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Diffusion spiders: Green kernel, excessive functions and optimal stopping

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Abstract

A diffusion spider is a strong Markov process with continuous paths taking values on a graph with one vertex and a finite number of edges (of infinite length). An example is Walsh's Brownian spider where the process on each edge behaves as a Brownian motion. In this paper we calculate firstly the density of the resolvent kernel in terms of the characteristics of the underlying diffusion. Excessive functions are studied via the Martin boundary theory. A crucial result is an expression for the representing measure of a given excessive function. These results are used to solve optimal stopping problems for diffusion spiders.
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1. Introduction

Diffusion on graphs has been a topic of intense research in the past decades. One of the earliest papers in this direction is by Walsh [37], where he introduces a generalization of the skew Brownian motion, later called the Walsh Brownian motion. This process lives on the half lines in $\mathbb{R}^2$ with only one common point, the origin of the plane, for instance. On the rays (or

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legs) the process behaves like a Brownian motion, and when this Brownian motion hits the origin, it chooses, roughly speaking, a new ray randomly with an angle having a probability distribution defined on $[0, 2\pi)$. A special case is when this distribution consists of finitely many atoms. If n is the number of atoms then the process is called a Brownian (or Walsh) spider with n legs. The Brownian spider can be seen as a diffusion on a star shaped graph.

Salisbury [32] gives a formal construction of the skew Brownian motion (the particular case of a spider with two legs) based on the excursion theory of the right processes, see also an earlier paper by Rogers [30]. The Walsh Brownian motion was considered by Barlow, Pitman and Yor in [2]. They construct the process using the semi-group theory. Freidlin and Wentzell [19] study diffusions on graphs through martingale problems, provide methods to verify the weak convergence of discrete processes to these diffusions, and exhibit several examples. Friedlin and Sheu [18] prove the Itô formula for diffusions on graphs and establish large deviation principles. Papanicolau et al. [26] compute exit probabilities and related quantities for a Brownian spider with equal leg probabilities. This is extended in Fitzsimmons and Kuter [17] for Brownian motion on more general graphs. Kostrykin et al. [22] construct Brownian motion on metric graphs identifying the generator of the process. Fitzsimmons and Kuter study the harmonic functions of the Walsh Brownian motion in [16] and, more generally, Brownian motion on metric graphs in [17].

For a more recent work see the paper by Karatzas and Yan [21] where they generalize the Walsh Brownian motion to a semimartingale framework, provide an Itô type formula and generalize other classical tools of stochastic calculus (as the removal of drift by a change of the measure). They furthermore characterize the solution of an optimal stopping problem in this framework, and discuss also stochastic control problems with discretionary stopping. An optimal stopping problem on a Brownian spider, related to maximal inequalities, has been considered by Ernst [15]. We refer also to Bayraktar and Zhang [4] for an embedding problem for Walsh Brownian motions. For local times and occupation times for random walks and Brownian spiders, see Csáki et al. [10,11]. In Dassios and Zhang [12] further results on hitting times for Brownian spiders can be found, with a potential application in mathematical finance, more precisely related to a liquidity risk problem in a system of banks.

In the present paper, following [2], we consider the case where instead of a Brownian motion on the legs a regular recurrent diffusion is used. We call such a process a *diffusion spider*.

The contribution of the paper is twofold: Firstly, we derive (hereby our basic reference is [30]) a formula for the Green function (resolvent density) of the process. This is used, in particular, to characterize the excessive functions of the diffusion spider. For results on excessive functions for Walsh Brownian motion, see also [4,21]. Secondly, we solve some optimal stopping problems. For the solution of these optimal stopping problems we use two closely related but different methods, inscribed in the so called *representation approach*: (i) when the stopping region is expected to be connected, the candidate for the representing measure of the value function is used in order to find the solution, (ii) when the stopping region is expected to be disconnected, a verification theorem is adapted from [6], based on the notion of the extension of the negative set of the problem (see also [9]). Both methods above require the computation of the Martin kernel, or equivalently the Green kernel. It is very well established that the resolvent kernel (i.e. the integration against the Green function) is the inverse operator to the infinitesimal generator of the process. This second operator is described locally inside the edges as a generalized differential operator obtained from the infinitesimal generator of the underlying diffusion. To complete the description of the infinitesimal generator (and the process) we should determine its domain. In our approach to optimal stopping we use

the relationship between the resolvent operator and the infinitesimal generator as a tool but circumvent the consideration of the domains of the operators.

The rest of the paper is organized as follows. In Section 2 we introduce the diffusion spider and discuss its existence as a strong Markov process. After this, in Section 3 some preliminaries on linear diffusions are presented. In Section 4 the first main result of the paper is given, that is, an expression for the density of the Green kernel of the spider. We proceed then in Section 5 by computing the Laplace transform of a hitting time associated with a diffusion spider. Section 6 contains central main results of the paper concerning the representing measure of a given excessive function. Other characterizations of excessive functions are also discussed. In Section 7 we proceed with general results on optimal stopping, applied to diffusion spiders, adapted from [6,9]. In Section 8 we solve three examples of optimal stopping for a Brownian spider. The first one has a connected stopping region and the solution is obtained by analyzing the representing measure formulas for the candidate value function. The other two examples have disconnected stopping regions and are solved once the existence and the uniqueness of the solution of the associated equation system is established using the results of Section 7. We conclude with four appendices containing the lengthy and technical proofs of some of the results in Section 7; in particular, the fourth one is about the smooth fit principle for diffusion spiders.

2. Definition and existence of a diffusion spider

Let $X = \{X_t : t \geq 0\}$ be a recurrent linear (Feller) diffusion taking values in $[0, \infty)$. It is assumed that 0 is a regular boundary point with reflection, and $+\infty$ is natural (for the boundary classification of a linear diffusion, see, e.g., [5]). The notations $\mathbb{P}_x$ and $\mathbb{E}_x$ are for the probability measure and the expectation operator, respectively, associated with X when initiated at x.

Consider a "star shaped" graph $\Gamma \subset \mathbb{R}^2$ with only one internal vertex taken to be the origin of $\mathbb{R}^2$, and n edges $\ell_1, \ldots, \ell_n$, also called legs, meeting each other the origin $\mathbf{0}$. Hence, the legs are of infinite length. For $x \geq 0$ and $i \in \{1, 2, \ldots, n\}$ we let $\mathbf{x} = (x, i)$ denote the point on Γ located on the leg ℓ_i at the distance x from the origin. For any i the point $(0, i)$ is identified as the origin. The topology on Γ is the relative Euclidean topology, and, hence, Γ is a locally compact Hausdorff space with a countable base.

On the graph Γ we consider a stochastic process for which we use the notation

$$\mathbf{X} := \{\mathbf{X}_t := (X_t, \rho_t) : t \geq 0\},$$

where $\rho_t \in \{1, 2, \ldots, n\}$ indicates the leg on which $\mathbf{X}_t$ is located at time t and X_t is the distance of $\mathbf{X}_t$ to the origin at time t measured along the leg ρ_t. On each edge $i = 1, \ldots, n$ before hitting zero, the process $\mathbf{X}$ behaves like the diffusion X. As a part of the definition of the process there is a vector $\mathbf{p} = (p_1, \ldots, p_n)$ such that $p_i > 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n p_i = 1$. When $\mathbf{X}$ hits $\mathbf{0}$, it continues, roughly speaking, with probability p_i on the leg ℓ_i. In Fig. 1 we have illustrated such a graph with 3 legs.

This process when X is a reflected Brownian motion was introduced by Walsh in [37], and is, hence, in this case, called the Walsh Brownian motion with discrete (finitely many atoms) distribution for the angles. A formal construction of the Walsh Brownian motion based on the semigroup theory was given by Barlow et al. in [2]. We refer to [3] also for earlier references. According to the remark on p. 281 in [2], a similar construction can be carried out in our case where the underlying process is X, i.e., a general recurrent Feller diffusion on $[0, \infty)$ reflected at zero. We call this process a (homogeneous) diffusion spider. It is a strong Markov process

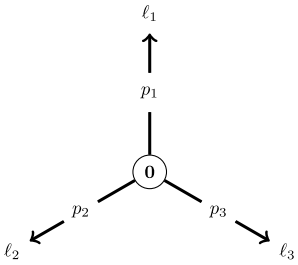


Fig. 1. A star graph with 3-legs.

with a Feller semigroup and continuous sample paths (in the relative Euclidean topology on Γ).

To indicate explicitly the number of legs and the probabilities for the excursions we write $\mathbf{X} = (X, n, \mathbf{p})$. The notations $\mathbb{P}_{(x,i)}$ and $\mathbb{E}_{(x,i)}$ are for the probability measure and the expectation operator, respectively, associated with $\mathbf{X}$ when initiated at (x, i). The term homogeneous refers to the property that on each leg the movement is governed by the same diffusion X. The non-homogeneous case is the one when the diffusion is not the same on each leg. This situation and also more general graphs has been considered in [19].

3. Preliminaries for linear diffusions

Consider the linear diffusion X introduced above. Recall that the generator of X is given in the form

$$\mathcal{G}u := \frac{d}{dm}\frac{d}{dS}u, \tag{1}$$

where m and S denote the speed measure and the scale function of X, respectively, and u is a function in the domain of the generator. Recall that 0 is a regular boundary point with reflection. For simplicity, it is assumed that $m(\{0\}) = 0$. We also put $S(0) = 0$.

In addition to the diffusion X, we need also the diffusion X^∂ with the same speed and scale as X but for which 0 is a killing boundary. For $r > 0$ let φ_r (φ_r^∂) and ψ_r (ψ_r^∂) be the decreasing and the increasing, respectively, fundamental solutions of the generalized differential equation

$$\frac{d}{dm}\frac{d}{dS}f = rf$$

associated with X (X^∂), see, e.g., [5]. These functions are non-negative, continuous and unique up to multiplicative constants. For simplicity, it is moreover assumed that φ_r and ψ_r^∂ are differentiable with respect to S. To shorten the notation, the subindex r is omitted and it is written, e.g., φ instead of φ_r.

Recall also that

$$\psi^\partial(0) = 0, \quad \frac{d\psi}{dS}(0+) = 0, \quad \varphi^\partial \equiv \varphi, \tag{2}$$

and for $x \geq 0$

$$\mathbb{E}_x(\mathrm{e}^{-rH_0}) = \frac{\varphi(x)}{\varphi(0)}, \tag{3}$$

where $H_y := \inf\{t : X_t = y\}$ with $y = 0$. Moreover, we normalize

$$\varphi(0) = \varphi^{\partial}(0) = 1, \quad \text{and} \quad \frac{d\psi^{\partial}}{dS}(0+) = 1, \tag{4}$$

and, consequently, the Wronskian w_r of X^{∂} is given by

$$w_r := \frac{d\psi^{\partial}}{dS}(x)\varphi(x) - \frac{d\varphi}{dS}(x)\psi^{\partial}(x) = \frac{d\psi^{\partial}}{dS}(0+) = 1, \tag{5}$$

where it is used that the Wronskian does not depend on x and that

$$\frac{d\varphi}{dS}(0+) > -\infty, \tag{6}$$

since 0 is assumed to be a regular boundary point. Finally, because $+\infty$ is taken to be natural, it holds

$$\lim_{x\to+\infty} \frac{d\varphi}{dS}(x) = 0. \tag{7}$$

4. Green kernel

Consider now a diffusion spider $\mathbf{X} = (X, n, \mathbf{p})$ with n legs and the corresponding probabilities $p_1, p_2, \ldots, p_n$. Define

$$\mathbf{m}(dx, i) := p_i m(dx), \quad i = 1, \ldots, n, \qquad \mathbf{m}(\{\mathbf{0}\}) := 0,$$
$$\mathbf{S}(dx, i) := \frac{1}{p_i} S(x).$$

We call $\mathbf{m}$ and $\mathbf{S}$ the speed measure and the scale function, respectively, of $\mathbf{X}$. Clearly, on every leg $\ell_i, i = 1, \ldots, n$, of the spider it holds

$$\frac{d}{d\mathbf{m}}\frac{d}{d\mathbf{S}} = \frac{d}{dm}\frac{d}{dS}.$$

The main contribution in this section is an explicit expression of the Green kernel (also called the resolvent kernel) for $\mathbf{X}$ defined as

$$\mathbf{G}_r f(x, i) := \int_0^{\infty} \mathrm{e}^{-rt} \mathbb{E}_{(x,i)} \left(f(\mathbf{X}_t) \right) dt, \tag{8}$$

where $r > 0$ and $f : \Gamma \to \mathbb{R}$ is a bounded measurable function. This is accomplished by finding its density.

Theorem 1. *The Green kernel of the diffusion spider* $\mathbf{X}$ *has a density* $\mathbf{g}_r$ *with respect to the speed measure* $\mathbf{m}$ *which is given for* $x \geq 0$ *and* $y \geq 0$ *by*

$$\mathbf{g}_r((x, i), (y, j)) = \begin{cases} \varphi(y)\tilde{\psi}(x, i), & x \leq y, \ i = j \\ \varphi(x)\tilde{\psi}(y, i), & y \leq x, \ i = j \\ c_r^{-1}\, \varphi(y)\varphi(x), & i \neq j, \end{cases} \tag{9}$$

where

$$\tilde{\psi}(x, i) := \frac{1}{p_i}\psi^{\partial}(x) + \frac{1}{c_r}\varphi(x), \tag{10}$$

and

$$c_r := -\frac{d}{dS}\varphi(0+) > 0. \tag{11}$$

Proof. We apply the strong Markov property of $\mathbf{X}$ at the first hitting time

$$\overline{H}_{\mathbf{0}} := \inf\{t \geq 0 : \mathbf{X}(t) = \mathbf{0}\}$$

of the origin to obtain the following equation for the Green kernel (see Lemma 1 in [30])

$$\mathbf{G}_r f(x,i) = \mathbf{G}_r^{\partial} f(x,i) + \mathbb{E}_{(x,i)}(\mathrm{e}^{-r\overline{H}_{\mathbf{0}}})\,\mathbf{G}_r f(\mathbf{0}), \quad x \geq 0, i \in \{1,2,\ldots,n\}, \tag{12}$$

where $\mathbf{G}_r^{\partial}$ is the Green kernel of $\mathbf{X}$ killed when it hits $\mathbf{0}$ and f is a real-valued, bounded and measurable function defined on Γ. Since $\mathbf{X}$ behaves on the legs before hitting $\mathbf{0}$ as the diffusion X before hitting 0 the Green kernel $\mathbf{G}_r^{\partial}$ coincides on a given leg with the Green kernel of X killed at 0. Hence, for all i

$$\mathbf{G}_r^{\partial} f(x,i) = \int_0^{\infty} g_r^{\partial}(x,y)\, f(y,i)\, m(dy),$$

where

$$g_r^{\partial}(x,y) = \begin{cases} \psi^{\partial}(y)\varphi(x), & 0 \leq y \leq x, \\ \psi^{\partial}(x)\varphi(y), & 0 \leq x \leq y, \end{cases} \tag{13}$$

see [5] p. 19 and recall (5). Consequently, $\mathbf{G}_r^{\partial}$ has a density $\mathbf{g}_r^{\partial}$ (with respect to $\mathbf{m}$) given by

$$\mathbf{g}_r^{\partial}((x,i),(y,j)) = \begin{cases} 0, & i \neq j \\ g_r^{\partial}(x,y)/p_i, & i = j \end{cases} \tag{14}$$

Consider now the second term on the right hand side of (12). Clearly, for all i

$$\mathbb{E}_{(x,i)}(\mathrm{e}^{-r\overline{H}_0}) = \mathbb{E}_x(\mathrm{e}^{-rH_0}) = \varphi(x), \tag{15}$$

where (3) is used. To find $\mathbf{G}_r f(\mathbf{0})$ recall formula (2) in [30] saying that

$$\mathbf{G}_r f(\mathbf{0}) = \frac{\boldsymbol{\nu}_r f}{r\boldsymbol{\nu}_r 1}. \tag{16}$$

Here $\boldsymbol{\nu}_r$ refers for any $r > 0$ to a non-negative σ-finite measure obtained as the Laplace transform of the excursion entrance law associated with $\mathbf{X}$ for excursions starting from $\mathbf{0}$. The excursions from $\mathbf{0}$ of the diffusion spider take place on the legs, and we let (ξ, i) denote such an excursion on the ith leg with ξ as an element in the space

$$\begin{aligned} U := \{\xi : [0,\infty) \to [0,\infty) \,:\, &\xi \text{ continuous, } \xi(0) = 0, \\ &\text{and } \exists\, \zeta(\xi) > 0 \text{ such that } \xi(0) > 0 \text{ for } 0 < t < \zeta(\xi) \text{ and} \\ &\xi(t) = 0 \text{ for } t \geq \zeta(\xi)\}. \end{aligned}$$

Let ν denote the Itô excursion measure of the linear diffusion X for excursions starting from 0, see Revuz and Yor [29] Chapter XII, that is

$$\nu^t(A) := \nu(\{\xi \in U : \xi(t) \in A, 0 < t < \zeta(\xi)\}), \;\; A \in \mathcal{B}([0,\infty)),$$

and it holds (see Rogers and Williams [31] and Salminen, Vallois and Yor [35])

$$\nu^t(A) = \int_A h(t,y)\, m(dy),$$

where

$$h(t, y) := \mathbb{P}_y(H_0 \in dt)/dt.$$

For the Laplace transform of $\nu^t(A)$ we have

$$\nu_r(A) := \int_0^\infty \mathrm{e}^{-rt}\nu^t(A)\,dt = \int_A m(dy)\int_0^\infty dt\,\mathrm{e}^{-rt}h(t, y) = \int_A \varphi(y)\,m(dy).$$

Let $\overline{A} := \cup_{i=1}^n (A_i, i)$ with $A_i \in \mathcal{B}([0,\infty))$, $i = 1, 2, \ldots, n$, be a subset of Γ. Then the excursion entrance law $\boldsymbol{\nu}^t$ of $\mathbf{X}$ has the representation

$$\boldsymbol{\nu}^t(\overline{A}) = \sum_{i=1}^n p_i\,\nu^t(A_i).$$

Consequently,

$$\boldsymbol{\nu}_r(\overline{A}) := \sum_{i=1}^n p_i\,\nu_r(A_i) = \sum_{i=1}^n p_i \int_{A_i} \varphi(y)\,m(dy),$$

and putting here $A_i = (0,\infty)$, $i = 1, 2, \ldots, n$, yields

$$\begin{aligned} r\boldsymbol{\nu}_r 1 &= r\sum_{i=1}^n p_i \int_0^\infty \varphi(y)m(dy) = r\int_0^\infty \varphi(y)m(dy) \\ &= \int_0^\infty \frac{d}{dm}\frac{d}{dS}\varphi(y)m(dy) = -\frac{d}{dS}\varphi(0+) = c_r, \end{aligned}$$

where (4) and (7) are used. Hence, for the real-valued, bounded and measurable function f defined on Γ we may now write, cf. (16),

$$\mathbf{G}_r f(\mathbf{0}) = \frac{1}{c_r}\sum_{i=1}^n p_i \int_0^\infty \varphi(y) f(y, i) m(dy),$$

and the density with respect to $\mathbf{m}(dx, i)$ is then

$$\mathbf{g}_r(\mathbf{0}, (y, i)) = \frac{1}{c_r}\varphi(y), \tag{17}$$

which does not depend on i. Combining (14), (15), and (17) results to the formula

$$\mathbf{g}_r((x, i), (y, j)) = \frac{1}{p_j}\psi^\partial(x)\varphi(y)\mathbf{1}_{\{i=j\}} + \frac{1}{c_r}\varphi(x)\varphi(y),$$

concluding the proof. □

It is interesting and useful to note that the function $\tilde{\psi}$ appearing in the formula for the Green function is increasing and can be seen as the increasing fundamental solution associated with a skew diffusion. This is made precise in the next proposition and its proof.

Proposition 1. *For all $i = 1, \ldots, n$, the function*

$$\tilde{\psi}(x) := \tilde{\psi}(x, i) = \frac{1}{p_i}\psi^\partial(x) + \frac{1}{c_r}\varphi(x), \quad x \geq 0,\ 0 < p_i < 1,$$

is positive and increasing.

Proof. The positivity of $\tilde{\psi}$ is immediate from the definitions of ψ^{∂} and φ. To prove that $\tilde{\psi}$ is increasing let i be fixed and consider a diffusion spider with 2 legs with $\mathbf{p} = (p_i, 1-p_i)$. This is probabilistically equivalent with a skew diffusion on $\mathbb{R}$ when we identify, for instance, the leg 2 with the negative half line. Let $X^{(sk)}$ denote this diffusion with the fundamental decreasing and increasing solutions $\varphi^{(sk)}$ and $\psi^{(sk)}$, respectively. Using the Green function given in Theorem 1 we can deduce the following expressions for $\varphi^{(sk)}$ and $\psi^{(sk)}$

$$\varphi^{(sk)}(x) = \begin{cases} c_r \tilde{\psi}(-x, 2), & x \le 0, \\ \varphi(x), & x \ge 0, \end{cases} \qquad \psi^{(sk)}(x) = \begin{cases} \varphi(-x), & x \le 0, \\ c_r \tilde{\psi}(x, 1), & x \ge 0, \end{cases} \tag{18}$$

where

$$\tilde{\psi}(x, 1) = \frac{1}{p_i}\psi^{\partial}(x) + \frac{1}{c_r}\varphi(x).$$

Because $\psi^{(sk)}$ is increasing so is also $\tilde{\psi}$. □

Remark 1. We show here how to verify that $\tilde{\psi}$ is increasing with elementary calculations in the special case when m and S have positive and continuous derivatives. The task is to check that

$$\frac{d\tilde{\psi}}{dS}(x) = \frac{1}{S'(x)}\frac{d\tilde{\psi}}{dx}(x) > 0. \tag{19}$$

For this, consider

$$\frac{d\tilde{\psi}}{dS}(x) = \frac{1}{p_i}\frac{d\psi^{\partial}}{dS}(x) + \frac{1}{c_r}\frac{d\varphi}{dS}(x) = \frac{1}{p_i}\frac{d\psi^{\partial}}{dS}(x) - \frac{1}{\frac{d\varphi}{dS}(0+)}\frac{d\varphi}{dS}(x) \tag{20}$$

Taking limit as $x \downarrow 0$, we obtain

$$\frac{d\tilde{\psi}}{dS}(0+) = \frac{1}{p_i}\frac{d\psi^{\partial}}{dS}(0+) - \frac{1}{\frac{d\varphi}{dS}(0+)}\frac{d\varphi}{dS}(0+) = \frac{1}{p_i} - 1 > 0.$$

Differentiate with respect to m in (20) to obtain

$$\frac{d}{dm}\frac{d\tilde{\psi}}{dS}(x) = \frac{1}{p_i}\frac{d}{dm}\frac{d\psi^{\partial}}{dS}(x) + \frac{1}{c_r}\frac{d}{dm}\frac{d\varphi}{dS}(x) = \frac{r}{p_i}\psi^{\partial}(x) + \frac{r}{c_r}\varphi(x) > 0.$$

This means that $x \mapsto \frac{d\tilde{\psi}}{dS}(x)$ is increasing. Therefore,

$$0 < \frac{d\tilde{\psi}}{dS}(0+) < \frac{d\tilde{\psi}}{dS}(x),$$

and the inequality in (19) follows. Hence, $\frac{d\tilde{\psi}}{dx}(x) > 0$ since $S'(x) > 0$ by the assumption.

Example 1. Consider a Brownian spider $\mathbf{X}$ with n legs and the corresponding probabilities $p_1, p_2, \ldots, p_n$. The underlying diffusion X is a reflecting Brownian motion and the scale function and speed measure of X are $S(x) = x$ and $m(dx) = 2dx$. Moreover,

$$\varphi(x) = \varphi^{\partial}(x) = \mathrm{e}^{-\theta x}, \qquad \psi(x) = \frac{1}{\theta}\cosh(\theta x), \qquad \psi^{\partial}(x) = \frac{1}{\theta}\sinh(\theta x),$$

where $\theta = \sqrt{2r}$. These functions meet the normalizations made in (4). We have

$$c_r = -\frac{d}{dS}\varphi(0+) = \theta,$$

and, hence,

$$\tilde{\psi}(x,i) = \frac{1}{p_i}\frac{1}{\theta}\sinh(\theta x) + \frac{1}{\theta}\mathrm{e}^{-\theta x}. \tag{21}$$

The Green function, see Theorem 1, is in this case given by

$$\mathbf{g}_r((x,i),(y,j)) = \begin{cases} \frac{1}{\theta}\mathrm{e}^{-\theta x}\left(\frac{1}{p_i}\sinh(\theta y) + \mathrm{e}^{-\theta y}\right), & 0 \le y \le x,\ i = j, \\ \frac{1}{\theta}\mathrm{e}^{-\theta y}\left(\frac{1}{p_i}\sinh(\theta x) + \mathrm{e}^{-\theta x}\right), & 0 \le x \le y,\ i = j, \\ \frac{1}{\theta}\mathrm{e}^{-\theta x}\mathrm{e}^{-\theta y}, & i \neq j. \end{cases} \tag{22}$$

Notice that $\mathbf{g}_r(\mathbf{0},\mathbf{0}) = 1/\theta$ and that the Green function is continuous everywhere (and in particular at $\mathbf{0}$). It is possible to obtain the density function of the Brownian spider with respect to the speed measure $\mathbf{m}$ by inverting the Laplace transforms above. In fact, as the Green kernel is a linear combination of the Green kernels of the killed BM and the standard BM, the corresponding densities can be obtained through the same linear combination. We obtain, when $i \neq j$, that

$$p(t;(x,i),(y,j)) = \frac{2p_j}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x+y)^2}{2t}}, \qquad i \neq j,$$

and for $i = j$

$$p(t;(x,i),(y,i)) = \frac{1}{\sqrt{2\pi t}}\left(\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \mathrm{e}^{-\frac{(x+y)^2}{2t}}\right) + \frac{2p_i}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x+y)^2}{2t}}.$$

5. Hitting times

Let $\mathbf{X} = (X, n, \mathbf{p})$ be a homogeneous diffusion spider and $\mathbf{y} = (y, 1)$ a fixed point on Γ. Define the hitting time

$$H_{\mathbf{y}} = \inf\{t \ge 0 : \mathbf{X}_t = \mathbf{y}\}.$$

The objective here is find $\mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right)$, the Laplace transform of $H_{\mathbf{y}}$. Let $\mathbf{x} = (x,i)$ be the starting state of $\mathbf{X}$ and assume first that $i \neq 1$. In this case we observe that by the strong Markov property it holds

$$\mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right) = \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r(H_{\mathbf{y}}\circ\theta_{H_{\mathbf{0}}}+H_{\mathbf{0}})}\right) = \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-rH_{\mathbf{0}}}\right)\mathbb{E}_{\mathbf{0}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right) = \frac{\varphi(x)}{\varphi(0)}\mathbb{E}_{\mathbf{0}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right),$$

where θ_t denotes the usual shift operator. To calculate $\mathbb{E}_{\mathbf{0}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right)$ we consider an auxiliary skew diffusion $X^{(sk)}$ defined on $\mathbb{R}$ (see the proof of Proposition 1), such that

$$\mathbb{P}_0(X_t^{(sk)} \le 0) = 1 - p_1, \qquad \mathbb{P}_0(X_t^{(sk)} \ge 0) = p_1.$$

Then, since the spider diffusion behaves as X on each leg before hitting $\mathbf{0}$ we have

$$\mathbb{E}_{\mathbf{0}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right) = \mathbb{E}_0^{(sk)}\left(\mathrm{e}^{-rH_y}\right) = \frac{\psi^{(sk)}(0)}{\psi^{(sk)}(y)}$$

where $\psi^{(sk)}$ is the increasing fundamental solution associated with the skew diffusion $X^{(sk)}$. We conclude that

$$\mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-rH_{\mathbf{y}}}\right)=\begin{cases}\dfrac{\varphi(x)}{\varphi(0)}\dfrac{\psi^{(sk)}(0)}{\psi^{(sk)}(y)}, & 0\le x \text{ on leg } i\neq 1,\\ \dfrac{\psi^{(sk)}(x)}{\psi^{(sk)}(y)}, & 0\le x\le y \text{ on leg } 1,\\ \dfrac{\varphi(x)}{\varphi(y)}, & 0\le y\le x \text{ on leg } 1.\end{cases} \tag{23}$$

Recall that $\psi^{(sk)}$ can be expressed in terms of the functions ψ^{∂} and φ as displayed in (18) taken therein $i=1$.

Example 2. For a Brownian spider we can write the Laplace transform in (23) explicitly using

$$\varphi(x)=\mathrm{e}^{-\theta x},\quad x>0,$$

$$\psi^{(sk)}(x)=\frac{1-2p_1}{p_1}\sinh(\theta x)+\mathrm{e}^{\theta x},\quad x>0,$$

where φ ($\psi^{(sk)}$) are, for $x>0$, the decreasing (the increasing) fundamental solution for the skew Brownian motion with parameter $\beta=p_1$ normalized as in [5] A1.12 p. 130. Observe that

$$\psi^{(sk)}(x)=\frac{1-2p_1}{p_1}\sinh(\theta x)+\mathrm{e}^{\theta x}=\frac{1}{p_1}\sinh(\theta x)-2\sinh(\theta x)+\mathrm{e}^{\theta x}=\frac{1}{p_1}\sinh(\theta x)+\mathrm{e}^{-\theta x}$$

and, hence, under this normalization, $\psi^{(sk)}$ equals with the function $\tilde{\psi}$ appearing in (18) when $i=1$.

6. Excessive functions

Recall that for a given $r\ge 0$ the measurable function $f:\Gamma\to[0,\infty)\cup\{+\infty\}$ is called r-excessive for $\mathbf{X}$ if for all $(x,i)\in\Gamma$

$$\mathrm{e}^{-rt}\mathbb{E}_{(x,i)}\left(f(\mathbf{X}_t)\right)\le f(x,i) \tag{24}$$

and

$$\lim_{t\to 0}\mathrm{e}^{-rt}\mathbb{E}_{(x,i)}\left(f(\mathbf{X}_t)\right)\to f(x,i). \tag{25}$$

Since $\mathbf{X}$ is regular in the sense that every point of Γ is hit with a positive probability it follows from (24) that if $(x,i)\in\Gamma$ is such that for an excessive function f it holds $f(x,i)=0$ ($f(x,i)=+\infty$) then $f\equiv 0$ ($f\equiv+\infty$). These functions are called trivial r-excessive functions.

We study the non-trivial r-excessive functions of $\mathbf{X}$ via the Martin boundary theory. Using the Martin metric induced by the Green function $\mathbf{g}_r$ the state space Γ of $\mathbf{X}$ is compactified to $\bar{\Gamma}:=\Gamma\cup\{(\infty,i):\ i=1,2,\dots,n\}$ (the Martin compactification). For every $(y,k)\in\bar{\Gamma}$ define

an r-excessive function by

$$(x,i)\mapsto \mathbf{u}_r((x,i);(y,k)) := \begin{cases} \dfrac{\mathbf{g}_r((x,i),(y,k))}{\mathbf{g}_r(\mathbf{0},(y,k))}, & 0\le y<\infty, \\ \lim\limits_{z\to\infty}\dfrac{\mathbf{g}_r((x,i),(z,k))}{\mathbf{g}_r(\mathbf{0},(z,k))}, & y=\infty. \end{cases}$$

These functions are called the minimal r-excessive functions of $\mathbf{X}$ attaining the value 1 at $\mathbf{0}$. Let σ be a probability measure on Γ and define for $(x,i)\in\Gamma$

$$h(x,i;\sigma) := \sum_{k=1}^{n}\int_{(0,\infty]}\mathbf{u}_r((x,i);(y,k))\,\sigma(dy,k) + \mathbf{u}_r((x,i);\mathbf{0})\,\sigma(\{\mathbf{0}\}). \tag{26}$$

Notice that $h(\mathbf{0};\sigma)=1$ since σ is assumed to be a probability measure. It can be proved directly that h is r-excessive for $\mathbf{X}$ but this also follows from the Martin boundary theory. In fact, there is a one-to-one correspondence between the probability measures on $\bar{\Gamma}$ and the non-trivial r-excessive functions of $\mathbf{X}$. For this, see [23] Theorem 4 p. 513 which is proved under Hypothesis (B) p. 498 valid in our case since the Green function is symmetric (and, hence, $\mathbf{X}$ is self-dual).

Consider next

$$\mathbf{u}_r((x,i);(\infty,k)) = \lim_{z\to\infty}\frac{\mathbf{g}_r((x,i),(z,k))}{\mathbf{g}_r(\mathbf{0},(z,k))} = \begin{cases} c_r\tilde{\psi}(x,i), & i=k, \\ \varphi(x), & i\neq k, \end{cases} \tag{27}$$

where $\tilde{\psi}$ is given in (10). The function $(x,i)\mapsto \mathbf{u}_r((x,i);(\infty,k))$ is r-invariant (and also r-harmonic) for $\mathbf{X}$ since its representing measure is supported by the infinite part of the Martin compactification. Moreover, all (positive) r-harmonic functions can be written as linear combinations with non-negative coefficients of $\mathbf{u}_r((x,i);(\infty,k))$, $k=1,2,\dots,n$. Therefore, given a positive r-harmonic function H there exists $a_i\ge 0$, $i=1,2,\dots,n$, such that

$$\begin{aligned} H(x,i) &= \sum_{k=1}^{n} a_k\,\mathbf{u}_r((x,i);(\infty,k)) = a_i c_r\tilde{\psi}(x,i) + \sum_{k=1,k\neq i}^{n} a_k\varphi(x) \\ &= \frac{a_i c_r}{p_i}\psi^{\partial}(x) + \varphi(x)\sum_{k=1}^{n} a_k. \end{aligned} \tag{28}$$

Clearly, $(\mathrm{e}^{-rt}H(\mathbf{X}_t))_{t\ge 0}$ is a positive martingale and $H(\mathbf{0}) = a_1+\cdots+a_n$. Notice also that since both ψ^{∂} and φ are r-harmonic for X^{∂}, it follows that H when restricted on the leg i is r-harmonic for X^{∂} on that leg. However, there are harmonic functions of X^{∂} which are not harmonic for $\mathbf{X}$, e.g., ψ^{∂}. In Theorem 5 this issue is taken up more in detail.

Example 3. In the particular case of the Brownian spider, the general form of a harmonic function is

$$H(x,i) = \frac{a_i c_r}{p_i}\psi^{\partial}(x) + \varphi(x)\sum_{k=1}^{n} a_k = \frac{a_i}{p_i}\sinh(\theta x) + \mathrm{e}^{-\theta x}\sum_{k=1}^{n} a_k, \tag{29}$$

since

$$\varphi(x) = \mathrm{e}^{-\theta x}, \qquad \psi^{\partial}(x) = \frac{1}{\theta}\sinh(\theta x), \qquad c_r = -\frac{d}{dx}\varphi(0+) = \theta = \sqrt{2r}.$$

In the next theorem formula (30) states a fundamental property of r-excessive functions found earlier in the literature, see, for instance, Freidlin and Sheu [18] p. 183 (where (30) is called the glueing condition), Theorem 4.1 in Fitzsimmons and Kuter [16], and formula (5.7) in Karatzas and Yan [21]. In [16,21] a more general process with uncountable many rays is considered. The proof below is based on the Martin representation (26) and extends the proof from the one-dimensional case as given in Salminen [33] (3.7) Corollary.

Theorem 2. *Let f be a non-trivial r-excessive function of $\mathbf{X}$. Assume that $f(\mathbf{0}) = 1$ and let σ be the probability measure such that the right hand side of (26) represents f. Then*

$$f^+(\mathbf{0}, i) := \lim_{\delta \downarrow 0} \frac{f(\delta, i) - f(\mathbf{0})}{S(\delta) - S(0)}$$

exists for every $i = 1, 2, \dots, n$, and

$$\mathbf{D}f(\mathbf{0}) := \sum_{i=1}^{n} p_i f^+(\mathbf{0}, i) \leq 0. \tag{30}$$

Moreover, for $x > 0$ and $i = 1, 2, \dots, n$

$$f^-(x, i) := \lim_{\delta \downarrow 0} \frac{f(x, i) - f(x - \delta, i)}{S(x) - S(x - \delta)} \quad \text{and} \quad f^+(x, i) := \lim_{\delta \downarrow 0} \frac{f(x + \delta, i) - f(x, i)}{S(x + \delta) - S(x)}.$$

exist, and satisfy

$$f^+(x, i) - f^-(x, i) = -\frac{c_r}{p_i} \frac{\sigma(\{x\}, i)}{\varphi(x)} \leq 0. \tag{31}$$

Proof. We assume that (26) holds and write

$$\begin{aligned} f(x, i) = &\int_0^{+\infty} \frac{\mathbf{g}_r((x, i), (y, i))}{\mathbf{g}_r(\mathbf{0}, (y, i))} \sigma(dy, i) \\ &+ \sum_{k \neq i} \int_0^{+\infty} \frac{\mathbf{g}_r((x, i), (y, k))}{\mathbf{g}_r(\mathbf{0}, (y, k))} \sigma(dy, k) + \frac{\mathbf{g}_r((x, i), \mathbf{0})}{\mathbf{g}_r(\mathbf{0}, \mathbf{0})} \sigma(\mathbf{0}) \\ =: &\, \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

Consider first Δ_1. Plug in the explicit expression for the resolvent density (Green function) from (9) and recall $\varphi(0) = 1$ to obtain

$$\begin{aligned} \Delta_1 &= \int_{(0,x]} \frac{\left(\frac{1}{p_i}\psi^\partial(y) + \frac{1}{c_r}\varphi(y)\right)\varphi(x)}{\frac{1}{c_r}\varphi(y)} \sigma(dy, i) + \int_{(x,\infty)} \frac{\left(\frac{1}{p_i}\psi^\partial(x) + \frac{1}{c_r}\varphi(x)\right)\varphi(y)}{\frac{1}{c_r}\varphi(y)} \sigma(dy, i) \\ &= \frac{c_r}{p_i} \left(\varphi(x) \int_{(0,x]} \frac{\psi^\partial(y)}{\varphi(y)} \sigma(dy, i) + \psi^\partial(x)\sigma((x, \infty), i) \right) + \varphi(x)\sigma((0, x], i) \\ &\quad + \varphi(x)\sigma((x, \infty), i). \end{aligned}$$

For Δ_2 it holds

$$\Delta_2 = \sum_{k \neq i} \int_0^{+\infty} \frac{\frac{1}{c_r}\varphi(y)\varphi(x)}{\frac{1}{c_r}\varphi(y)} \sigma(dy, k) = \varphi(x) \sum_{k \neq i} \sigma((0, \infty), k),$$

and for Δ_3

$$\Delta_3 = \frac{\frac{1}{c_r}\varphi(x)}{\frac{1}{c_r}}\sigma(\mathbf{0}) = \varphi(x)\sigma(\mathbf{0}).$$

Consequently, since σ is a probability measure, we have

$$f(x,i) = \Delta_1 + \Delta_2 + \Delta_3 = \frac{c_r}{p_i}\varphi(x)v(x,i) + \varphi(x), \tag{32}$$

where

$$v(x,i) := \int_{(0,x]} \frac{\psi^\partial(y)}{\varphi(y)}\,\sigma(dy,i) + \frac{\psi^\partial(x)}{\varphi(x)}\sigma((x,\infty),i) \tag{33}$$

We prove next that $f^+(\mathbf{0},i)$ exists and find its value. For this it is enough to calculate

$$v^+(0,i) := \lim_{\delta\downarrow 0}\frac{v(\delta,i)-v(0,i)}{S(\delta)-S(0)} = \lim_{\delta\downarrow 0}\frac{v(\delta,i)}{S(\delta)},$$

where we used $v(0,i)=0$ and $S(0)=0$. Consider now the limit of the first term on the right hand side of (33):

$$0 \le \frac{1}{S(\delta)}\int_{(0,\delta]}\frac{\psi^\partial(y)}{\varphi(y)}\,\sigma(dy,i) \le \frac{1}{S(\delta)}\frac{\psi^\partial(\delta)}{\varphi(\delta)}\sigma((0,\delta],i) \to 0,$$

as $\delta\to 0$, since (cf. (5))

$$\frac{1}{S(\delta)}\frac{\psi^\partial(\delta)}{\varphi(\delta)} \to \frac{w_r}{\varphi(0)^2} = 1,$$

and $\sigma((0,\delta),i)\to 0$. Hence, $v^+(0,i)=\sigma((0,\infty),i)$ and we have (cf. (11))

$$f^+(\mathbf{0},i) = \frac{c_r}{p_i}\sigma((0,\infty),i) + \frac{d\varphi}{dS}(0) = \frac{c_r}{p_i}\sigma((0,\infty),i) - c_r. \tag{34}$$

Consequently,

$$\begin{aligned}\mathbf{D}f(\mathbf{0}) &= \sum_{i=1}^n p_i f^+(\mathbf{0},i) = \sum_{i=1}^n\left(p_i\left(\frac{c_r}{p_i}\sigma((0,\infty),i) - c_r\right)\right)\\ &= c_r\sum_{i=1}^n(\sigma((0,\infty),i) - p_i)\\ &= -c_r\left(1 - \sum_{i=1}^n\sigma((0,\infty),i)\right)\\ &= -c_r\,\sigma(\{\mathbf{0}\}) \le 0,\end{aligned} \tag{35}$$

where it is used that σ is a probability measure. This proves (30). Statement (31) can be verified with similar calculations. We skip the details. □

In the next two theorems, the first one with some overlapping with Theorem 2, we give explicit forms of the representation measure of a given excessive function. These formulas are our key tools in solving optimal stopping problems for diffusion spiders. In the first theorem we consider excessive functions taking value 1 at the origin $\mathbf{0}$. In the second theorem general formulas are presented when the value 1 is attained at some other point on Γ.

Theorem 3. *Let f be a non-trivial r-excessive function of $\mathbf{X}$ such that $f(\mathbf{0}) = 1$, and σ its representing measure. Then for all $x \geq 0$ and $i = 1, 2, \ldots, n$*

$$\sigma((x,\infty), i) = \frac{p_i}{c_r}\left(f^+(x,i)\varphi(x) - \varphi^+(x)f(x,i)\right), \tag{36}$$

$$\sigma(\{x\}, i) = \frac{p_i}{c_r\varphi(x)}\left(f^-(x,i) - f^+(x,i)\right), \tag{37}$$

and

$$\sigma(\{\mathbf{0}\}) = -\frac{1}{c_r}\sum_{i=1}^{n} p_i f^+(\mathbf{0}, i) = -\frac{1}{c_r}\mathbf{D}f(\mathbf{0}). \tag{38}$$

Proof. Recall that f has the representation (32). Our first task is, hence, to find the derivative of v defined as

$$v^+(x,i) := \lim_{\delta\downarrow 0}\frac{v(x+\delta,i) - v(x,i)}{S(x+\delta) - S(x)}.$$

It holds for $\delta \geq 0$

$$\begin{aligned} &v(x+\delta,i) - v(x,i)\\ &\quad= \int_{(x,x+\delta]}\frac{\psi^\partial(y)}{\varphi(y)}\sigma(dy,i) + \frac{\psi^\partial(x+\delta)}{\varphi(x+\delta)}\sigma((x+\delta,\infty),i) - \frac{\psi^\partial(x)}{\varphi(x)}\sigma((x,\infty),i). \end{aligned}$$

Since

$$\lim_{\delta\downarrow 0}\frac{1}{S(x+\delta) - S(x)}\left(\int_{(x,x+\delta]}\frac{\psi^\partial(y)}{\varphi(y)}\sigma(dy,i) - \frac{\psi^\partial(x)}{\varphi(x)}\sigma((x,x+\delta],i)\right) = 0,$$

we have

$$\begin{aligned} v^+(x,i) &= \lim_{\delta\downarrow 0}\frac{1}{S(x+\delta) - S(x)}\left(\frac{\psi^\partial(x+\delta)}{\varphi(x+\delta)} - \frac{\psi^\partial(x)}{\varphi(x)}\right)\sigma((x+\delta,\infty),i)\\ &= \frac{1}{(\varphi(x))^2}\sigma((x,\infty),i), \end{aligned}$$

where (5) is used. Consequently, cf. (32),

$$\begin{aligned} f^+(x,i) &= \frac{c_r}{p_i}\varphi^+(x)v(x,i) + \frac{c_r}{p_i}\varphi(x)v^+(x,i) + \varphi^+(x)\\ &= \frac{c_r}{p_i}\varphi^+(x)v(x,i) + \frac{c_r}{p_i}\frac{\sigma((x,\infty),i)}{\varphi(x)} + \varphi^+(x), \end{aligned}$$

and (36) results after a straightforward computation. Recalling that $\varphi(0) = 1$, $c_r = -\varphi^+(0)$, and $f(\mathbf{0}) = 1$ formula (38) is obtained from (36) when taking $x = 0$, also cf. (34) and (35). Finally, (38) follows directly from (35). □

We consider now an r-excessive function f taking value 1 at a point $\mathbf{x}_o = (x_o, i_o) \in \Gamma\setminus\{\mathbf{0}\}$. Then there exists a probability measure σ_{x_o} such that for all $(x,i) \in \Gamma$ it holds

$$f(x,i) = \sum_{k=1}^{n}\int_0^{+\infty}\frac{\mathbf{g}_r((x,i),(y,k))}{\mathbf{g}_r(\mathbf{x}_o,(y,k))}\sigma_{x_o}(dy,k) + \frac{\mathbf{g}_r((x,i),\mathbf{0})}{\mathbf{g}_r(\mathbf{x}_o,\mathbf{0})}\sigma_{x_o}(\{\mathbf{0}\}). \tag{39}$$

Theorem 4. *Let f be an r-excessive function with the representation as in (39). Then for $x \geq x_o$ on the leg i_o*

$$\sigma_{\mathbf{x}_o}((x,\infty),i_o) = p_{i_o}\tilde{\psi}(\mathbf{x}_o)\left(f^+(x,i_o)\varphi(x) - \varphi^+(x)f(x,i_o)\right), \tag{40}$$

and for $0 \leq x \leq x_o$

$$\begin{aligned}\sigma_{\mathbf{x}_o}((0,x),i_o) + \sigma_{\mathbf{x}_o}(\{\mathbf{0}\}) + \sum_{k\neq i_0}\sigma_{\mathbf{x}_o}((0,\infty),k)\\ = p_{i_o}\varphi(x_o)\left(f(x,i_o)\tilde{\psi}^-(x,i_o) - f^-(x,i_o)\tilde{\psi}(x,i_o)\right).\end{aligned} \tag{41}$$

Proof. Let $x \geq x_o$ on the leg i_o. Then

$$\begin{aligned}f(x,i_0) &= \int_0^{+\infty}\frac{\mathbf{g}_r((x,i_0),(y,i_0))}{\mathbf{g}_r(\mathbf{x}_o,(y,i_0))}\sigma_{\mathbf{x}_o}(dy,i_0) + \sum_{k\neq i_0}\int_0^{+\infty}\frac{\mathbf{g}_r((x,i),(y,k))}{\mathbf{g}_r(\mathbf{x}_o,(y,k))}\sigma_{\mathbf{x}_o}(dy,k)\\ &\quad + \frac{\mathbf{g}_r((x,i_0),\mathbf{0})}{\mathbf{g}_r(\mathbf{x}_o,\mathbf{0})}\sigma_{\mathbf{x}_o}(\{\mathbf{0}\}) := \Delta_1 + \Delta_2 + \Delta_3.\end{aligned}$$

Using (9), one readily verifies that

$$\begin{aligned}\Delta_1 &= \frac{\varphi(x)}{\varphi(x_0)}\sigma_{x_0}((0,x_0],i_0) + \int_{(x_0,x]}\frac{\tilde{\psi}(y,i_0)\varphi(x)}{\tilde{\psi}(x_0,i_0)\varphi(y)}\sigma_{\mathbf{x}_0}(dy,i_0) + \frac{\tilde{\psi}(x,i_0)}{\tilde{\psi}(x_0,i_0)}\sigma_{\mathbf{x}_0}((x,\infty),i_0),\\ \Delta_2 &= \sum_{k\neq i_0}\frac{\varphi(x)}{\varphi(x_0)}\sigma_{\mathbf{x}_0}((0,\infty),k),\\ \Delta_3 &= \frac{\varphi(x)}{\varphi(x_0)}\sigma_{\mathbf{x}_0}(\{\mathbf{0}\}).\end{aligned}$$

Since σ_{x_0} is a probability measure, a simplification yields

$$\begin{aligned}f(x,i_0) &= \frac{\varphi(x)}{\varphi(x_0)}\sigma_{\mathbf{x}_0}((0,x_0],i_0) + \int_{(x_0,x]}\frac{\tilde{\psi}(y,i_0)\varphi(x)}{\tilde{\psi}(x_0,i_0)\varphi(y)}\sigma_{\mathbf{x}_0}(dy,i_0)\\ &\quad + \frac{\tilde{\psi}(x,i_0)}{\tilde{\psi}(x_0,i_0)}\sigma_{\mathbf{x}_0}((x,\infty),i_0)\\ &\quad + \frac{\varphi(x)}{\varphi(x_0)}(1-\sigma_{\mathbf{x}_0}((0,\infty),i_0))\\ &= \varphi(x)\left(\int_{(x_0,x]}\frac{\tilde{\psi}(y,i_0)}{\tilde{\psi}(x_0,i_0)\varphi(y)}\sigma_{\mathbf{x}_0}(dy,i_0) + \frac{\tilde{\psi}(x,i_0)}{\varphi(x)\tilde{\psi}(x_0,i_0)}\sigma_{\mathbf{x}_0}((x,\infty),i_0)\right)\\ &\quad + \varphi(x)\frac{1-\sigma_{\mathbf{x}_0}((x_0,\infty),i_0)}{\varphi(x_0)}\\ &:= \varphi(x)v(x,i_0) + \varphi(x)\frac{1-\sigma_{\mathbf{x}_0}((x_0,\infty),i_0)}{\varphi(x_0)}.\end{aligned}$$

The aim is to calculate the derivative of f. For this, consider

$$\begin{aligned}v(x+\delta,i_0) - v(x,i_0) &= \int_{(x,x+\delta]}\frac{\tilde{\psi}(y,i_0)}{\tilde{\psi}(x_0,i_0)\varphi(y)}\sigma_{\mathbf{x}_0}(dy,i_0)\\ &\quad + \frac{\tilde{\psi}(x+\delta,i_0)}{\varphi(x+\delta)\tilde{\psi}(x_0,i_0)}\sigma_{\mathbf{x}_0}((x+\delta,\infty),i_0)\\ &\quad - \frac{\tilde{\psi}(x,i_0)}{\varphi(x)\tilde{\psi}(x_0,i_0)}\sigma_{\mathbf{x}_0}((x,\infty),i_0).\end{aligned}$$

This yields

$$\begin{aligned} v^+(x,i_0) &= \lim_{\delta\downarrow 0}\frac{1}{S(x+\delta)-S(x)}\left(\frac{\tilde\psi(x+\delta,i_0)}{\varphi(x+\delta)}-\frac{\tilde\psi(x,i_0)}{\varphi(x)}\right)\frac{\sigma_{\mathbf{x}_0}((x+\delta,\infty),i_0)}{\tilde\psi(x_0,i_0)}\\ &= \lim_{\delta\downarrow 0}\frac{1}{S(x+\delta)-S(x)}\left(\frac{\psi^\partial(x+\delta)}{\varphi(x+\delta)}-\frac{\psi^\partial(x)}{\varphi(x)}\right)\frac{\sigma_{\mathbf{x}_0}((x+\delta,\infty),i_0)}{p_i\tilde\psi(x_0,i_0)}\\ &= \frac{1}{p_i\varphi^2(x)}\frac{\sigma_{\mathbf{x}_0}((x,\infty),i_0)}{\tilde\psi(x_0,i_0)}. \end{aligned}$$

Finally, we obtain by substituting and differentiating the following expression

$$f^+(x,i_0)=\varphi^+(x)v(x,i_0)+\varphi(x)\frac{1}{p_i\varphi^2(x)}\frac{\sigma_{\mathbf{x}_0}((x,\infty),i_0)}{\tilde\psi(x_0,i_0)}+\varphi^+(x)\frac{1-\sigma_{\mathbf{x}_0}((x_0,\infty),i_0)}{\varphi(x_0)},$$

which implies (40). The expression (41) is proved similarly, we omit the details. □

We can also characterize r-excessive functions in the following way (cf. [4,16], and [21]).

Theorem 5. *A measurable function $h:\Gamma\to[0,\infty)$ is r-excessive for $\mathbf{X}$ if and only if*

(a) the function $x\mapsto h(x,i)$, $x>0$, $i=1,2,\dots,n$, is r-excessive for X^∂,
(b) the gluing condition holds:

$$\mathbf{D}h(\mathbf{0}) := \sum_{i=1}^n p_i h^+(\mathbf{0},i)\le 0.$$

Proof. Assume first that h is r-excessive for $\mathbf{X}$. Then (b) holds by (30) in Theorem 2. To verify (a), we recall from (32) and (33) that on the leg i

$$\begin{aligned} h(x,i) &= \frac{c_r}{p_i}\varphi(x)v(x,i)+\varphi(x)\\ &= \frac{c_r}{p_i}\left(\int_{(0,x]}\frac{\varphi(x)\psi^\partial(y)}{\varphi(y)}\sigma(dy,i)+\psi^\partial(x)\sigma((x,\infty),i)\right)+\varphi(x). \end{aligned}$$

Using herein the explicit form of the Green function g_r^∂ associated with X^∂ as given in (13) yields

$$h(x,i)=\frac{c_r}{p_i}\int_0^\infty g_r^\partial(x,y)\,\mu(dy,i)+\varphi(x), \tag{42}$$

where $\mu(dy,i):=\sigma(dy,i)/\varphi(y)$ is a finite measure on $\mathbb{R}_+$. Consequently, h is r-excessive for X^∂ since the first term on the right hand side of (42) is r-excessive by the Riesz representation and, clearly, also the second term is such a one.

Assume next that (a) and (b) hold. We remark that (a) alone does not imply the claim. Indeed, the function $h(x,i):=\psi^\partial(x)+1$ is continuous, strictly positive and r-excessive for X^∂ but not r-excessive for $\mathbf{X}$. Indeed, if h were r-excessive then the representing measure should put a non-negative mass at $\mathbf{0}$. But $h^+(\mathbf{0},i)=1$ for very i and, hence, from (38), the mass is strictly negative, which contradicts the assumption of the r-excess. Since it is assumed that $x\mapsto h(x,i)$ is for every i r-excessive for X^∂ its representing probability measure is given for every i by

$$\sigma^\partial((x,\infty],i)=C_1\left(h^+(x,i)\varphi(x)-\varphi^+(x)h(x,i)\right),\qquad x>x_o, \tag{43}$$

and

$$\sigma^{\partial}([0,x),i) = C_2\left(h(x,i)(\psi^{\partial})^{-}(x,i) - h^{-}(x,i)\psi^{\partial}(x,i)\right), \qquad 0 < x < x_o, \tag{44}$$

where $x_o > 0$ is a point such that $h(x_o, i) = 1$, and C_1 and C_2 are constants (which can be calculated explicitly and do not depend on x). Notice that (43) and (44) induce probability measures in fact for any $x_o > 0$ since given an arbitrary $x_o > 0$ we may consider the representation of the function $x \mapsto h(x,i)/h(x_o,i)$. Consequently, the expression inside the parenthesis in (43) is positive and decreasing as a function of $x > 0$. Moreover, it is bounded by (b) and the assumptions on φ, cf. (4) and (6). We assume now without loss of generality that $h(\mathbf{0}) = 1$. It is also assumed that h has a representation as given on the right hand side of (26). The derivation of the formulas (36) and (38) in Theorem 3 is based only on the assumption that the representation (26) is valid. From the discussion above we conclude that the expression on the right hand side of (36) induces a measure. By the assumption (b) the mass at $\mathbf{0}$ is non-negative. Therefore, the function h has the representation given on the right hand side of (26) with σ as a probability measure. Consequently, h is r-excessive for $\mathbf{X}$. □

The proof of Theorem 5 yields also the following characterization of excessive functions expressed solely in terms of h and its derivatives.

Theorem 6. *A measurable function $h : \Gamma \to [0,\infty)$ is r-excessive for $\mathbf{X}$ if and only if*

(a) for every $x > 0$ and $i = 1, 2, \ldots, n$ the derivatives $h^{+}(x,i)$ and $h^{-}(x,i)$ exist,
(b) for every $i = 1, 2, \ldots, n$ the function

$$x \mapsto h^{+}(x,i)\varphi(x) - \varphi^{+}(x)h(x,i), \quad x > 0.$$

is bounded, positive and decreasing,
(c) the gluing condition holds:

$$Dh(\mathbf{0}) := \sum_{i=1}^{n} p_i h^{+}(\mathbf{0}, i) \le 0.$$

7. Optimal stopping

Consider a continuous function $g : \Gamma \to [0,\infty)$ called hereby the payoff function. We assume moreover that g is differentiable (with respect to the scale) on the interior of the edges, and has the right derivatives at 0 along the edges. Let $\mathcal{M}$ denote the set of stopping times adapted to the filtration generated by $\mathbf{X}$. Our optimal stopping problem consists of finding a function V, the value function, and a stopping time τ^*, an optimal stopping time, such that

$$V(\mathbf{x}) = \sup_{\tau\in\mathcal{M}} \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tau} g(\mathbf{X}(\tau))\right) = \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tau^*} g(\mathbf{X}(\tau^*))\right). \tag{45}$$

In order for the value function to be finite, we assume the following condition to hold:

$$\mathbb{E}_{\mathbf{x}}\left(\sup_{t\ge 0}(\mathrm{e}^{-rt} g(\mathbf{X}(t)))\right) < \infty. \tag{46}$$

In (45), when $\tau(\omega) = +\infty$ we define

$$\mathrm{e}^{-r\tau} g(\mathbf{X}(\tau)) := \limsup_{t\to\infty} \mathrm{e}^{-rt} g(\mathbf{X}(t)). \tag{47}$$

Let us mention that, as there exist several approaches to solve optimal stopping problems for linear diffusions, it is natural to expect that some of them can be reasonably extended to the

described probabilistic framework of a diffusion spider. In the present paper we choose two of them. The first one is in the core of many posterior approaches, we call it a representation approach, and consists in the examination of the representing measure of the candidate excessive value function, as initiated in [33]; the second one analyzes the negative set of the problem computed from the generator and the payoff function, initiated in [8]. Other approaches, as the one based on harmonic functions in [7], the one based on convex functions in [13], or the one via linear programming in [20], and, of course, the classical approach based on the martingale theory and the smooth fit-principle can hopefully also be applied successfully.

We begin with a criterion to check the finiteness of the value function for a given payoff function g. Since we have an explicit description of the harmonic functions in (28) we may and do apply this criterion in Section 8 when studying examples of optimal stopping.

Proposition 2. *Assume that there exists a positive r-harmonic function H of $\mathbf{X}$ such that g/H is bounded. Then, the value function in (45) is finite.*

Proof. We use the argument presented, e.g., in [6] Proposition 3.3. p. 2570. For this, recall that $\{\mathrm{e}^{-rt}H(\mathbf{X}(t))\}_{t\geq 0}$ is a positive martingale. Letting C be a bound of g/H then it holds for an arbitrary stopping time τ

$$\begin{aligned}\mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tau}g(\mathbf{X}_\tau)\right) &= \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tau}\frac{g(\mathbf{X}_\tau)}{H(\mathbf{X}_\tau)}H(\mathbf{X}_\tau)\right)\\ &\leq C\mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tau}H(\mathbf{X}_\tau)\right)\\ &\leq CH(\mathbf{x}) < \infty,\end{aligned}$$

where in case $\tau(\omega)=\infty$

$$\mathrm{e}^{-r\tau}H(\mathbf{X}_\tau) := \lim_{t\to\infty}\mathrm{e}^{-rt}H(\mathbf{X}_t). \quad \square$$

Theorem 7. *The value function V is the smallest r-excessive majorant of the payoff function g. An optimal stopping time of the problem (45) is given by*

$$\tau^* = \inf\{t\geq 0 : \mathbf{X}(t)\in\Sigma\},$$

where

$$\Sigma := \{\mathbf{x}\in\Gamma : g(\mathbf{x}) = V(\mathbf{x})\}. \tag{48}$$

Proof. Observe that the state space Γ is a locally compact Hausdorff space with a countable basis (i.e. semi compact), the process has continuous trajectories and is strong Markov. Furthermore the function g is assumed to be continuous and to satisfy the condition (46). Then the first claim follows from Theorem 1 p. 124 in [36] and the second one from Theorem 3 p. 127 in ibid. $\square$

The set Σ introduced in (48) is called the stopping region and its complement $C := \Sigma^c$ the continuation region of the OSP (45).

The next result is essentially Corollary on p. 124 in [36], and can be used to verify that a candidate value function is indeed the true value (see, e.g., [1,24], and [25]). We apply this, in particular, in our first example below. Recall that the underlying diffusion is assumed to be recurrent and, hence, the first hitting time H_A defined below is a.s. finite.

Theorem 8. *Let $A \subset \Gamma$ be a (non-empty) Borel subset of Γ and*

$$H_A := \inf\{t \geq 0 : \mathbf{X}_t \in A\}.$$

Assume that the function

$$\hat{V}(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}\left[e^{-rH_A} g(\mathbf{X}_{H_A})\right]$$

is r-excessive and dominates g. Then, $\hat{V}$ coincides with the value function of OSP (45) and H_A is an optimal stopping time.

We have the following sufficient condition for $\mathbf{0}$ to belong to the continuation region.

Proposition 3. *If*

$$\mathbf{D}g(\mathbf{0}) := \sum_{i=1}^{n} p_i g^+(\mathbf{0}, i) > 0, \tag{49}$$

then $\mathbf{0} \in C$.

Proof. Notice that $\mathbf{D}g(\mathbf{0})$ is well defined by the assumptions on g. Suppose that (49) holds. If $\mathbf{0} \in \Sigma$ then, from (48), $g(0) = V(0)$. Since V is r-excessive we have $\mathbf{D}V(\mathbf{0}) \leq 0$ by Theorem 5. But since $V(\mathbf{x}) \geq g(\mathbf{x})$ for all $\mathbf{x} \in \Gamma$ it follows that $0 < \mathbf{D}g(\mathbf{0}) \leq \mathbf{D}V(\mathbf{0}) \leq 0$ which is a contradiction. This concludes the proof of the proposition. □

We proceed now to characterize the stopping set as a unique solution to an integral equation. It is seen that these results can be derived by modifying/converting the approach presented in [6] for multidimensional diffusions to the present case with diffusion spiders, but, of course, the rôle played by the node point must be analyzed carefully. The first result, Theorem 9, is a verification theorem presenting the key integral equation for the stopping region, and the second one, Theorem 10, is for the uniqueness of the solution of this equation. The proof of the uniqueness uses ideas presented in Peskir [27] and [14]. Examples in Sections 8.2 and 8.3 below illustrate the approach based on these results. The proofs of the results are given in Appendices A and B, respectively.

For this program, recall the notation, cf. (8) and (9),

$$\begin{aligned}\mathbf{G}_r f(x,i) &:= \mathbb{E}_{(x,i)}\left[\int_0^\infty \mathrm{e}^{-rt} f(\mathbf{X}_t) dt\right] \\ &= \sum_{k=1}^{n} \int_0^{+\infty} \mathbf{g}_r((x,i),(y,k)) f(y,k) p_k m(dy),\end{aligned} \tag{50}$$

with a measurable and bounded $f : \Gamma \mapsto \mathbb{R}$. In fact, if f is "only" locally bounded and satisfies

$$\sum_{i=1}^{n} \int_0^\infty \varphi(y)|f(y,i)| p_i m(dy) < \infty \tag{51}$$

then we still have for all $(x,i) \in \Gamma$

$$|\mathbf{G}_r f(x,i)| < \infty.$$

Using the strong Markov property the following decomposition for the Green kernel is seen to hold

$$\mathbf{G}_r f(x,i) = \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-r\tau}\mathbf{G}_r f(\mathbf{X}_\tau)\mathbf{1}_{\{\tau<\infty\}} + \int_0^\tau \mathrm{e}^{-rt} f(\mathbf{X}_t) dt\right], \tag{52}$$

where $f : \Gamma \mapsto \mathbb{R}$ is a locally bounded and measurable function satisfying (51) and τ a stopping time in the filtration generated by $\mathbf{X}$.

Theorem 9. *Assume there exist a locally bounded and measurable function $f : \Gamma \mapsto \mathbb{R}$ and a constant $\Delta_{\mathbf{0}}$ (both depending on the payoff function g) such that (51) holds, and g can be written for all $(x, i) \in \Gamma$ as*

$$g(x,i) = \mathbf{G}_r f(x,i) + \mathbf{g}_r((x,i),\mathbf{0})\,\Delta_{\mathbf{0}}. \tag{53}$$

Assume also that there exists a (non-empty) closed Borel subset O of Γ such that

(a) *if $(x, i) \in O$ then $f(x, i) \geq 0$,*

(b) *if $\Delta_{\mathbf{0}} < 0$ then $\mathbf{0} \in O^c := \Gamma \setminus O$,*

and that the function $\tilde{V} : \Gamma \to \mathbb{R}$ defined by

$$\tilde{V}((x,i)) := \mathbf{G}_r(\mathbf{1}_O f)(x,i) + \mathbf{g}_r((x,i),\mathbf{0})\,\mathbf{1}_O(\mathbf{0})\,\Delta_{\mathbf{0}} \tag{54}$$

satisfies the following two conditions:

(1) *$\tilde{V}((x, i)) \geq g((x, i))$ for all $(x, i) \in O^c$,*

(2) *$\tilde{V}((x, i)) = g((x, i))$ for all $(x, i) \in \partial O$, or, equivalently, for all $(x, i) \in \partial O$*

$$g(x,i) - \tilde{V}(x,i) = \mathbf{G}_r(\mathbf{1}_{O^c} f)(x,i) + \mathbf{g}_r((x,i),\mathbf{0})\,\mathbf{1}_{O^c}(\mathbf{0})\,\Delta_{\mathbf{0}} = 0. \tag{55}$$

Then, in the problem (45), the stopping set $\Sigma = O$, $\tau^* = H_O := \inf\{t \geq 0 : \mathbf{X}_t \in O\}$ is an optimal stopping time and the value function $V = \tilde{V}$.

Theorem 10. *Let g, f, O, $\Delta_{\mathbf{0}}$ and $\tilde{V}$ satisfy the assumptions in Theorem 9. Furthermore, let $\mathcal{U}$ be a class of nonempty closed subsets of Γ containing O such that for all $U, U' \in \mathcal{U}$*

(a) *$U \cap U' \neq \emptyset$,*

(b) *$(x, i) \in U \Rightarrow f(x, i) \geq 0$,*

(c) *$\mathbf{0} \in U^c$,*

(d) *the following implications hold true:*

(i) *if $\mathbb{P}_{(x,i)}(\mathrm{Leb}(\{t \leq H_{U^c} : \mathbf{X}_t \notin U'\}) = 0) = 1$ for all $(x, i) \in U \cap U'$, then $U \subseteq U'$,*

(ii) *if $\mathbb{P}_{(x,i)}(\mathrm{Leb}(\{t \leq H_U\ \mathbf{X}_t \in U'\}) = 0) = 1$ for all $(x, i) \in U^c$, then $U' \subseteq U$,*

where Leb *denotes the Lebesgue measure. Then O is the unique set in the class $\mathcal{U}$ which satisfies Eq. (55).*

To apply Theorem 9 it is helpful to express f and $\Delta_{\mathbf{0}}$ in terms of the function g and its derivatives. This is achieved with similar ideas as in the proof of Theorem 3 but since g is not r-excessive more care is needed to analyze the integral representation (53). The proof is in Appendix C.

Proposition 4. *Assume that the payoff function g has the representation (53). Then*

(a) *$\Delta_{\mathbf{0}} = -\mathbf{D}g(\mathbf{0})$,*

(b) *for all $x \geq 0$ and $i = 1, 2, \ldots, n$*

$$\int_{(x,\infty)} \varphi(y)\, f(y,i)\, m(dy) = g^+(x,i)\varphi(x) - g(x,i)\varphi^+(x). \tag{56}$$

Remark 2. As is seen from the proof, the existence of the derivatives of g, in fact, follows from the assumption that g satisfies (53). Notice also that if the speed measure m has a continuous derivative m' with respect to the Lebesgue measure, the scale function S is continuously differentiable and g is two times continuously differentiable then the formula (56) yields

$$f(x,i) = -\left(\frac{d}{dm}\frac{d}{dS}g(x,i) - rg(x,i)\right) = (r-\mathcal{G})\,g(x,i). \tag{57}$$

Hence, (53) expresses, in a sense, the classical fact that the infinitesimal operator is the inverse of the resolvent operator. However, to make this precise, g should be in the domain of the infinitesimal operator. We refer to [8,9] for this approach for one-dimensional diffusions. Instead of giving conditions for the reverse of Proposition 4 to hold we calculate in the examples below $\Delta_{\mathbf{0}}$ as given in (a) and f as given in (57) and verify that (53) is satisfied.

8. Examples of optimal stopping

In our examples the underlying diffusion $\mathbf{X}$ is a Brownian spider with n legs. In Section 8.1 the stopping region is connected and the problem is solved using the explicit expressions for the representing measure of an excessive function displayed in Theorems 3 and 4. Here it is also studied how the stopping region changes when the discounting parameter r is varying. In Sections 8.2 and 8.3 the stopping region is disconnected. In these cases, we analyze Eq. (55) in Theorem 9 and deduce using Theorem 10 that there is a unique solution. The resulting equations for the boundary points of the stopping region are then solved numerically.

8.1. Example

Consider the optimal stopping problem (45) for the Brownian spider $\mathbf{X}$ with 3 legs and the following payoff function

$$g(x,1) = 1+x, \qquad g(x,2) = \left(1-\frac{x}{2}\right)^+, \qquad g(x,3) = (1-2x)^+.$$

It is also assumed that $p_1 = p_2 = p_3 = 1/3$. This assumption slightly simplifies the analysis but does not diminish the understanding of more general cases. That the problem has a finite value follows from Proposition 2 evoking the explicit form of the harmonic function given in (29). The solution of the problem is given in the next proposition. It is seen that the stopping region is connected for all values of the discounting parameter r.

Proposition 5. *The stopping region Σ of OSP introduced above is as follows:*

(a) in case $r > 2$ there exist $x_2^* \in (0,2)$ and $x_3^* \in (0,1/2)$ such that

$$\Sigma = \{(x,1) : 0 \le x\} \cup \{(x,2) : 0 \le x \le x_2^*\} \cup \{(x,3) : 0 \le x \le x_3^*\}, \tag{58}$$

(b) in case $1/8 < r \le 2$ there exists $y_2^* \in (0,2)$ such that

$$\Sigma = \{(x,1) : 0 \le x\} \cup \{(x,2) : 0 \le x \le y_2^*\}, \tag{59}$$

(c) in case $0 < r \le 1/8$ there exists $z_1^* \ge 0$ such that

$$\Sigma = \{(x,1) : z_1^* \le x\}, \tag{60}$$

where any point $(0,i)$, $i = 1,2,3$, is identified as the origin $\mathbf{0}$.

Proof. The idea of the proof is to study the expressions (36) and (38) for g and deduce what is the largest set such that the induced measure is a probability measure. Firstly, recalling that the scale function is $S(x) = x$ we have

$$\mathbf{D}g(\mathbf{0}) = \frac{1}{3}\left(g'(0,1) + g'(0,2) + g'(0,3)\right) = \frac{1}{3}\left(1 - \frac{1}{2} - 2\right) = -\frac{1}{2}. \tag{61}$$

This means that the payoff function satisfies the condition (30) which should hold for excessive functions at $\mathbf{0}$ and, hence, it is possible that $\mathbf{0} \in \Sigma$. Notice that if $\mathbf{0} \in \Sigma$ then $V(\mathbf{0}) = g(\mathbf{0}) = 1$. Consider next the function on the right hand side of (36) for leg 2 evaluated for $f = g$, that is, for $x \in (0,2)$, $i = 2$, and $\theta := \sqrt{2r}$

$$\frac{p_2}{c_r}\left(g^+(x,2)\varphi(x) - \varphi^+(x)g(x,2)\right) = \frac{1}{3\theta}\,\mathrm{e}^{-\theta x}\left(-\frac{1}{2} + \theta\left(1 - \frac{x}{2}\right)\right) =: F_2(x) \tag{62}$$

Clearly, F_2 is decreasing and $F_2(2) < 0$. Hence, if $F_2(0) > 0$ there is a unique zero. This is the case when $r > 1/8$. We study similarly F_3 defined for $x \in (0, 1/2)$ by

$$F_3(x) := \frac{p_3}{c_r}\left(g'(x,3)\varphi(x) - \varphi^+(x)g(x,3)\right) = \frac{1}{3\theta}\,\mathrm{e}^{-\theta x}\left(-2 + \theta\,(1 - 2x)\right).$$

It holds that F_3 is decreasing and $F_3(1/2) < 0$. Hence, if $F_3(0) > 0$ there is a unique zero. This is the case when $r > 2$. Finally, consider for $x \geq 0$

$$F_1(x) := \frac{p_1}{c_r}\left(g'(x,1)\varphi(x) - \varphi^+(x)g(x,1)\right) = \frac{1}{3\theta}\,\mathrm{e}^{-\theta x}\left(1 + \theta\,(1 + x)\right).$$

It is immediately seen that F_1 is positive everywhere and decreasing.

We prove now (a). For $r > 2$ let $x_2^* \in (0,2)$ be the zero of F_2 and $x_3^* \in (0,1/2)$ the zero of F_3. Let Σ_a be the set defined by the right hand side of (58) and introduce

$$\tilde{H}_a := \inf\{t : \mathbf{X}_t \in \Sigma_a\}$$

and

$$\tilde{V}_a(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tilde{H}_a} g\left(\mathbf{X}_{\tilde{H}_a}\right)\right) = \begin{cases} g(\mathbf{x}), & \mathbf{x} \in \Sigma_a, \\ g(x_2^*, 2)\dfrac{\varphi(x)}{\varphi(x_2^*)}, & x \geq x_2^* \text{ on leg 2}, \\ g(x_3^*, 3)\dfrac{\varphi(x)}{\varphi(x_3^*)}, & x \geq x_3^* \text{ on leg 3}. \end{cases}$$

Recall, cf. (3), that for $x \geq y \geq 0$

$$\frac{\varphi(x)}{\varphi(y)} = \mathbb{E}_x\left(\mathrm{e}^{-rH_y}\right),$$

where $H_y := \inf\{t : X_t = y\}$. Clearly, $\tilde{V}_a$ dominates g. Moreover, it can be checked that $\tilde{V}_a$ is r-excessive by calculating the expressions in (36) on the legs (in fact, these are multiples of the functions F_1, F_2, and F_3) and in (38). Notice that $\mathbf{D}\tilde{V}(\mathbf{0}) = \mathbf{D}g(\mathbf{0}) = -\frac{1}{2}$ by (61). Evoking Theorem 8 completes the proof of (a).

Consider next the case (b). Analogously as above, we let Σ_b denote the right hand side of (59) where x_2^* is the unique zero of F_2. Furthermore,

$$\tilde{H}_b := \inf\{t : \mathbf{X}_t \in \Sigma_b\}$$

and

$$\tilde{V}_b(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tilde{H}_b} g\left(\mathbf{X}_{\tilde{H}_b}\right)\right) = \begin{cases} g(\mathbf{x}), & \mathbf{x} \in \Sigma_b, \\ g(x_2^*, 2)\dfrac{\varphi(x)}{\varphi(x_2^*)}, & x \geq x_2^* \text{ on leg 2}, \\ \varphi(x), & x \geq 0 \quad \text{on leg 3}. \end{cases}$$

Clearly, $\tilde{V}_b$ dominates g on the legs 1 and 2. Since $1/8 < r \leq 2$ it is easily seen that $\tilde{V}_b$ dominates g also on the leg 3. We can check the r-excessivity of $\tilde{V}_b$ by calculating the representing measure. Notice, in particular, that $\mathbf{D}\tilde{V}_b(\mathbf{0}) = \frac{1}{3}\left(1 - \frac{1}{2} - \theta\right) \leq 0$.

We consider now case (c). The three functions F_1, F_2 and F_3 above have no roots, so we should consider the function on the l.h.s. of (41) with $i_o = 1$. The function is proportional to

$$\bar{F}_1(x) = (1+x)\tilde{\psi}^-(x,1) - \tilde{\psi}(x,1)$$

with $\tilde{\psi}$ in (21). After some computations we obtain

$$\bar{F}_1(x) = \frac{\theta(1+x)}{2}\left(3\mathrm{e}^{\theta x} + \mathrm{e}^{-\theta x}\right) - \frac{1}{2}\left(3\mathrm{e}^{\theta x} - \mathrm{e}^{-\theta x}\right)$$

For $r \leq 1/8$ we have $\theta \leq 1/2$ and $\bar{F}_1(0) \leq 0$ (with equality only at $r = 1/8$). Furthermore, by differentiation we obtain that the function is increasing and it follows that it has a unique non-negative root z_1^*. Consider now Σ_c the set in the l.h.s. of (60) and the stopping time

$$\tilde{H}_c := \inf\{t : \mathbf{X}_t \in \Sigma_c\}.$$

Applying the hitting time in (23), we obtain the candidate value function for the problem:

$$\tilde{V}_c(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}\left(\mathrm{e}^{-r\tilde{H}_{z_1^*}} g\left(\mathbf{X}_{\tilde{H}_{z_1^*}}\right)\right) = \begin{cases} g(\mathbf{x}), & \mathbf{x} \in \Sigma_c, \\ \dfrac{g(z_1^*,1)}{\tilde{\psi}(z_1^*,1)}\tilde{\psi}(x,1), & 0 \leq x \leq z_1^* \text{ on leg 1}, \\ \dfrac{g(z_1^*,1)}{\tilde{\psi}(z_1^*,1)}\dfrac{\varphi(x)}{\theta}, & x \geq 0 \text{ on legs 2, 3}. \end{cases} \tag{63}$$

We now apply Theorem 8. The function $\tilde{V}_c$ is r-excessive because its representing measure is positive on the set (z_1^*, ∞) on leg 1 according to the selection of z_1^* and (41), and vanishes on the set $(0, z_1^*)$ on leg 1, and also on the other legs since the chosen function in (63) is harmonic on the complement of Σ_c. Let us now prove that

$$\tilde{V}_c \geq g. \tag{64}$$

It is clear that $\tilde{V}_c(\mathbf{x}) = g(\mathbf{x})$ for $\mathbf{x} = (x, 1)$ and $x \geq z_1^*$. The condition $\bar{F}_1(z_1^*) = 0$ implies

$$\tilde{\psi}'(z_1^*, 1) = \frac{\tilde{\psi}(z_1^*, 1)}{g(z_1^*, 1)},$$

and then according to (63) we have that $\tilde{V}_c'(z_1^*, 1) = 1$. As the function $\tilde{\psi}$ is strictly convex, it holds that $\tilde{V}_c(\mathbf{x}) \geq g(\mathbf{x})$ everywhere on the leg 1. As a consequence, this yields that

$$\tilde{V}_c(\mathbf{0}) = \frac{g(z_1^*, 1)}{\theta\tilde{\psi}(z_1^*, 1)} =: K > 1.$$

Introduce the tangent line to $\tilde{V}_c$ at $x = 0$ on the leg 2. As $\tilde{V}_c(x, 2) = K\mathrm{e}^{-\theta x}$, the tangent is

$$t(x) = K(1 - \theta x).$$

As $t(2) = K(1 - 2\theta) \geq 0$ we conclude that $\tilde{V}_c(x, 2) \geq t(x) \geq g(x, 2)$. The fact that $g(x, 3) < g(x, 2)$ concludes the proof of (64). □

Remark 3 (*OSP for Skew Brownian Motion*)**.** In case $n = 2$, process $\mathbf{X}$ is the usual skew Brownian motion with parameter $\beta = p_1$, identifying the first leg with the positive half-line and the second leg with the negative half-line, and denoting by x the distance from the origin. For a general payoff function g, considered as a function on the line $\mathbb{R}$, the application of Proposition 3 gives that when

$$\beta g'(0+) - (1 - \beta)g'(0-) > 0,$$

the origin $\mathbf{0}$ belongs for all values of the discounting parameter r to the continuation region. This result can also be found in [1] Proposition 1.

8.2. Example

Consider next the optimal stopping problem for a Brownian spider with n legs and payoff function

$$g(x, i) = A_i x, \;\; x > 0, \;\; i = 1, 2, \ldots, n,$$

where $A_i > 0$. Straightforward but a bit lengthy calculations, show that the representation (53) for g holds (cf. Proposition 4) when

$$\Delta_{\mathbf{0}} = -\mathbf{D}g(\mathbf{0}) = -\sum_{i=1}^{n} p_i A_i$$

and for $x > 0$, see (57),

$$f(x, i) = -\left(\frac{1}{2}\frac{d^2}{dx^2}(A_i x) - r A_i x\right) = r\, A_i x.$$

Applying Proposition 2, with the harmonic function given in (29), it is seen that the value function V of the problem is finite. From Theorem 7 we know that V is the smallest r-excessive majorant of g and the stopping set Σ is given by (48). Since g and V are continuous it follows that Σ is closed and, in particular, $V(z, i) = g(z, i)$ when $(z, i) \in \partial\Sigma$. Moreover, notice that $\mathbf{0} \notin \Sigma$ because $g(\mathbf{0}) = 0$ (cf. also Proposition 3). Recall that V is r-harmonic on $C := \Sigma^c$ and $V = g$ on Σ. Hence V can be represented for all $(x, i) \in \Gamma$ as

$$V(x, i) = \sum_{k=1}^{n} \int_0^{+\infty} \mathbf{g}_r((x, i), (y, k))\, \mathbf{1}_{\Sigma}((y, k))\; f(y, k)\, dy, \tag{65}$$

where the Green function $\mathbf{g}_r$ is given in (22). We remark that (65) is the Riesz representation of the r-excessive function V. Here it is also used that, due to the smooth pasting property (valid in this case, see Theorem 11 in Appendix B), the representing measure of V does not put mass on the boundary points of Σ, see (37), and is, therefore, absolutely continuous with respect to the Lebesgue measure. The density is f since $V = g$ on Σ. Consequently, it holds (cf. (55)) for all $(z, i) \in \partial\Sigma$

$$\sum_{k=1}^{n} \int_0^{+\infty} \mathbf{g}_r((z, i), (y, k))\, \mathbf{1}_{\Sigma^c}((y, k))\, 2r p_k\, A_k y\, dy = \mathbf{g}_r((z, i), \mathbf{0}) \sum_{k=1}^{n} p_k A_k. \tag{66}$$

Proposition 6. *There exist* $z_i^* > 0$, $i = 1, 2, \dots, n$, *such that* $\Sigma = \{(x, i) : x \geq z_i^*, i = 1, 2, \dots, n\}$.

Proof. Let

$$A_* = \min\{A_1, \dots, A_n\} \quad \text{and} \quad A^* = \max\{A_1, \dots, A_n\}.$$

Let V_* (V^*) be the value function of OSP with payoff $g_*(x, i) = A_* x$ ($g^*(x, i) = A^* x$), $i = 1, 2, \dots, n$. Then, because for all x and i

$$g_*(x, i) \leq g(x, i) \leq g^*(x, i),$$

it follows

$$V_*(x, i) \leq V(x, i) \leq V^*(x, i). \tag{67}$$

By symmetry, for all x and i

$$V_*(\mathbf{x}) = \sup_\tau \mathbb{E}_{\mathbf{x}} \left(\mathrm{e}^{-r\tau}\, A_* \, \mathbf{X}(\tau)\right) = A_* \sup_\eta \mathbb{E}_x \left(\mathrm{e}^{-r\eta}\, X_\eta\right) \tag{68}$$

and

$$V^*(\mathbf{x}) = \sup_\tau \mathbb{E}_{\mathbf{x}} \left(\mathrm{e}^{-r\tau}\, A^* \, \mathbf{X}(\tau)\right) = A^* \sup_\eta \mathbb{E}_x \left(\mathrm{e}^{-r\eta}\, X_\eta\right), \tag{69}$$

where $X = (X_t)_{t\geq 0}$ denotes a reflecting Brownian motion, $\mathbb{E}_x$ the expectation operator associated with X when started at x and the supremum in case of X is taken over all stopping times η in the filtration of X. For OSP (68) and for (69) there exists z_* and z^*, respectively, such that the corresponding stopping regions are given by $\Sigma_* = \{(x, i) : x \geq z_*, i = 1, 2, \dots, n\}$ and $\Sigma^* = \{(x, i) : x \geq z^*, i = 1, 2, \dots, n\}$. In particular, on every leg i it holds for $x \geq z^*$

$$V(x, i) \leq A^* x. \tag{70}$$

Consequently, Σ^c does not contain sets of the form $O_{i,z} := \{(x, i) : x > z\}$ for any i and $z > 0$. Indeed, if there is i and z such $O_{i,z} \subset \Sigma^c$ then V is r-harmonic on $O_{i,z}$ and increases exponentially, see (29), but this violates (70). Next, (67) also yields that $\Sigma \subseteq \{(x, i) : x \geq z_*, i = 1, 2, \dots, n\}$. Finally, assume that Σ is not connected on some leg k Then there exist $x_k \in \Sigma^c$ such that $V(x_k, k) > g(x_k, k) = A_k x_k$, and, by continuity, x_{k_1} and x_{k_2} in Σ such that $x_{k_1} < x_k < x_{k_2}$ and $V(x, k) > A_k x$ for all in $x \in (x_{k_1}, x_{k_2})$. Let

$$H_{1,2} := \inf\{t : \mathbf{X}_t \notin (x_{k_1}, x_{k_2})\}.$$

Since V is r-harmonic on (x_{k_1}, x_{k_2}) it holds

$$V(x, k) = \mathbb{E}_{(x,k)} \left(\mathrm{e}^{-rH_{1,2}}\, A_k \, \mathbf{X}(H_{1,2})\right) = A_k \mathbb{E}_x \left(\mathrm{e}^{-rH_{1,2}}\, X_{H_{1,2}}\right).$$

Using [5] 3.3.0.5 (a) and (b) p. 218 gives

$$\begin{aligned}
\mathbb{E}_x \left(\mathrm{e}^{-rH_{1,2}}\, X_{H_{1,2}}\right) &= x_{k_1} \mathbb{E}_x \left(\mathrm{e}^{-rH_{1,2}}\,;\, X_{H_{1,2}} = x_{k_1}\right) + x_{k_2} \mathbb{E}_x \left(\mathrm{e}^{-rH_{1,2}}\,;\, X_{H_{1,2}} = x_{k_2}\right) \\
&= \frac{1}{\sinh\left((x_{k_2} - x_{k_1})\sqrt{2r}\right)} \left(x_{k_1} \sinh\left((x_{k_2} - x)\sqrt{2r}\right)\right. \\
&\quad \left. +\; x_{k_2} \sinh\left((x - x_{k_1})\sqrt{2r}\right)\right).
\end{aligned}$$

Consequently, V is convex on (x_{k_1}, x_{k_2}) and, hence, $V(x, k) < g(x, k)$ for all $x \in (x_{k_1}, x_{k_2})$. This is a contradiction, and the proof is complete. □

Let $z_i^*, i = 1, 2, \ldots, n$, be the boundary points of Σ, as introduced in Proposition 6. Then $(z_1^*, \ldots, z_n^*)$ constitutes a solution to the system of the equations (66), i.e., for $i = 1, 2, \ldots, n$

$$\sum_{k=1}^{n} \int_0^{z_k^*} \mathbf{g}_r((z_i^*, i), (y, k))\, 2r p_k\, A_k y\, dy = \mathbf{g}_r((z_i^*, i), \mathbf{0}) \sum_{k=1}^{n} p_k A_k. \tag{71}$$

The uniqueness of the solution follows from Theorem 10. Indeed, it is easily verified that letting $\mathcal{U}$ consist of all sets U of the form

$$U = U(\alpha_1, \ldots, \alpha_n) = \{(x_i, i) : x_i \geq \alpha_i, i = 1, \ldots, n\}, \quad \alpha_i > 0, i = 1, 2, \ldots, n,$$

then the properties **(a)**, **(b)**, **(c)**, and **(d)** listed therein hold.

To give some numerical flavor, let $n = 3$ and $r = 1/2$. Then the system (71) becomes

$$A_i \int_0^{z_i} y \sinh(y) dy + \sum_{k=1}^{3} A_k p_k \int_0^{z_k} y \mathrm{e}^{-y} dy = \sum_{k=1}^{3} A_k p_k, \quad i = 1, 2, 3,$$

and integrating yields

$$A_i(z_i \cosh(z_i) - \sinh(z_i)) = \sum_{k=1}^{3} A_k p_k \mathrm{e}^{-z_k}(1 + z_k), \quad i = 1, 2, 3. \tag{72}$$

Putting, moreover, $p_1 = p_2 = p_3 = 1/3$, $A_1 = 1$, $A_2 = 2$ and $A_3 = 3$ and solving numerically Eqs. (72) we obtain the stopping region of the OSP that has the respective critical threshold values

$$z_1 = 1.4816, \quad z_2 = 1.2041, \quad z_3 = 1.0628.$$

8.3. Example

We conclude by considering in the same setting as in Section 8.2 the optimal stopping problem with the payoff function

$$g(x, i) = A_i x^2, \ x > 0, \ i = 1, 2, 3,$$

where $A_i > 0$. Again, applying Proposition 2 it is seen that the value function of the problem is finite. Straightforward but a bit lengthy calculations show the representation (53) for g holds (cf. Proposition 4) when

$$\Delta_{\mathbf{0}} = -\mathbf{D}g(\mathbf{0}) = 0$$

and for $x > 0$, see (57),

$$f(x, i) = -\left(\frac{1}{2}\frac{d^2}{dx^2}(A_i x^2) - r A_i x^2\right) = A_i\left(r x^2 - 1\right).$$

The corresponding equations obtained from (55) then are:

$$\sum_{k=1}^{n} \int_0^{+\infty} \mathbf{g}_r((z, i), (y, k))\, \mathbf{1}_{\Sigma^c}((y, k))\, A_k(r y^2 - 1) 2 p_k\, dy = 0. \tag{73}$$

For simplicity we take $r = 1/2$. We use the Green kernel expression (22) and proceed similarly as in Section 8.2, (in particular, it is seen that the continuation region is connected, and that the solution of the associated equation system is unique). Introducing the auxiliary functions

$$F(z) = \int_0^z \sinh(y)(y^2 - 1) dy = z^2 \cosh(z) - 2z \sinh(z)$$

$$H(z) = \int_0^z \mathrm{e}^{-y}(y^2 - 1)dy = -\mathrm{e}^{-z}z(z+2).$$

the equation system to find the boundary points of the continuation region is as follows

$$A_1 F(z_1^*) = A_2 F(z_2^*) = A_3 F(z_3^*)$$

and

$$A_1\big(F(z_1^*) + H(z_1^*)\big) + A_2\big(F(z_2^*) + H(z_2^*)\big) + A_3\big(F(z_3^*) + H(z_3^*)\big) = 0.$$

To solve numerically, we take $A_1 = 1$, $A_2 = 2$ and $A_3 = 3$ obtaining

$$z_1^* = 2.16987, \qquad z_2^* = 2.06543, \qquad z_3^* = 2.02250.$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 9

The proof consists of checking that the function $\tilde{V}$ fulfills the conditions in Theorem 8. Firstly, notice that $\tilde{V}$ is r-excessive since it is a linear combination of the Green kernel at $\mathbf{0}$ and an integral of the Green kernel with respect to a positive measure. Since, by assumption **(1)**, $\tilde{V}$ dominates g on O^c it remains to show that for all $(x,i) \in \Gamma$

$$\tilde{V}(x,i) = \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O} g(\mathbf{X}_{H_O})\right]. \tag{74}$$

Recall that $\mathbf{X}$ is recurrent and, hence, $H_O < \infty$ a.s. for all starting states. Moreover, $\mathbf{X}_{H_O} \in \partial O \subseteq O$ since O is assumed to be closed and $\mathbf{X}$ has continuous sample paths. Suppose now that $(x,i) \in O^c$ and $\mathbf{0} \in O^c$. Then, using (52),

$$\begin{aligned}
\tilde{V}(x,i) = \mathbf{G}_r(\mathbf{1}_O f)(x,i) &= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{G}_r(\mathbf{1}_O f)(\mathbf{X}_{H_O}) + \int_0^{H_O} \mathrm{e}^{-rH_O}(\mathbf{1}_O f)(\mathbf{X}_t)dt\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{G}_r(\mathbf{1}_O f)(\mathbf{X}_{H_O})\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\tilde{V}(\mathbf{X}_{H_O})\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O} g(\mathbf{X}_{H_O})\right],
\end{aligned}$$

where, in the last step, assumption (2) is applicable since $\mathbf{X}_{H_O} \in \partial O$. If $\mathbf{0} \in O$ we have

$$\begin{aligned}
\tilde{V}(x,i) &= \mathbf{G}_r(\mathbf{1}_O f)(x,i) + \mathbf{g}_r((x,i),\mathbf{0})\,\Delta_{\mathbf{0}} \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{G}_r(\mathbf{1}_O f)(\mathbf{X}_{H_O})\right] + \mathbf{g}_r((x,i),\mathbf{0})\,\Delta_{\mathbf{0}}.
\end{aligned} \tag{75}$$

To analyze the second term on RHS of (75) recall from (9) in Theorem 1 that

$$\mathbf{g}_r((x,i),\mathbf{0}) = c_r^{-1}\varphi(x).$$

Hence, the function $\mathbf{g}_r(\cdot, \mathbf{0})$ is r-harmonic on the legs of Γ; in other words, if A is a Borel subset of $[0, +\infty)$ including 0, $\mathbf{X}_0 = (x, i) \in (A^c, i)$ and $H_A := \inf\{t : \mathbf{X}_t \in (A, i)\}$ then

$$\mathbf{g}_r((x, i), \mathbf{0}) = \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_A}\mathbf{g}_r(\mathbf{X}_{H_A}, \mathbf{0})\right].$$

Consequently, if $(x, i) \in O^c$ and $\mathbf{0} \in O$

$$\mathbf{g}_r((x, i), \mathbf{0}) = \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{g}_r(\mathbf{X}_{H_O}, \mathbf{0})\right],$$

and (75) can be developed as follows

$$\begin{aligned}
\tilde{V}(x, i) &= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{G}_r(\mathbf{1}_O f)(\mathbf{X}_{H_O})\right] + \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{g}_r(\mathbf{X}_{H_O}, \mathbf{0})\right] \Delta_{\mathbf{0}} \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\left(\mathbf{G}_r(\mathbf{1}_O f)(\mathbf{X}_{H_O}) + \mathbf{g}_r(\mathbf{X}_{H_O}, \mathbf{0})\,\Delta_{\mathbf{0}}\right)\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\,\tilde{V}(\mathbf{X}_{H_O})\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\,g(\mathbf{X}_{H_O})\right]
\end{aligned}$$

proving (74) also in this case. To show that (74) holds when $(x, i) \in O$ we show that $\tilde{V} = g$ on O. This clearly implies the claim since if $(x, i) \in O$ then a.s. $H_O = 0$ in (74). Subtracting (54) from (53) yields for all (x, i)

$$g(x, i) - \tilde{V}(x, i) = \mathbf{G}_r(\mathbf{1}_{O^c} f)(x, i) + \mathbf{g}_r((x, i), \mathbf{0})\,\mathbf{1}_{O^c}(\mathbf{0})\,\Delta_{\mathbf{0}} \tag{76}$$

Let $(x, i) \in O$ and $\mathbf{0} \in O$. Then, evoking (52),

$$\begin{aligned}
g(x, i) - \tilde{V}(x, i) &= \mathbf{G}_r(\mathbf{1}_{O^c} f)(x, i) \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{G}_r(\mathbf{1}_{O^c} f)(\mathbf{X}_{H_{O^c}}) + \int_0^{H_{O^c}} \mathrm{e}^{-rH_{O^c}}(\mathbf{1}_{O^c} f)(\mathbf{X}_t)dt\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{G}_r(\mathbf{1}_{O^c} f)(\mathbf{X}_{H_{O^c}})\right] \\
&= 0,
\end{aligned}$$

where in the last step assumption 2 is applicable since $\mathbf{X}_{H_{O^c}} \in \partial O$. Finally, suppose $(x, i) \in O$ and $\mathbf{0} \in O^c$. Then, with analogous arguments as above,

$$\begin{aligned}
g(x, i) - \tilde{V}(x, i) &= \mathbf{G}_r(\mathbf{1}_{O^c} f)(x, i) + \mathbf{g}_r((x, i), \mathbf{0})\Delta_{\mathbf{0}} \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{G}_r(\mathbf{1}_{O^c} f)(\mathbf{X}_{H_{O^c}})\right] + \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{g}_r(\mathbf{X}_{H_{O^c}}, \mathbf{0})\Delta_{\mathbf{0}}\right] \\
&= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\left(\mathbf{G}_r(\mathbf{1}_{O^c} f)(\mathbf{X}_{H_{O^c}}) + \mathbf{g}_r(\mathbf{X}_{H_{O^c}}, \mathbf{0})\Delta_{\mathbf{0}}\right)\right] \\
&= 0.
\end{aligned}$$

Appendix B. Proof of Theorem 10

Let $U \in \mathcal{U}$ be such that for all $(x, i) \in \partial U$

$$\mathbf{G}_{\mathbf{r}}(f\mathbf{1}_{U^c})(x, i) + \mathbf{g}_r((x, i), \mathbf{0})\mathbf{1}_{U^c}(\mathbf{0})\Delta_{\mathbf{0}} = 0, \tag{77}$$

i.e., (55) holds when O is replaced with U. The claim is that $U = O$. To prove this, define

$$W(x, i) := \mathbf{G}_{\mathbf{r}}(f\mathbf{1}_U)(x, i) + \mathbf{g}_r((x, i), \mathbf{0})\mathbf{1}_U(\mathbf{0})\Delta_{\mathbf{0}}.$$

Using the assumed representation of the payoff function g, see (53), yields

$$g(x, i) - W(x, i) = \mathbf{G}_{\mathbf{r}}(f\mathbf{1}_{U^c})(x, i) + \mathbf{g}_r((x, i), \mathbf{0})\mathbf{1}_{U^c}(\mathbf{0})\Delta_{\mathbf{0}}.$$

From (77) it follows that for all $(x,i)\in\partial U$

$$g(x,i)-W(x,i)=0, \tag{78}$$

and arguing analogously as in the proof of Theorem 9 it is seen that (78) holds for all $(x,i)\in U$. Recall that $\tilde V$ is the value function of the problem. We show now that for all $(x,i)\in\Gamma$

$$W(x,i)\le\tilde V(x,i). \tag{79}$$

Firstly, for $(x,i)\in U$ it holds $W(x,i)=g(x,i)\le V(x,i)$ since $\tilde V$ is a majorant of g. Secondly, let $(x,i)\in U^c$. Proceeding similarly as when proving (74) in the proof of Theorem 9 it is seen that

$$\begin{aligned}W(x,i)&=\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_U}W(\mathbf{X}_{H_U})\right]\\&=\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_U}g(\mathbf{X}_{H_U})\right]\\&\le\tilde V(x,i),\end{aligned}$$

where the second equality is valid since $\mathbf{X}_{H_U}\in\partial U$, and the inequality holds since $\tilde V$ is the value of the problem. Hence, (79) is verified. Next we show that $O\subseteq U$. Recall the assumption **(a)**, and let $(x,i)\in O\cap U$. Using (52), the fact that the function $\mathbf{g}_r(\cdot,\mathbf{0})$ is r-harmonic on the legs of Γ, and the inequality (79), yield

$$\begin{aligned}g(x,i)&=W(x,i)\\&=\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{G}_r(f\mathbf{1}_U)(\mathbf{X}_{H_{O^c}})\right]+\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{g}_r(\mathbf{X}_{H_{O^c}},\mathbf{0})\mathbf{1}_U(\mathbf{0})\Delta_{\mathbf{0}}\right]\\&\qquad+\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right]\\&=\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}W(\mathbf{X}_{H_{O^c}})\right]+\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right]\\&\le\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\tilde V(\mathbf{X}_{H_{O^c}})\right]+\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right].\end{aligned} \tag{80}$$

On the other hand, since $\mathbf{0}\in O^c$ by assumption **(c)** we have (see also (54))

$$\begin{aligned}g(x,i)&=\tilde V(x,i)\\&=\mathbf{G}_r(f\mathbf{1}_O)(x,i)+\mathbf{g}_r((x,i),\mathbf{0})\,\mathbf{1}_O(\mathbf{0})\,\Delta_{\mathbf{0}}\\&=\mathbf{G}_r(f\mathbf{1}_O)(x,i)\\&=\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}\mathbf{G}_r(f\mathbf{1}_O)(\mathbf{X}_{H_{O^c}})\right]+\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)\mathbf{1}_O(\mathbf{X}_s)ds\right]\\&=\mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_{O^c}}V(\mathbf{X}_{H_{O^c}})\right]+\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)ds\right].\end{aligned} \tag{81}$$

Subtracting (81) from (80) yields

$$\begin{aligned}0&\le\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right]-\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)ds\right]\\&=-\mathbb{E}_{(x,i)}\left[\int_0^{H_{O^c}}\mathrm{e}^{-rs}f(\mathbf{X}_s)\mathbf{1}_{U^c}(\mathbf{X}_s)ds\right]\le 0.\end{aligned}$$

Thus $\mathbb{P}_{(x,i)}(\mathrm{Leb}(\{t\le H_{O^c}|\mathbf{X}_t\in U^c\})=0)=1$ for all $(x,i)\in O\cap U$, and it follows from assumption **c(i)** that $O\subseteq U$. Finally, we prove the opposite inclusion, i.e., $U\subseteq O$. For this, let

$(x,i) \in O^c$ and recall that $H_O = \inf\{t \geq 0 | X_t \in O\}$ is an optimal stopping time. Calculating similarly as above we have

$$\begin{aligned} V(x,i) &\geq W(x,i) \\ &= \mathbf{G_r}(f\mathbf{1}_U)(x,i) + \mathbf{g}_r((x,i),\mathbf{0})\mathbf{1}_U(\mathbf{0})\Delta_{\mathbf{0}} \\ &= \mathbf{G_r}(f\mathbf{1}_U)(x,i) \\ &= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O}\mathbf{G}_r(f\mathbf{1}_U)(\mathbf{X}_{H_O})\right] + \mathbb{E}_{(x,i)}\left[\int_0^{H_O} \mathrm{e}^{-rs} f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right] \\ &= \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O} W(\mathbf{X}_{H_O})\right] + \mathbb{E}_{(x,i)}\left[\int_0^{H_O} \mathrm{e}^{-rs} f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right]. \end{aligned}$$

Since $O \subseteq U$ and $W = g$ on U, it holds that $W(\mathbf{X}_{H_O}) = g(\mathbf{X}_{H_O})$. Thus

$$V(x,i) \geq \mathbb{E}_{(x,i)}\left[\mathrm{e}^{-rH_O} g(X_{H_O})\right] + \mathbb{E}_{(x,i)}\left[\int_0^{H_O} \mathrm{e}^{-rs} f(\mathbf{X}_s)\mathbf{1}_U(\mathbf{X}_s)ds\right] \geq V(x,i).$$

This implies $\mathbb{P}_{(x,i)}(\mathrm{Leb}(\{t \leq H_O | \mathbf{X}_t \in U\}) = 0) = 1$ for all $(x,i) \in O^c$. Now by assumption **d(ii)** we conclude $U \subseteq O$.

Appendix C. Proof of Proposition 4

We prove first (a). For this, consider for $x \geq 0$

$$\begin{aligned} g(x,i) &= \int_0^{+\infty} \mathbf{g}_r((x,i),(y,i))\, f(y,i)\, p_i m(dy) \\ &\qquad + \sum_{k=1}^{n} \int_0^{+\infty} \mathbf{g}_r((x,i),(y,k))\, f(y,k)\, p_k m(dy) \\ &\quad + \mathbf{g}_r((x,i),\mathbf{0})\, \Delta_{\mathbf{0}} \\ &= \int_{(0,x]} \varphi(x)\tilde{\psi}(y,i)\, f(y,i)\, p_i m(dy) + \int_{(x,\infty)} \varphi(y)\tilde{\psi}(x,i)\, f(y,i)\, p_i m(dy) \\ &\qquad + \frac{1}{c_r}\sum_{k\neq i} \int_0^{+\infty} \varphi(x)\varphi(y)\, f(y,k)\, p_k m(dy) + \varphi(x)\tilde{\psi}(\mathbf{0})\, \Delta_{\mathbf{0}}, \end{aligned}$$

where the explicit form of the Green kernel is used, in particular, $\tilde{\psi}(y,i) = p_i^{-1}\psi^{\partial}(y) + c_r^{-1}\varphi(y)$ and $\tilde{\psi}(\mathbf{0}) = c_r^{-1}$ since $\psi^{\partial}(0) = 0$ and $\varphi(0) = 1$). Introduce a signed measure on Γ via

$$\begin{aligned} \mu(dy,i) &:= \varphi(y) f(y,i)\, p_i m(dy), \quad y > 0, i = 1,2,\ldots,n, \\ \mu(\mathbf{0}) &:= \Delta_{\mathbf{0}}. \end{aligned}$$

Notice that, by assumption (51), μ is of finite total variation. Straightforward manipulations (cf. (32) and (33)) yield

$$g(x,i) = \frac{1}{p_i}\varphi(x)u(x,i) + \frac{1}{c_r}\varphi(x)\mu(\Sigma), \tag{82}$$

where

$$u(x,i) = \int_{(0,x]} \frac{\psi^{\partial}(y)}{\varphi(y)}\, \mu(dy,i) + \psi^{\partial}(x)\, \mu((x,\infty),i). \tag{83}$$

Next we calculate the scale derivative of g at $(0,i)$. Since $\varphi^+(0) = -c_r$ and $u(0,i) = 0$ we have from (82)

$$g^+(0,i) = \frac{1}{p_i} u^+(0,i) - \mu(\Gamma),$$

and $g^+(0,i)$ exists if and only if $u^+(0,i)$ exists. Hence consider

$$\lim_{\delta \downarrow 0} \frac{u(y,i) - u(0,i)}{S(y) - S(0)} = \lim_{\delta \downarrow 0} \frac{u(\delta,i)}{S(\delta)}.$$

The scale derivative of the first term on the r.h.s. of (83) is zero. To show this, introduce for $y > 0$ and $i = 1, 2, \dots, n$

$$\begin{aligned}\mu_+(dy,i) &:= \varphi(y) f(y,i) \mathbf{1}_{\{f(y,i) \geq 0\}} p_i m(dy) \quad \text{and} \quad \mu_-(dy,i) \\ &:= -\varphi(y) f(y,i) \mathbf{1}_{\{f(y,i) < 0\}} p_i m(dy),\end{aligned}$$

and then

$$\lim_{\delta \downarrow 0} \left| \frac{1}{S(\delta)} \int_{(0,\delta]} \frac{\psi^\partial(y)}{\varphi(y)} \mu(dy,i) \right| \leq \lim_{\delta \downarrow 0} \left(\frac{1}{S(\delta)} \frac{\psi^\partial(\delta)}{\varphi(\delta)} \left(\mu_+((0,\delta],i) + \mu_-((0,\delta],i) \right) \right) = 0,$$

since μ_+ and μ_- are finite measures and $\lim_{\delta \downarrow 0} \psi^\partial(\delta)/S(\delta) = 1$ (see (4)). Consequently,

$$u^+(0,i) = \lim_{\delta \downarrow 0} \frac{1}{S(\delta)} \mu((\delta,\infty),i) = \mu((0,\infty),i),$$

and, hence,

$$\begin{aligned}\mathbf{D}g(\mathbf{0}) &= \sum_{i=1}^n p_i \, g^+(0,i) = \sum_{i=1}^n p_i \left(\frac{1}{p_i} u^+(0,i) - \mu(\Gamma) \right) = \sum_{i=1}^n \mu((0,\infty),i) - \mu(\Gamma) \\ &= -\mu(\mathbf{0}) = -\Delta_0\end{aligned}$$

proving (a).

Consider now the claim (b). To deduce formula (56) the scale derivative of $g(\cdot,i)$ is needed. From (82) it is seen then that we should calculate the scale derivative of $u(\cdot,i)$. It holds for $\delta \geq 0$

$$\begin{aligned}u(x+\delta,i) - u(x,i) &= \int_{(x,x+\delta]} \frac{\psi^\partial(y)}{\varphi(y)} \mu(dy,i) + \frac{\psi^\partial(x+\delta)}{\varphi(x+\delta)} \mu((x+\delta,\infty),i) \\ &\quad - \frac{\psi^\partial(x)}{\varphi(x)} \mu((x,\infty),i) \\ &= \int_{(x,x+\delta]} \frac{\psi^\partial(y)}{\varphi(y)} \mu(dy,i) - \frac{\psi^\partial(x)}{\varphi(x)} \mu((x,x+\delta],i) \\ &\quad + \left(\frac{\psi^\partial(x+\delta)}{\varphi(x+\delta)} - \frac{\psi^\partial(x)}{\varphi(x)} \right) \mu((x+\delta,\infty),i) \\ &= \int_{(x,x+\delta]} \left(\frac{\psi^\partial(y)}{\varphi(y)} - \frac{\psi^\partial(x)}{\varphi(x)} \right) \mu(dy,i) \\ &\quad + \left(\frac{\psi^\partial(x+\delta)}{\varphi(x+\delta)} - \frac{\psi^\partial(x)}{\varphi(x)} \right) \mu((x+\delta,\infty),i) \\ &=: F_1(x,\delta) + F_2(x,\delta).\end{aligned}$$

Since $x \mapsto \psi^{\partial}(x)/\varphi(x)$ is non-decreasing and (cf. (5))

$$\lim_{\delta \downarrow 0} \frac{1}{S(x+\delta) - S(x)} \left(\frac{\psi^{\partial}(x+\delta)}{\varphi(x+\delta)} - \frac{\psi^{\partial}(x)}{\varphi(x)} \right) = \frac{1}{\varphi^2(x)}$$

we have

$$\lim_{\delta \downarrow 0} \frac{F_1(x,\delta)}{S(x+\delta) - S(x)} = 0$$

and

$$\lim_{\delta \downarrow 0} \frac{F_2(x,\delta)}{S(x+\delta) - S(x)} = \frac{\mu((x,\infty),i)}{\varphi^2(x)}.$$

Straightforward manipulations yield

$$g^+(x,i)\varphi(x) - g(x,i)\varphi^+(x) = \frac{1}{p_i}\mu((x,\infty),i),$$

as claimed in (b).

Appendix D. Smooth fit theorem for diffusion spiders

To make the paper more self contained, we discuss here the smooth fit property for diffusion spiders. In the formulation below the differentiation is taken with respect to an arbitrary continuous and increasing function F; the "usual" choices being $F = S$ (the scale function) or $F(x) = x, x > 0$. Notice that we consider the smooth fit "only" on the legs of the spider (and not in the vertex). In fact, in this case, the result is as in the diffusion case presented in Salminen and Ta [34] (and earlier in [33]), see also Peskir [28].

Theorem 11. *Let $(z,i), z \neq 0$ be on the leg i a left boundary point of the stopping set Σ, i.e., $([z, z+\varepsilon_1), i) \subset \Sigma \setminus \{\mathbf{0}\}$ and $((z-\varepsilon_2, z), i) \subset \Sigma^c \setminus \{\mathbf{0}\}$ for some positive ε_1 and ε_2. Let F be a continuous and increasing function on $\mathbb{R}_+$ and assume that the payoff function g and the functions φ_α and $\tilde{\psi}_\alpha$, $\alpha \geq 0$, are F-differentiable at (z,i). Then the value function V in (45) is F-differentiable at (z,i) and the smooth fit with respect to F holds:*

$$\frac{d^+V}{dF}(z,i) = \frac{d^-V}{dF}(z,i) = \frac{dg}{dF}(z,i). \tag{84}$$

Proof. Since $V > g$ on Σ^c and $V = g$ on Σ we have

$$\begin{aligned}\frac{d^+V}{dF}(z,i) &:= \lim_{\delta \to 0+} \frac{V(z+\delta,i) - V(z,i)}{F(z+\delta) - F(z)} = \lim_{\delta \to 0+} \frac{g(z+\delta,i) - g(z,i)}{F(z+\delta) - F(z)} \\ &= \frac{d^+g}{dF}(z,i) = \frac{dg}{dF}(z,i)\end{aligned}$$

and

$$\begin{aligned}\frac{d^-V}{dF}(z,i) &:= \lim_{\delta \to 0+} \frac{V(z-\delta,i) - V(z,i)}{F(z-\delta) - F(z)} = \lim_{\delta \to 0+} \frac{V(z,i) - V(z-\delta,i)}{F(z) - F(z-\delta)} \\ &\leq \lim_{\delta \to 0+} \frac{g(z,i) - g(z-\delta,i)}{F(z) - F(z-\delta)} \\ &= \frac{d^-g}{dF}(z,i) = \frac{dg}{dF}(z,i).\end{aligned}$$

Consequently,

$$\frac{d^- V}{dF}(z,i) \leq \frac{d^+ V}{dF}(z,i),$$

and, hence, (31) implies

$$\frac{d^- V}{dF}(z,i) = \frac{d^+ V}{dF}(z,i),$$

proving the claim. □

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