A note on pricing options on interest rates in Uruguay

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In this paper, we provide a dynamic model for derivative pricing on interest rate denominated Gaussian Affine Term Structure multidimensional model. For the model we describe the underlying mathematics and the procedure statistics using historical bond price data. For implementation, it is necessary to apply a Kalman filter with an EM algorithm In the application, we select different financial assets price series in dollars and CPI-Indexed. After the pricing, we compare our results with the benchmark BDT-model. Results show some differences with the classical model and suggest the need to apply multidimensional models to measure the risk of interest rate increases or decreases.
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JEL classification. H63, C58, G12, G13, G15.

1. Introduction

Despite the strong growth of the Uruguayan market debt, a derivatives market on interest rate was not developed. The problems in the generation and implementation of the financial derivatives market are based on the existence of several risks that are difficult to hedge, weaknesses in the market structure and in the regulatory framework. However, some derivatives on interest rate are traded but they are mainly designed according to the request of the agents. From our perspective, the discussion and analysis of different pricing methods for derivatives on interest rate with Uruguayan bond price allows us to understand and value the risks associated with this class of instruments. Moreover, there is empirical evidence that the development of these assets facilitates and increases the flow of capital between different countries as well as increases the liquidity of the secondary market bond and provides the opportunity of hedging certain financial risks for important local agents of large market share. For

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instance, the main local agents in the market debt in Uruguay are the Pension Fund Asset Management(PFAM). The development of certain financial assets on the interest rate improves the financial risk management of the Uruguayan pension system because hedging strategies involve the use of contracts that depend on the state of the economy.

In this paper, our objective is to introduce the debate on the different pricing methods of derivatives on interest rate in Uruguay. The literature based on term structure models has evolved especially in continuous time, where stochastic calculus tools and partial differential equations are of great relevance. In this case, we applied a multidimensional dynamic model because it has a good analytical tractability. The paper is organized as follows. Section 2 provides the theoretical framework that discusses in detail the econometric model and the optimal strategy to estimate the parameters by using sovereign bonds prices traded in the market. This section is completed by setting the price of the call and put options. In section 3, we present the data and we expose the requirements for an asset to be used in the estimation. We establish which assets and which currency we will use in the model estimation and options pricing. In section 4 we present the results and we discuss their performance. Section 5 sets out the conclusions.

2. Theoretical Framework

2.1 *Econometric Models*

In analyzing the models to valuation of derivatives, the estimation of yield curves in sovereign debt must satisfy relevant properties. As a fundamental requirement for the management of the risks associated with this asset class, the model must require non-arbitrage properties. Besides, in this paper we are interested in using flexible models and calibrate with historical information on traded bond prices in order to analyze the volatility in the market. With this principle in mind, the interest rate model range from simple one-factor model to more complex models based on the arbitrage free Heath-Jarrow-Morton framework and popular multi-factor market model. There is a large number of models proposed in the literature, some of which can be found at Bjork (2020) and Brigo and Mercurio (2006).

The proposal of this paper is to use a dynamic econometric model in several dimensions denominated *Gaussian Affine Term Structure model (GATS)*. This class of models plays an important role in the valuation of financial derivatives because they allow viable dynamic structures with the evolution of interest rates observed in the market. This model specifies that the short rate is a deterministic function of an N-dimensional vector of states satisfying a stochastic differential equation via multidimensional Brownian motion. The selection of this class is based on the trade-off between generality and computational tractability, because the sovereign bond yield rate formula is a linear function of the state vectors. The proposed model is N-dimensional, however, in order not to make understanding more difficult, its respective equations are established in the most general case applied in the paper, which corresponds to three dimensions.

The dynamics of the vector of latent states is given by

$$dy_t = A(B - y_t)dt + \Sigma dW_t;$$

where B it is a vector $N\times 1$, A y Σ are matrices $N\times N$, and $\{W_t : t\geq 0\}$ is a brownian motion N-dimensional defined in a probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbf{Q})$. In the paper, we denote $B=(b_i)$ and the matrices $A=\text{diag}(a_i)$ y $\Sigma=\text{diag}(\sigma_i)$ are chosen diagonally for parsimony's sake.

If the process y_t is discretized into equal intervals of length $\Delta_t = t_{i+1}-t_i$, the Euler approximation verifies that

$$\underbrace{\begin{pmatrix} y^1_{t_i}\\ y^2_{t_i}\\ y^3_{t_i}\end{pmatrix}}_{y_{t_i}} = \underbrace{\begin{pmatrix} b_1(1-e^{-a_1\Delta_t})\\ b_2(1-e^{-a_2\Delta_t})\\ b_3(1-e^{-a_3\Delta_t})\end{pmatrix}}_{\alpha} + \underbrace{\begin{pmatrix} e^{-a_1\Delta_t} & 0 & 0\\ 0 & e^{-a_2\Delta_t} & 0\\ 0 & 0 & e^{-a_3\Delta_t}\end{pmatrix}}_{\phi}\underbrace{\begin{pmatrix} y^1_{t_{i-1}}\\ y^2_{t_{i-1}}\\ y^3_{t_{i-1}}\end{pmatrix}}_{y_{t_{i-1}}} + \underbrace{\begin{pmatrix} \mu^1_{t_i}\\ \mu^2_{t_i}\\ \mu^3_{t_i}\end{pmatrix}}_{\mu_{t_i}}; \tag{1}$$

where $\mu_{t_i}|F_{t_{i-1}} \approx N(0,Q)$ and

$$Q=\begin{pmatrix} \frac{\sigma_1^2}{2a_1}(1-e^{-2a_1\Delta_t}) & 0 & 0\\ 0 & \frac{\sigma_2^2}{2a_2}(1-e^{-2a_2\Delta_t}) & 0\\ 0 & 0 & \frac{\sigma_3^2}{2a_3}(1-e^{-2a_3\Delta_t})\end{pmatrix}.$$

In the econometric model, it is obtained that the yield return of the bond price is a linear function of the vector of states of the form

$$z(\tau) := -\frac{\log P(\tau)}{\tau} = -\frac{A(\tau)}{\tau} + \frac{\sum_{i=1}^{3} B_i(\tau)y^i_t}{\tau},$$

where

$$B_i(\tau) = \frac{1}{a_i}(1-e^{-a_i\tau});$$
$$A(\tau) = \sum_{i=1}^{i=3}(b_i - \frac{\sigma_i^2}{2a})\big(B_i(\tau)-\tau\big) - \frac{\sigma_i^2 B_i^2(\tau)}{4a_i}.$$

The model assumes that there are $n>N$ sovereign bonds in the market with maturities $T^1,\ldots,T^n$, and respective yield rates $z(\tau_t^1),z(\tau_t^2),\ldots,z(\tau_t^n)$, where $\tau_t^j = T^j - t$ is the time to maturity. In order to avoid identification problems it is needed to add gaps between the observed yield and the estimated yields by introducing an error variable (v_{t_i}). This technique was first proposed in Chen and Scott (1993) by introducing more sovereign bond information than dimensions in the vector of states. The measurement equation is given

$$\underbrace{\begin{pmatrix} z(\tau^1_{t_i})\\ z(\tau^2_{t_i})\\ \vdots\\ z(\tau^n_{t_i})\end{pmatrix}}_{z_{t_i}} = \underbrace{\begin{pmatrix} \frac{-A(\tau^1_{t_i})}{\tau^1_{t_i}}\\ \frac{-A(\tau^2_{t_i})}{\tau^2_{t_i}}\\ \vdots\\ \frac{-A(\tau^n_{t_i})}{\tau^n_{t_i}}\end{pmatrix}}_{A_{t_i}} + \underbrace{\begin{pmatrix} \frac{B_1(\tau^1_{t_i})}{\tau^1_{t_i}} & \frac{B_2(\tau^1_{t_i})}{\tau^1_{t_i}} & \frac{B_3(\tau^1_{t_i})}{\tau^1_{t_i}}\\ \frac{B_1(\tau^2_{t_i})}{\tau^2_{t_i}} & \frac{B_2(\tau^2_{t_i})}{\tau^2_{t_i}} & \frac{B_3(\tau^2_{t_i})}{\tau^2_{t_i}}\\ \vdots & \vdots & \vdots\\ \frac{B_1(\tau^n_{t_i})}{\tau^n_{t_i}} & \frac{B_2(\tau^n_{t_i})}{\tau^n_{t_i}} & \frac{B_3(\tau^n_{t_i})}{\tau^n_{t_i}}\end{pmatrix}}_{B_{t_i}}\underbrace{\begin{pmatrix} y^1_{t_i}\\ y^2_{t_i}\\ y^3_{t_i}\end{pmatrix}}_{y_{t_i}} + \underbrace{\begin{pmatrix} v^1_{t_i}\\ v^2_{t_i}\\ \vdots\\ v^n_{t_i}\end{pmatrix}}_{v_{t_i}}; \tag{2}$$

where $\nu_{t_i} \approx N(0,R)$ and

$$R = \begin{pmatrix} \varepsilon_1^2 & 0 & \cdots & 0 \\ 0 & \varepsilon_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varepsilon_n^2 \end{pmatrix}.$$

The equation (1) and the equation (2) represent the state-space form of the proposed econometric model. It is possible to set them in the system

$$\begin{cases} y_{t_i} = \alpha + \phi y_{t_{i-1}} + \mu_{t_i}; \\ z_{t_i} = A_{t_i} + B_{t_i} y_{t_i} + \nu_{t_i}; \end{cases} \tag{3}$$

where y_{t_i}, α, μ_{t_i}, z_{t_i} and ν_{t_i} are vectors of dimension N, ϕ is a matrix of dimension $N \times N$ and B_{t_i} is a matrix of dimension $n \times N$.

2.2 *Estimation Procedure*

The methodology must be able to solve the problem of estimating the unobservable variables and model parameters in the system (3). The estimation is to be made using the prices of sovereign bonds traded in the primary market and in the secondary market. The method is the Kalman Filter proposed in the work Kalman (1960). This methodology satisfies that it is optimal in the GATS models, for more information see Harvey (1990).

The Kalman Filter is a set of inductive equations that implement a predictor-corrector system that is updated by the input of new information. For the derivation of the equations in time t_i it is assumed that the estimate of the unobservable variable $y_{t_{i-1}}$ at instant t_{i-1} is known and is denoted as $y_{t_{i-1}|t_{i-1}}$. By using the equation (1), we obtain the variable y_{t_i} is a normal vector of mean and error covariance matrix

$$y_{t_i|t_{i-1}} = \alpha + \phi y_{t_{i-1}|t_{i-1}}; \tag{4}$$

$$P_{t_i|t_{i-1}} = \phi P_{t_{i-1}|t_{i-1}} \phi^{'} + Q. \tag{5}$$

The system formed by the equation (4) and the equation (5) is called *prediction system* and forms the first equations of the methodology. In order to adapt these estimates to the arrival of the information of the observable variable z_{t_i} at time t_i, it is deduced in the method that

$$y_{t_i|t_i} = y_{t_i|t_{i-1}} + K_{t_i}\left(z_{t_i} - A_{t_i} - B_{t_i} y_{t_i|t_{i-1}}\right); \tag{6}$$

$$P_{t_i|t_i} = \left(Id - K_{t_i} B_{t_i}\right) P_{t_i|t_{i-1}}; \tag{7}$$

where the matrix K_{t_i} is known as *Kalman's gain*, with dimension $N \times n$ and it is expressed

$$K_{t_i} = P_{t_i|t_{i-1}} B_{t_i}^{'} \underbrace{\left(B_{t_i} P_{t_i|t_{i-1}} B_{t_i}^{'} + R\right)}_{H_{t_i}}{}^{-1}. \tag{8}$$

The equation (8) is interpreted as the relative importance given to the prediction error in the estimation of the unobservable variable. That is, a low Kalman gain implies a higher weighting in the predictions, while a high gain means that the system is more sensitive to the measurement equation. The equations (6), (7) and (8) are called the "updating system".

For the estimation of the parameters, it is established that under these two systems of equations the likelihood function

$$\rho(z|\theta) = \prod_{i=1}^{i=m} \rho(z_{t_i}|\theta);$$

therefore

$$\log \rho(z|\theta) = \sum_{t=1}^{S} \log \rho(z_{t_i}|\theta) = -\frac{nS}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{S} \log|H_{t_i}| - \frac{1}{2}\sum_{t=1}^{S} \zeta_t^T H_{t_i}^{-1} \zeta_{t_i}.$$

Non-linear techniques must be applied for the optimization. In this paper, we use the algorithm called Expectation-Maximization(EM), to find maximum likelihood parameters of the model. This procedure has unbounded stability regions, greater flexibility on the choice of the stepsize, and hence lead to improved numerical efficiency. Besides of the structure of the problem, we examine how this algorithm can be used for missing data imputation. For more information see Mader et al. (2014).

2.3 *Pricing Options on Interest Rate*

The econometric model has a closed formula for pricing European call options and European put options. In these financial assets, the underlying asset is a sovereign bond of maturity in time T and provide the right to buy (sell) at a certain time S (maturity of the option) that satisfies $S < T$ the sovereign bond at a fixed price K (strike price). In Jamshidian (1989), the author derived the option pricing formula for the classical Vasicek model by using arguments similar to those established in Black and Scholes (1973). With similar arguments, it is possible to establish that in this econometric model, the price of call options ($Call\ Op.$) and put options ($Put\ Op.$) satisfy

$$Call\ Op.(t,S,K,T) = P(t,T)N(h) - KP(t,S)N(h-\sigma_m);$$

$$Put\ Op.(t,S,K,T) = KP(t,S)N(\sigma_m - h) - P(t,T)N(-h);$$

where

$$h = \frac{1}{\sigma_m}\log\left(\frac{P(t,T)}{KP(t,S)}\right) + \frac{\sigma_m}{2}; \qquad \text{and} \qquad \sigma_m = \sum_{i=1}^{3} \frac{\sigma_i}{a_i}\left(1 - e^{-a_i(T-S)}\right)\sqrt{\frac{1-e^{-2a_i(S-t)}}{2a_i}}.$$

3. Data

The search of suitable data to estimate the models needs to balance concerns about measurement errors, sample length, observation frequency, non-linearities in pricing

formulas and even the documentation quality of different data sets. In this paper, we select two currencies for discuss the results; on the one hand, the dollar debt is the largest share in the foreign currency category in Uruguay and on the other hand, we take CPI-Indexed in the domestic currency category which accounts for approximately 50% of the outstanding. Estimations are carried out using the information available in the market, which are the prices of financial assets issued in the primary market or traded in the secondary market. The databases are the daily prices of sovereign bonds in the time period from January 2, 2014 to August 31, 2016.

The dollar market is the most diversified due to the large amount of financial assets (mostly sovereign bonds) with medium-term and long-term maturities. In the analysis, the six global bonds (international issues) that are outstanding throughout the analysis period are given preference. In addition, two local bonds with maturity in 2019 and 2020 are incorporated with the aim to obtain more precision in the short-term maturities. Sovereign bonds used in the dollar market are shown in the table 1. In turn in CPI-Indexed market, we use all available global bonds and two Treasury Notes are added for the short term estimate. The two Treasury Notes are the ones with the highest cash flow in the period under analysis. Financial assets used in CPI-Indexed market are shown in the table 2.

Table 1. Financial Assets in dollars market

Financial Assets	Issue	Maturity	Coupon	Amortizations
Local Bond 2019	03/29/03	03/23/19	7.500	(100%) 03/23/19
Local Bond 2020	05/29/03	02/28/20	9.750	(100%) 02/28/20
Global Bond 2022	11/18/05	11/18/22	8.000	(33.3%) 11/18/20 - 11/18/21 - 11/18/22
Global Bond 2024	08/14/13	08/14/24	4.500	(33.3%) 08/14/22 - 08/14/23 - 08/14/24
Global Bond 2025	09/28/09	09/28/25	6.875	(33.3%) 09/28/23 - 09/28/24 - 09/28/25
Global Bond 2033	05/29/03	01/15/33	7.875	(100%) 01/15/33
Global Bond 2036	03/21/06	03/21/36	7.625	(33.3%) 03/21/34 - 03/21/35 - 03/21/36
Global Bond 2045	11/20/12	11/20/45	4.125	(33.3%) 11/20/43 - 11/20/44 - 11/20/45

Table 2. Financial Assets in CPI-Indexed market

Financial Assets	Issue	Maturity	Coupon	Amortizations
Treasury Notes Serie 14	06/10/10	06/10/20	4.000	(100%) 06/10/20
Treasury Notes Serie 16	01/27/11	01/27/19	4.000	(100%) 01/27/19
Global Bond 2018	09/14/06	09/14/18	5.000	(100%) 09/14/18
Global Bond 2027	05/03/07	05/04/27	4.250	(33.3%) 05/04/25 - 05/04/26 - 05/04/27
Global Bond 2028	12/15/11	12/15/28	4.375	(33.3%) 12/15/25 - 12/15/27 - 12/15/28
Global Bond 2030	07/10/08	07/10/30	4.000	(33.3%) 07/10/28 - 07/10/29 - 07/10/30
Global Bond 2037	06/26/07	06/26/37	3.700	(33.3%) 06/26/35 - 06/26/36 - 06/26/37

It should be noted that all the assets used have semi-annual coupons. Therefore, the future flows of each asset are decomposed in order to obtain the price of the equivalent zero coupon bond.

4. Results

To assess the relevance of our proposal, we now consider the models with one, two, and three dimension on a data set in dollar market and CPI-Indexed market. First, we estimate the model parameters with the available data in each currency. Secondly, we value European call and put options for different strikes and maturities.

The case of one-dimensional models it is important because present a significant interpretation according to the magnitude of the estimated parameters. However, its disadvantage is that yield rates are perfectly correlated across maturities and this property is not satisfied in the market debt. For this reason it is necessary to add more dimensions to the econometric model. Generally, adding more dimensions to the model results in a better fit to the market prices, but overfitting problems may arise by include more parameters. This situation is equivalent to find a trade off between the fit of the data and the number of parameters of the model. For this reason, we use the two usual information criteria: Akaike (AIC) and Bayesian (BIC). These criteria are composed of two terms: the one related to the goodness of fit in the maximum likelihood estimators and the one that penalizes the number of parameters. The parameter estimates for both currencies, the likelihood function and the two information criteria are shown in table 3.

Table 3. Parameter estimates in all dimensions in both currencies

	Dollar			CPI-Indexed		
	1D-Mod	**2D-Mod**	**3D-Mod**	**1D-Mod**	**2D-Mod**	**3D-Mod**
a_1	0.2162	0.6646	0.7386	0.6837	0.4781	0.5484
a_2	—	0.2535	0.2555	—	0.0350	0.2436
a_3	—	—	0.1132	—	—	0.0717
b_1	0.0698	0.0654	0.0519	0.0453	0.0330	0.0394
b_2	—	0.0038	0.0169	—	0.0186	0.0071
b_3	—	—	0.0011	—	—	0.0043
σ_1	0.0123	0.0154	0.0137	0.0168	0.0157	0.0146
σ_2	—	0.0123	0.0126	—	0.0128	0.0153
σ_3	—	—	0.0088	—	—	0.0099
ε_1	0.0053	0.0019	0.0014	0.0065	0.0064	0.0001
ε_2	0.0047	0.0014	0.0021	0.0023	0.0021	0.0001
ε_3	0.0040	0.0013	0.0026	0.0024	0.0003	0.0002
ε_4	0.0035	0.0025	0.0008	0.0023	0.0015	0.0003
ε_5	0.0038	0.0034	0.0009	0.0017	0.0015	0.0005
ε_6	0.0033	0.0038	0.0012	0.0009	0.0007	0.0008
ε_7	0.0039	0.0033	0.0018	0.0031	0.0007	0.0009
ε_8	0.0031	0.0030	0.0017	—	—	—
$\ln L$	3404.48	3660.60	3861.42	8022.55	11077.84	11636.06
AIC	-6784.96	-7293.20	-7688.84	-16025.09	-22129.68	-23240.12
BIC	-6713.01	-7201.63	-7577.65	-15961.02	-22046.38	-23137.60

Regarding the interpretation of the results in the one-dimensional model, the speed of reversion to the mean is higher in the CPI-Indexed market and the dollar market is less volatile. Finally, the estimated parameters for the long-term rate are ordered in relation to what is observed in the market, with the CPI-Indexed market having the lowest value, closely followed by the dollar market. The parameter interpretations obtained are in accordance with expectations.

Since the coefficients associated with $y_{t_{i-1}}$ in the autoregressive process (equation (1)) are $e^{-a_i\Delta_t}$, it is satisfied that if the parameter a_i is close to zero the coefficient is close to one. This is equivalent to having roots close to the unit circle. Therefore, the parameters a_i of reduced values present a behavior in the process similar to that of a random walk, being the main responsible for explaining the variations in the long-term return. In the model estimated in several dimensions, the first dimension is the greatest reversion to the mean, being the one that captures the dynamics in the level of the short-term rate of return. The other dimensions explain the impact in the medium-term and long-term structure. The b_i parameters in the case of several dimensions agree with their respective models in one dimension.

We present the call and put options prices on August 31, 2016 with the results in table 3. The strikes of pricing options correspond to fluctuation in 0%, 1% and 2% in the underlying asset's yield rates and the maturities are 3, 6 and 12 months. In Uruguay, the options on interest rate are not liquid, so we must select the benchmark model to compare the results. The BDT-model is one of the most widely used in the financial industry for pricing European and American options (proposed in Black et al. (1990)). This model is arbitrage-free in discrete time. The binomial tree is calibrated in order to fit the market yield curve and the historical volatility structure. For the calibration of BDT-model in Uruguay, we used data published on the BEVSA website.

In the case of the dollar market the underlying asset is the zero coupon bond equivalent to the Local Bond 2019 maturing on May 23, 2019. This bond is priced at 94.84 (yield rate of 1.94%) on August 31, 2016. In the case of the CPI-Indexed (UI) market the underlying asset is the zero coupon bond equivalent to the Global Bond 2018 maturing on September 14, 2018. This bond is priced at 89.97 (yield rate of 5.19%) on August 31, 2016. Option prices for the dollar market can be found in the table 4 and option prices for CPI-Indexed in the table 5.

Although option call prices and option put prices are valued using their respective formulas, we verified that the well-known "call-put parity formula" is satisfied. Therefore, we only discuss the results in call option prices. In the dollar market, the most significant difference between the models is that options with higher strike prices (decrease in the rate of return) in the BDT-model generate option prices close to zero and in the econometric model this is not true in any dimension. It is also verified that the price of the options in the proposed model increases when an additional dimension is added. It is possible that the BDT-model is not perceiving that risk but models with a larger number of dimensions do capture it. In the CPI-Indexed market, the one-dimensional model generates option prices similar to those generated by the BDT-model. In turn, these values differ from the prices generated in the two-dimension and three-dimension model (similar to each other), which are lower in all the situations analyzed.

5. Conclusions

In the international market debt, pricing developments in financial derivatives bring about even greater modeling challenges. The essence of our paper is to impose a discussion on valuation methodologies for interest rate derivatives in Uruguay. In the first

Table 4. Call and Put Option in dollar market

Option Price 3 month								
Str.	Call Option				Put Option			
	1D-Mod	**2D-Mod**	**3D-Mod**	**BDT**	**1D-Mod**	**2D-Mod**	**3D-Mod**	**BDT**
97.69	0.003	0.012	0.029	0.000	2.649	2.658	2.675	2.646
96.48	0.048	0.095	0.149	0.006	1.487	1.533	1.587	1.444
95.48	0.317	0.408	0.494	0.256	0.578	0.669	0.755	0.526
94.13	1.029	1.098	1.170	1.043	0.122	0.191	0.262	0.136
92.97	2.076	2.096	2.126	2.096	0.011	0.032	0.062	0.032
Option Price 6 month								
Str.	Call Option				Put Option			
	1D-Mod	**2D-Mod**	**3D-Mod**	**BDT**	**1D-Mod**	**2D-Mod**	**3D-Mod**	**BDT**
97.92	0.017	0.046	0.088	0.000	2.611	2.640	2.682	2.594
96.83	0.097	0.167	0.247	0.009	1.619	1.690	1.770	1.533
95.76	0.355	0.464	0.574	0.223	0.809	0.919	1.029	0.679
94.70	0.901	1.007	1.113	0.850	0.301	0.407	0.512	0.250
93.65	1.721	1.784	1.858	1.732	0.077	0.139	0.213	0.089
Option Price 12 month								
Str.	Call Option				Put Option			
	1D-Mod	**2D-Mod**	**3D-Mod**	**BDT**	**1D-Mod**	**2D-Mod**	**3D-Mod**	**BDT**
98.39	0.043	0.091	0.155	0.000	2.369	2.418	2.482	2.327
97.54	0.131	0.215	0.310	0.003	1.628	1.712	1.806	1.497
96.70	0.330	0.444	0.562	0.101	0.997	1.111	1.229	0.768
95.87	0.682	0.806	0.929	0.473	0.530	0.652	0.776	0.320
95.05	1.201	1.304	1.413	1.085	0.238	0.341	0.451	0.122

instance, we propose to use a well-known econometric model motivated by simplicity and parsimony: Gaussian Affine term Structure Model. Mathematically, bond prices in the model are derived by solving the Feynman-Kac partial differential equation. These models owe their popularity to the fact that this partial differential equation decomposes into a system of ordinary differential equations, which can be very well approximated in discrete time by a state space representation, even with a large number of dimensions. In the paper, we use options on zero bond for which analytical solutions in all these models exist and vary only the maturity of the options and the yields returns.

In the paper, we apply the econometric models to a historical data set of sovereign bond prices in Uruguay. We attempt to analyze and discuss the nature of the models' adjustments and the pricing options. We applied the models to two different currencies: the dollar and the CPI-Indexed to test the results. The parameters of the models were estimated using daily data of bonds traded in the market. We implemented a procedure for maximum likelihood estimation where the function is known in closed form. However, we propose an Expectation Maximization algorithm based on the Kalman Filter for the parameter estimator.

Table 5. Call and Put Option in CPI-Indexed market

Option Price 3 month								
Str.	Put Option				Op. de Venta			
	1D-Mod	**2D-Mod**	**3D-Mod**	**BDT**	**1D-Mod**	**2D-Mod**	**3D-Mod**	**BDT**
92.79	0.003	0.063	0.088	0.003	1.631	1.692	1.717	1.631
91.96	0.048	0.204	0.248	0.053	0.861	1.018	1.062	0.867
91.14	0.284	0.506	0.558	0.316	0.291	0.513	0.565	0.309
90.33	0.842	0.999	1.043	0.874	0.049	0.206	0.249	0.081
89.52	1.589	1.649	1.675	1.601	0.003	0.064	0.089	0.016
Option Price 6 month								
Str.	Call Option				Put Option			
	1D-Mod	**2D-Mod**	**3D-Mod**	**BDT**	**1D-Mod**	**2D-Mod**	**3D-Mod**	**BDT**
93.75	0.021	0.159	0.203	0.012	1.415	1.5553	1.597	1.382
93.03	0.106	0.334	0.393	0.095	0.799	1.028	1.086	0.599
92.32	0.348	0.619	0.682	0.349	0.346	0.617	0.681	0.351
91.61	0.796	1.023	1.080	0.807	0.104	0.331	0.388	0.115
90.91	1.397	1.532	1.576	1.403	0.020	0.156	0.199	0.026
Option Price 12 month								
Str.	Call Option				Put Option			
	1D-Mod	**2D-Mod**	**3D-Mod**	**BDT**	**1D-Mod**	**2D-Mod**	**3D-Mod**	**BDT**
95.75	0.052	0.233	0.283	0.002	1.012	1.193	1.243	0.958
95.26	0.144	0.379	0.436	0.107	0.634	0.869	0.927	0.383
94.77	0.321	0.579	0.639	0.292	0.344	0.602	0.663	0.269
94.28	0.598	0.835	0.893	0.589	0.156	0.394	0.452	0.148
93.79	0.962	1.147	1.197	0.969	0.058	0.243	0.293	0.065

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