Psi(z) = r$ that has two roots $-r_I < 0 < r_S$, with

$$r_I = \frac{\sqrt{(c\alpha_1 - \lambda_1 - r)^2 + 4cr\alpha_1} + c\alpha_1 - \lambda_1 - r}{2c},$$
$$r_S = \frac{\sqrt{(c\alpha_1 - \lambda_1 - r)^2 + 4cr\alpha_1} - (c\alpha_1 - \lambda_1 - r)}{2c}.$$

The Wiener–Hopf factorization to determine the law of S and I is

$$\frac{r}{r - \Psi(z)} = \left(\pi_I + (1 - \pi_I)\frac{r_I}{r_I + z}\right)\left(\frac{r_S}{r_S - z}\right),$$

where $\pi_I = r/(r_S c)$, and we used that $r_I r_S = r\alpha_1/c$. We observe that the random variable S follows an exponential distribution with parameter r_S, whereas the random

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variable $-I$ follows a defective exponential distribution with parameter r_I, withan atom at zero of mass π_I. The respective densities are

$$f_I(x) = \pi_I \delta_0(x) + (1-\pi_I) r_I e^{r_I x}, \quad x \le 0,$$
$$f_S(x) = r_S e^{-r_S x}, \quad x \ge 0,$$

where $\delta_0(x)dx$ denotes the Dirac mass measure at $x = 0$. Consequently, we obtain (4.3) and (4.4) with $\pi_I > 0$ and $\pi_S = 0$. We then obtain the following result.

COROLLARY 4.4. *The DG of Definition* 2.1 *for the Cramér–Lundberg process defined in* (4.25)*, and payoff functions* $G_1(x) = x - \delta$, $G_2(x) = x + \delta$, *has an optimal stopping rule given by hitting times of the levels given by* (4.9) *and* (4.10) *with* $\pi_S = 0$, *where* u *is the unique solution of* (4.11). *The corresponding value function is given in* (4.12).

Moreover, as a consequence of the discussion in subsection (3.2)*, since only the infimum has an atom, we have smooth pasting at* x_S *but the value function at* x_I *is not differentiable. The jump of the derivative can be computed by* (3.18).

Numerical examples. To illustrate our results in this process we consider the Cramér–Lundberg with $c = 1$, $\lambda = 1$, and $\alpha_1 = 1$. We consider $r = 1$ and $\delta = 1$. The critical thresholds are $x_I = -1.6127$ and $x_S = 1.4931$.

In Figure 2 we show the corresponding value function.

4.3. Compound Poisson process. We consider the compound Poisson process $X = \{X_t : t \ge 0\}$ with double-sided exponential jumps, given by

$$X_t = x - \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} + \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}, \tag{4.26}$$

where $N^{(1)} = \{N_t^{(1)} : t \ge 0\}$ and $N^{(2)} = \{N_t^{(2)} : t \ge 0\}$ are two Poisson process with respective positive intensities λ_1, λ_2, and the two sequences $Y^{(1)} = \{Y_i^{(1)} : i \ge 1\}$ and $Y^{(2)} = \{Y_i^{(2)} : i \ge 1\}$ are of independent identically distributed exponential random variables with respective positive parameters α_1, α_2. The four processes $N^{(1)}$, $N^{(2)}$, $Y^{(1)}$, $Y^{(2)}$ are independent.

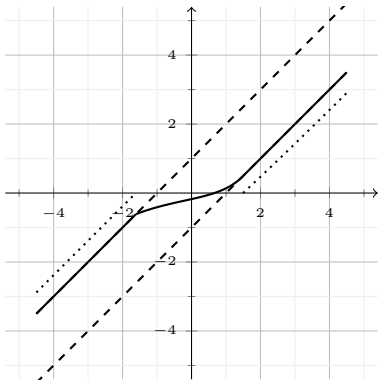


FIG. 2. *Cramér–Lundberg process. Value function* $V(x)$ *(solid),* $G_1(x)$*,* $G_2(x)$ *(dashed), and* $Q_I(x)$*,* $Q_S(x)$ *(dotted).*

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The characteristic exponent of the process X is given by

$$\Psi(z) = -\lambda_1 \frac{z}{\alpha_1 + z} + \lambda_2 \frac{z}{\alpha_2 - z},$$

and therefore the denominator in the Wiener–Hopf factorization is zero when $\Psi(z) = r$, that has two roots $-r_I$, r_S. These roots are the solution of the equation

$$(r + \lambda_1 + \lambda_2)z^2 + (\alpha_1(\lambda_2 + r) - \alpha_2(\lambda_1 + r))z - r\alpha_1\alpha_2 = 0,$$

which satisfy

$$-\alpha_1 < -r_I < 0 < r_S < \alpha_2.$$

The Wiener–Hopf factorization to determine the law of S and I is

$$\begin{aligned}\frac{r}{r - \Psi(z)} &= \frac{r(z - \alpha_2)(z - \alpha_1)}{(r + \lambda + \mu)(z - r_S)(z - r_I)} = \frac{r_I r_S(\alpha_1 + z)(\alpha_2 - z)}{\alpha_1\alpha_2(r_I + z)(r_S - z)} \\ &= \left(\frac{r_I}{\alpha_1} + \frac{\alpha_1 - r_I}{\alpha_1}\frac{r_I}{r_I + z}\right)\left(\frac{r_S}{\alpha_2} + \frac{\alpha_2 - r_S}{\alpha_2}\frac{r_S}{r_S - z}\right).\end{aligned}$$

In conclusion, due to the uniqueness of the factorization (see Thm. 5(ii), Chap. VI, of [7]), we obtain that the random variables $-I$ and S have (strictly) defective exponential distributions with parameters r_I and r_S, and atoms at zero of respective sizes $\pi_I = r_I/\alpha_1$ and $\pi_S = r_S/\alpha_2$, corresponding to (4.1) and (4.2), giving then the following result.

COROLLARY 4.5. *For the DG introduced in Definition* 2.1*, where the underlying process is the compound Poisson process with double-sided exponential jumps given in* (4.26)*, and the payoff functions are* $G_1(x) = x - \delta, G_2(x) = x + \delta$*, the optimal stopping rules are the hitting times of the levels defined in* (4.9) *and* (4.10)*, with* $\pi_I = r_I/\alpha_1$ *and* $\pi_S = r_S/\alpha_2$*, where* u *is the unique solution of* (4.11)*. The corresponding value function is given in* (4.12)*.*

Moreover, as a consequence of the discussion in subsection 3.2*, since both the infimum and the supremum have atoms, the value function is not differentiable at* x_S *and* x_I*. The jumps of the derivative at these points can be computed by* (3.18)*.*

Numerical examples. To illustrate our results we consider two examples. In the first one we choose a symmetric case with $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (1, 1, 1, 1)$ and in the second example an asymmetric case with $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (1, 3, 3, 1)$. We consider $r = 1$ and $\delta = 1$. The results obtained are $x_I = -1.5901$, $x_S = 1.5901$ and $x_I = -3.7750$, $x_S = 0.0834$, respectively.

In Figure 3 we show the corresponding value functions $V(x)$. In both cases we observe that in x_I and x_S the value function is not differentiable.

5. Callable futures under the Kou model. We consider the mathematical model of a financial market introduced by Kou [23]. There are two assets, a riskless asset given by $B_t = B_0 e^{rt}$, $t \geq 0$, and risky asset $S = \{S_t : t \geq 0\}$ given by

$$S_t = e^{X_t} = S_0 \exp\left(at + \sigma W_t - \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} + \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}\right), \quad S_0 = e^x,$$

where a Lévy process $X = \{X_t : t \geq 0\}$ consists in a drifted Brownian motion plus double-sided exponential jumps. More precisely, $W = \{W_t : t \geq 0\}$ is a standard

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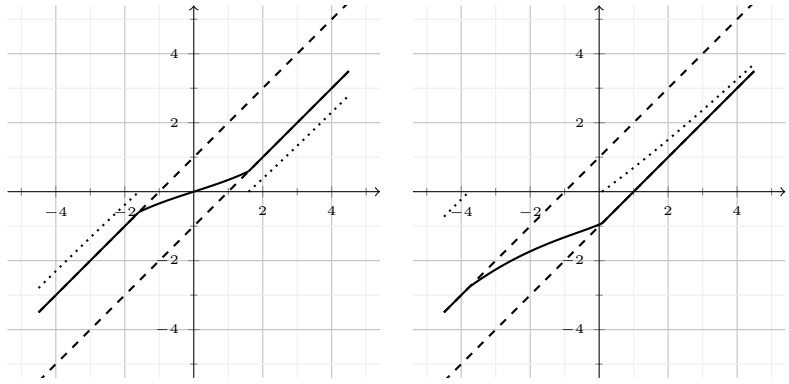


FIG. 3. *Compound Poisson process. Value function $V(x)$ (solid), $G_1(x)$, $G_2(x)$ (dashed), and $Q_I(x)$, $Q_S(x)$ (dotted). Left: symmetric case. Right: asymmetric case.*

Wiener process, $a \in \mathbb{R}$ and $\sigma > 0$, $N^{(1)} = \{N_t^{(1)} : t \geq 0\}$ and $N^{(2)} = \{N_t^{(2)} : t \geq 0\}$ are two Poisson process with respective positive intensities λ_1, λ_2, and the two sequences $Y^{(1)} = \{Y_i^{(1)} : i \geq 1\}$ and $Y^{(2)} = \{Y_i^{(2)} : i \geq 1\}$ are of independent identically distributed exponential random variables with respective positive parameters α_1, α_2. The five processes W, $N^{(1)}$, $N^{(2)}$, $Y^{(1)}$, $Y^{(2)}$ are independent.

In this market model, a perpetual callable American futures contract is introduced, defined in the following way. Consider two positive strikes $k < K$. If the holder (here is the max player) chooses the execution time τ, the contract is struck at K, and he then receives $S_\tau - K$. The contract has the additional feature that the issuer (the min player), by paying a penalty $K - k$, can also choose the execution time σ, of the same contract struck at k (i.e., is a callable contract). The issuer then pays $S_\sigma - k$ to the holder. The resulting contract is a discounted DG for the process X with payoff functions

$$G_1(x) = e^x - K < G_2(x) = e^x - k.$$

The characteristic exponent of the Lévy process X is

$$\Psi(z) = az + \sigma^2 \frac{z^2}{2} - \lambda_1 \frac{z}{\alpha_1 + z} + \lambda_2 \frac{z}{\alpha_2 - z},$$

and therefore the denominator in the Wiener–Hopf factorization is zero when $\psi(z) = r$, which has four real roots $-n_2 < -\alpha_1 < -n_1 < 0 < p_1 < \alpha_2 < p_2$. Moreover, the Wiener–Hopf factorization (2.2) gives the densities

$$f_I(x) = \pi_I n_2 e^{n_2 x} + (1 - \pi_I) n_1 e^{n_1 x}, \qquad x \leq 0, \tag{5.1}$$

$$f_S(x) = \pi_S p_1 e^{-p_1 x} + (1 - \pi_S) p_2 e^{-p_2 x}, \qquad x \geq 0. \tag{5.2}$$

It is important to note that the condition $p_1 > 1$ in necessary for the integrability condition (2.3) to hold. The coefficients in (5.1) and (5.2) are

$$\pi_I = \frac{n_1(\alpha_1 - n_1)}{\alpha_1(n_2 - n_1)} \qquad \text{and} \qquad \pi_S = \frac{p_2(\alpha_2 - p_1)}{\alpha_2(p_2 - p_1)}$$

(see (11) with $n = 1$ in [31]).

5.1. Averaging functions and thresholds. In order to find functions $Q_I(x)$ and $Q_S(x)$ s.t. (3.1) holds, we compute $\mathbf{E}_x\, Q_I(I)$ for $x \leq x_I$ and $\mathbf{E}_x\, Q_S(S)$ for $x \geq x_S$ (using (5.1) and (5.2)),

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(5.3)
$$\begin{aligned}\mathbf{E}_x\, Q_I(I) &= \int_{-\infty}^{0} Q_I(x+y) f_I(y)dy = \int_{-\infty}^{x\wedge x_I} Q_I(z) f_I(z-x)dz \\ &= \pi_I e^{-n_2 x}\int_{-\infty}^{x\wedge x_I} n_2 e^{n_2 z} Q_I(z)dz + (1-\pi_I)\, e^{-n_1 x}\int_{-\infty}^{x\wedge x_I} n_1 e^{n_1 z} Q_I(z)dz.\end{aligned}$$

Analogously

(5.4)
$$\begin{aligned}\mathbf{E}_x\, Q_S(S) &= \int_{0}^{\infty} Q_S(x+y) f_S(y)dy = \int_{x\vee x_S}^{\infty} Q_S(z) f_S(z-x)dz \\ &= \pi_S e^{p_1 x}\int_{x\vee x_S}^{\infty} p_1 e^{-p_1 z} Q_S(z)dz + (1-\pi_S)\, e^{p_2 x}\int_{x\vee x_S}^{\infty} p_2 e^{-p_2 z} Q_S(z)dz.\end{aligned}$$

For convenience, we introduce the notation

$$\begin{aligned} N_2 &= \pi_I e^{-n_2 x_S}\int_{-\infty}^{x_I} n_2 e^{n_2 z} Q_I(z)\, dz, \quad & N_1 &= (1-\pi_I) e^{-n_1 x_S}\int_{-\infty}^{x_I} n_1 e^{n_1 z} Q_I(z)\, dz, \\ P_1 &= \pi_S e^{p_1 x_I}\int_{x_S}^{\infty} p_1 e^{-p_1 z} Q_S(z)\, dz, & P_2 &= (1-\pi_S) e^{p_2 x_I}\int_{x_S}^{\infty} p_2 e^{-p_2 z} Q_S(z)\, dz.\end{aligned}$$

Two continuity equations are obtained evaluating (3.1) at the points x_I and x_S, with the computations in (5.3) and (5.4). With the above notation, letting $u = x_S - x_I$, we obtain

$$\text{(I)} \qquad e^{n_2 u} N_2 + e^{n_1 u} N_1 + P_1 + P_2 = e^{x_I} - k,$$
$$\text{(II)} \qquad N_2 + N_1 + e^{p_1 u} P_1 + e^{p_2 u} P_2 = e^{x_S} - K.$$

In view of (3.2), our guess for the value function for $x \in (x_I, x_S)$ is

(5.5)
$$V(x) = \begin{cases} e^x - k, & x \le x_I, \\ N_2 e^{-n_2(x-x_S)} + N_1 e^{-n_1(x-x_S)} + P_1 e^{p_1(x-x_I)} + P_2 e^{p_2(x-x_I)}, & x_I \le x \le x_S, \\ e^x - K, & x \ge x_S. \end{cases}$$

As $\mathbf{P}(S=0) = \mathbf{P}(I=0) = 0$, in view of Proposition 3.6, we expect the smooth pasting property to hold, at both contact points. We then impose the continuity of derivatives, by differentiation of (5.5) from the left and the right at the points x_I and x_S. The resulting equations are

$$\text{(III)} \qquad -n_2 e^{n_2 u} N_2 - n_1 e^{-n_1 u} N_1 + p_1 P_1 + p_2 P_2 = e^{x_I},$$
$$\text{(IV)} \qquad -n_2 N_2 - n_1 N_1 + p_1 e^{p_1 u} P_1 + p_2 e^{p_2 u} P_2 = e^{x_S}.$$

In view of (3.1) and our computations in (5.3) and (5.4), we conclude that each averaging function should be a linear combination of three exponentials,

$$\begin{aligned} Q_I(z) &= B_1 e^{p_1(z-x_I)} + B_2 e^{p_2(z-x_I)} + B_3 e^z + B_4, \\ Q_S(z) &= C_1 e^{-n_1(z-x_S)} + C_2 e^{-n_2(z-x_S)} + C_3 e^z + C_4. \end{aligned}$$

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After plugging these formulas in (3.1) to find the coefficients, and imposing the continuity of the averaging functions at x_I and x_S, we arrive at

$$\text{(V)} \qquad -\frac{(n_2+p_1)(n_1+p_1)}{n_1n_2+p_1\bar{n}}P_1 - \frac{(n_2+p_2)(n_1+p_2)}{n_1n_2+p_2\bar{n}}P_2 + \frac{(n_2+1)(n_1+1)}{n_1n_2+\bar{n}}e^{z_I} = k,$$

$$\text{(VI)} \qquad -\frac{(n_2+p_1)(n_2+p_2)}{p_1p_2+n_2\bar{p}}N_2 - \frac{(n_1+p_1)(n_1+p_2)}{p_1p_2+n_1\bar{p}}N_1 + \frac{(p_1-1)(p_2-1)}{p_1p_2-\bar{p}}e^{z_S} = K,$$

where

$$\bar{p} = \pi_S p_1 + (1-\pi_S)p_2 \qquad \text{and} \qquad \bar{n} = \pi_I n_2 + (1-\pi_I)n_1.$$

Equations (I)–(VI) constitute a nonlinear system of six equations to find the six unknowns $N_2, N_1, P_1, P_2, x_I, x_S$. Once a solution is found, we should check $x_I < x_S$ and that Q_I is negative, Q_S positive. Condition (3.1) holds (because it was imposed to obtain the system), but condition (3.3) should also be checked.

Numerical example. To illustrate our result, we consider parameters $a = 0$, $\sigma^2 = 2$, $\alpha_1 = \alpha_2 = 8/3$, $\lambda_1 = \lambda_2 = 35/18$, and $r = 9$. Besides, we take the values $k = 1$, $K = 2$ in the perpetual callable futures contracts. The Wiener–Hopf factorization is

$$\frac{r}{r-\Psi(z)} = \frac{64-9z^2}{z^4-20z^2+64} = \left(\frac{1}{2}\frac{2}{2+z} + \frac{1}{2}\frac{4}{4+z}\right)\left(\frac{1}{2}\frac{2}{2-z} + \frac{1}{2}\frac{4}{4-z}\right),$$

and therefore $n_2 = p_2 = 4,\ n_1 = p_1 = 2,\ \pi_I = \pi_S = 1/2$.

To solve the system we implement a numerical scheme that, departing from $x_I^{(0)} = k$ and $x_S^{(0)} = K$, computes $N_2^{(0)}, N_1^{(0)}, P_1^{(0)}, P_2^{(0)}$ by solving (I)–(IV) that is a linear system in the four unknowns. With these four values we compute $x_I^{(1)}$ and $x_S^{(1)}$ using (V) and (VI). Afterward we compute , $N_2^{(1)}, N_1^{(1)}, P_1^{(1)}, P_2^{(1)}$ and so on. This scheme

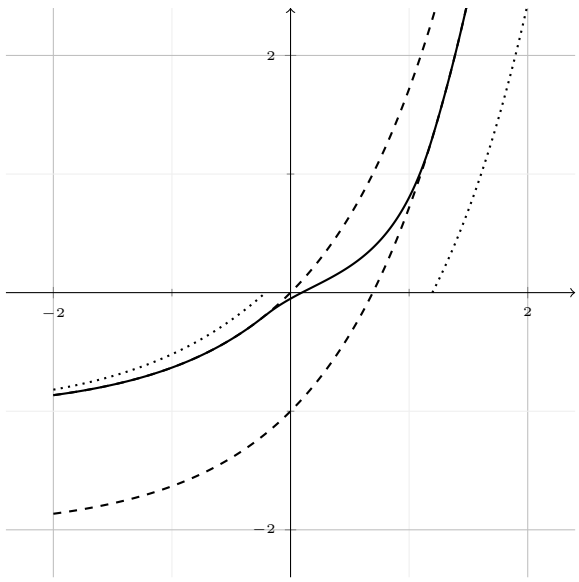


FIG. 4. *Callable future under the Kou model. Value function $V(x)$ (solid), $G_1(x)$, $G_2(x)$ (dashed), and $Q_I(x)$, $Q_S(x)$ (dotted).*

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quickly converges to a solution such that the value function in (5.5) verifies all the requirements of Theorem 3.1 (see Figure 4). The solution is $x_I = -0.211$ and $x_S = 1.195$, with $(N_2, N_1, P_1, P_2) = (-3.347 \times 10^{-4},\ -9.473 \times 10^{-3},\ 5.928 \times 10^{-2},\ 1.181 \times 10^{-3})$.

6. Conclusions. We present a verification theorem to solve a DG driven by Lévy processes. The theorem requires finding two averaging functions that give an explicit form of the value function in terms of the infimum and the supremum of the process. The optimal stopping times result in being the entry times of the support sets of these averaging functions. The main challenge in this problem is managing the characteristic overshoot caused by the jumps of the Lévy processes, rendering traditional solution techniques, used for diffusions, ineffective. Notably, in some instances, when Wiener–Hopf factors can be computed explicitly, we are able to identify the averaging functions and solve the problem completely. We first consider Brownian motion with drift, the Cramér–Lundberg process, and the compound Poisson process with affine payoffs. Second, we present a financial application, a perpetual callable American futures in the Kou's market model. In these examples we discuss the smooth pasting property, which does not always hold.

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