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On the finiteness of the moments of the measure of level sets of random fields

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Abstract. General conditions on smooth real valued random processes and fields are given to ensure the finiteness of the moments of the measure of their level sets. As a by product, a new generalized Kac–Rice formula for the expectation of the measure of these level sets in the one-dimensional case is obtained when the second moment is uniformly bounded. The conditions involve: (i) the differentiability of the trajectories up to a certain order k; (ii) the finiteness of the moments of the k-th partial derivatives of the field up to another order; and (iii) the boundedness of the field's joint density and some of its derivatives. Particular attention is given to the shot noise processes and fields. Other applications include stationary Gaussian processes, Chi-square processes and regularized diffusion processes. We also sketch the application of these tools to study critical points of random fields.

1 Introduction

Level sets of random fields play a key role in several branches of mathematics such as random algebraic geometry, probability, and mathematical physics. The focus is on different geometric characteristics of the level sets, typically their geometric measure. Depending on the dimensions, these characteristics can be the number of crossings of a stochastic process, the length of level curves of a random field, etc. Since the actual distribution of these random variables is usually out of reach, it is natural to study their moments and asymptotic distributions. See Adler and Taylor (2007), Azaïs and Wschebor (2009) and Berzin, Latour and León (2022) for a general picture of the field.

The present paper addresses the following two issues: (i) assessing the finiteness of the moments and (ii) their computation or estimation. For (ii) the main tool is the Kac–Rice formula (KRF for short) which gives the expected value and the higher order moments of the measure of level sets of smooth random fields. These two issues are highly connected, for instance:

- The KRF of order one is, in general, valid under conditions that imply (without further hypotheses) that the expectation is finite. Furthermore, in some cases, necessary and sufficient conditions for the finiteness of the expectation of the measure of level sets can be obtained from the KRF (see Armentano et al. (2019) and the references therein).
- For stationary Gaussian random fields, the KRF of order two can be used to obtain the finiteness of the second moment of the measure of level sets as in the works by Geman (1972), Kratz and León (2006) and Azaïs and León (2020). Some complicated studies have

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been carried out for higher moments in Belyaev (1966). In other cases, the calculations remain intractable.
- In the other direction, as explained in Section 8, the finiteness of the second moment is a tool to establish the validity of the KRF.

The seminal works in this field are those of Kac (1944) and Rice (1945). The KRFs were first established for Gaussian stochastic processes, profiting from the fact that the Gaussian framework allows not only to obtain conditions under which the formulas are valid but also permits some explicit computations. Adler (1981) obtained KRF for Gaussian fields. Recent works of Angst and Poly (2018), Armentano et al. (2019), Ancona and Letendre (2021) and Gass (2021) are also worth mentioning. In Armentano et al. (2019) and Angst and Poly (2018), the finiteness of the moments of the measure of nodal sets of real-valued Gaussian stationary fields defined on $\mathbb{R}^d$ is established. In the former, all the moments are shown to be finite when the field is C^∞ and in the latter the moments up to the order $\frac{d+1}{2}$ are shown to be finite. In Ancona and Letendre (2021) the authors study carefully the Rice function ρ_k, $k \geq 2$ (i.e., the integrand in KRF for the k-th moment) and produce estimates for the k-th moment, $k \geq 2$, of smooth Gaussian processes with fast decaying correlations. As a consequence, they are able to deduce strong laws of large numbers, central limits theorems for the number of zeros (and linear statistics of the zero set) and the asymptotic equidistribution of the zeros. In the way, the authors identify the Rice function ρ_k with the k-point function of the zeros viewed as a point process and they study the order of vanishing and the continuity of ρ_k. These results are improved in Gass (2021) weakening the hypothesis and quantifying the convergence by considering the cumulants instead of the moments. For a panoramic and contemporary view of these matters, we refer to the books by Adler and Taylor (2007), Azaïs and Wschebor (2009) and Berzin, Latour and León (2022).

Mainly motivated by the applications, there has been an interest in studying such formulas for non-Gaussian processes. The first successful attempt was that of Marcus (1977), who provided a formula for the expected number of crossings of a process whose trajectories are absolutely continuous. Concerning the finiteness of moments, by using properties of the trajectories of regular processes, Nualart and Wschebor (1991) show that the expectation of the number of crossings and its moments of order greater than one can be bounded. This result is based on the idea that a "nice function" cannot have too many zeros. Unfortunately, the proof heavily relies on the intermediate value theorem, and for this reason, it applies only to stochastic processes. Wschebor (1985) establishes the KRF for the measure of the level sets of fields (not necessarily Gaussian). However, it is important to point out that the hypotheses of these formulas are difficult to check. An important exception is when the field is a nonlinear transformation of a Gaussian one, as is the case of χ^2, t, or F random fields. The books by Adler and Taylor (2007) and Azaïs and Wschebor (2009) contain a comprehensive update of these subjects. The papers by Biermé and Desolneux (2012a), Borovkov and Last (2012) and Dalmao and Mordecki (2015) consider crossings for discontinuous processes, and the two last works include KRFs. Moreover, Biermé and Desolneux (2012b) and Biermé and Desolneux (2016) study crossings problems and KRF for shot noise processes.

Within the applications of the KRF we mention the random sea surface modeling and the articles by Machado and Rychlik (2003), Podgorki, Rychlik and Wallin (2015), Baxevani, Podgorki and Rychlik (2014) and Podgorki and Rychlik (2008) that contain Gaussian and general KRFs. Worsley (1995b) computes the expectations of some level sets characteristic in the context of medical image processing, and Worsley (1995a) considers similar applications in astrophysics. Other applications and KRFs for fields can be found in the recent monograph by Berzin, Latour and León (2022).

The finiteness of moments considered in this paper remains an open problem, except for the stationary Gaussian case, limited to the first and second moments. This finiteness can be

a first step in order to establish speed of convergence results for ergodic processes or fields, when observations are made on an increasing subset of the parameter space. In the particular case of η-dependent random processes the finiteness of the $2+\delta$ moment for some $\delta > 0$ gives directly a central limit theorem, see Example 6. This finiteness gives also a central limit theorem in the case of increasing number of independent observations of the random field.

The rest of the paper is organized as follows. The main result in dimension one is given by Theorem 1 in Section 2. Section 3 considers the application of Theorem 1 to different examples with new results, with the exception of Example 1, even in the case of Gaussian or χ^2 processes. Section 4 contains the study of shot noise processes, this theme, together with shot noise random fields, constitutes the main application of our results. In Section 5, Theorem 1 is extended to real-valued random fields defined on $\mathbb{R}^d$ and shot noise random fields are studied in this framework. Section 6 adapts the previous section to the case of real-valued random fields defined on the sphere S^d including an application to shot noise random fields defined on the sphere. In Section 7, we sketch how these tools can be used to assess the finiteness of the moments of the number of critical points of random fields. Finally, non-Gaussian KRFs are considered in Section 8.

2 Processes from $\mathbb{R}$ to $\mathbb{R}$

The main object of this section is the number of crossings through a level u over a compact time interval, by the trajectories of a smooth stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. As usual, $\mathbf{E}$ denotes the mathematical expectation w.r.t. the probability $\mathbf{P}$.

Definition 1. Assume that $X = \{X(t) : t \in I\}$ is a real valued stochastic process, where $I \subset \mathbb{R}$ is a compact interval, with k times continuous derivatives, that is, with trajectories in the class $C^k(I)$. We define the number of crossings through level u by the process X over I by

$$N_u = N_u(I) := \#\big\{t \in I : X(t) = u\big\}.$$

Our objective is to obtain conditions under which the random variable N_u has finite moments, and the basic idea is the following. Consider for example $f : [0,1] \to \mathbb{R}$. Assume that the kth derivative of f is bounded by M and that f has k zeros on $[0,1]$. Then f satisfies

$$\big|f(1/2)\big| \le M\frac{(1/2)^k}{k!}.$$

We now replace the function f by the paths of our random process X, assuming that it admits a bounded density at $t = 1/2$, denote by C the bound. Then, the probability of the event $\{|X(1/2)| \le M(1/2)^k/k!\}$ is smaller than

$$2CM\frac{(1/2)^k}{k!}.$$

Theorem 1 below is just a systematic exploitation of this method with some generalization considering the joint density of $X(t)$ and some of its higher order derivatives. In order to formulate the following result, denote by $|I|$ the length of a compact interval I and by $\bar{I}$ its middle point. As usual, we interpret the zeroth derivative $X^{(0)}$ as X itself. For each trajectory of the process X, denote

$$|X|_\infty = \sup_{t\in I}\big|X(t)\big|$$

Theorem 1. *Consider a stochastic process $X = \{X(t) : t \in I\}$ and let I, u and N_u be as above. Assume that X satisfies*

(H1) *The sample paths of X are $C^k(I)$ for some $k \geq 2$.*
(H2) *For some $m = 1, 2, \ldots$, there exists a constant D_m such that*

$$\mathbf{E}(|X^{(k)}|_\infty^m) \leq D_m.$$

(H3) *There exist $0 \leq h \leq k$ and a constant $C > 0$ such that the joint density of $X(t)$, $X'(t)$, $\ldots$, $X^{(h)}(t)$ is bounded by C uniformly in $t \in I$, and on a neighborhood of $(u, 0, \ldots, 0)$.*

Then, for $p = 1, 2, \ldots$, such that

$$p < \left(k - \frac{h}{2} - \frac{1}{1+h}\right)\left(\frac{1}{m} + \frac{1}{1+h}\right)^{-1}, \tag{2.1}$$

the pth moment of the number of crossings N_u is finite.

If in addition, we assume

(H4) *the density in* (H3) *is bounded by C uniformly in $t \in I$ and $(u_1, \ldots, u_{h+1}) \in \mathbb{R}^{h+1}$.*

Then

$$\mathbf{E}(N_u^p) \leq (k-1)^p + D_m \cdot E_{\alpha,k,p} + C|I|^{(h+1)(k-h/2)} \cdot D_{\alpha,k,h,p}, \tag{2.2}$$

where α is any real number such that $\frac{p}{m} < \alpha < k - \frac{h}{2} - \frac{1+p}{1+h}$, and the constants are given by

$$E_{\alpha,k,p} = p(k-1)^{p-1} \sum_{a=1}^{\infty} \frac{(a+1)^{p-1}}{a^{m\alpha}},$$

$$D_{\alpha,k,h,p} = \frac{2^{(h+1)(1+h/2-k)}}{k! \cdot (k-h)!} \sum_{a=1}^{\infty} (a+1)^{p-1} a^{1-(h+1)(k-h/2-\alpha)}.$$

Some remarks are in order. First, note that $h = 0$ and $m = 1$ are allowed, thus, the hypotheses are no so restrictive as they may seem at first sight. Naturally, the more information is available the more precise results are obtained.

Remark 1 (Large m). The limit case, when m can be chosen arbitrarily large (as is the case for Gaussian processes), corresponds to α close to zero. The limit condition (2.1) then becomes

$$p < \left(k - \frac{h}{2}\right)(h+1) - 1.$$

A particular case of this situation is when, for some $\beta > 0$ it holds

$$\mathbf{E}(e^{\beta|X^{(k)}|_\infty}) \leq D_\beta,$$

for some constant D_β, that is, we have exponential moments. In this case, the constant $E_{\alpha,k,p}$ in (2.2) can be replaced by

$$\tilde{E}_{\alpha,k,p} = p(k-1)^{p-1} \sum_{a=1}^{\infty} \frac{(a+1)^{p-1}}{e^{\beta a^\alpha}}.$$

Remark 2 (Large k). When the process X has $C^\infty(I)$ paths, if (H2) holds true for $m = 1$ with arbitrary k, and (H3) holds true for some $h = 0, 1, \ldots$, then, all the moments of N_u are finite.

Remark 3 ($h = 0$)**.** The case $h = 0$ and $m = 1$ corresponds to (3.23) in p. 82 in Azaïs and Wschebor (2009), obtained in Lemma 1.2 in Nualart and Wschebor (1991). When $h = 0$ and $m > 1$ Theorem 1 assures that the pth moment is finite for

$$p < \frac{m}{m+1}(k-1).$$

Before proving the theorem, we establish three preliminary lemmas. Recall that $f^{(0)} = f$.

Lemma 1. *Consider a function $f : I \to \mathbb{R}$ of class $C^k(I)$ for a compact interval I and $k \geq 1$. If the function $f - u$ has k roots in I and $0 \leq h \leq k$, we have*

$$|f(\bar{I}) - u| \leq \frac{|f^{(k)}|_\infty}{k!}\left(\frac{|I|}{2}\right)^k, \qquad |f^{(\ell)}(\bar{I})| \leq \frac{|f^{(k)}|_\infty}{(k-\ell)!}\left(\frac{|I|}{2}\right)^{k-\ell}, \quad \ell = 1, \ldots, h.$$

Proof. The proof is based on the Lagrange remainder form for polynomial interpolation. That is, let $g : I \to \mathbb{R}$ be C^k, $t_1, \ldots, t_k \in I$ and P be the only polynomial of degree $k-1$ such that $g(t_j) = P(t_j)$, $j = 1, \ldots, k$. Then, for $t \in I$ we have

$$g(t) - P(t) = \frac{1}{k!}\prod_{j=1}^{k}(t - t_j) \cdot g^{(k)}(\xi),$$

for some ξ such that $\min\{t_1, \ldots, t_k, t\} < \xi < \max\{t_1, \ldots, t_k, t\}$, see Lemma 5.2, p. 135 in Azaïs and Wschebor (2009). Now, for $g = f - u$, $t = \bar{I}$ and using as $t_1, \ldots, t_k$ the roots of $f - u$, we get $P = 0$ and so the first inequality ($h = 0$) follows. The proofs of the other inequalities follow in a similar way. □

Lemma 2. *Let Z be a random variable taking non negative integer values. Then,*

$$\mathbf{E}(Z^p) \leq (k-1)^p + p\sum_{\ell=k}^{\infty} \ell^{p-1}\mathbf{P}(Z \geq \ell).$$

Proof. We have

$$\begin{aligned}
\mathbf{E}(Z^p) &= \sum_{\ell=1}^{\infty} \ell^p \mathbf{P}(Z = \ell) = \sum_{\ell=1}^{k-1} \ell^p \mathbf{P}(Z = \ell) + \sum_{\ell=k}^{\infty} \ell^p \mathbf{P}(Z = \ell) \\
&= \sum_{\ell=1}^{k-1} \ell^p \mathbf{P}(Z = \ell) + \sum_{\ell=k}^{\infty} \ell^p \big[\mathbf{P}(Z \geq \ell) - \mathbf{P}(Z \geq \ell + 1)\big] \\
&= \sum_{\ell=1}^{k-1} \ell^p \mathbf{P}(Z = \ell) + (k-1)^p \mathbf{P}(Z \geq k) + \sum_{\ell=k}^{\infty} \big[\ell^p - (\ell-1)^p\big]\mathbf{P}(Z \geq \ell) \\
&\leq (k-1)^p + p\sum_{\ell=k}^{\infty} \ell^{p-1}\mathbf{P}(Z \geq \ell),
\end{aligned}$$

concluding the proof. □

Lemma 3. *Assume that conditions* (H1), (H2), (H3) *and* (H4) *of Theorem 1 hold true. Let $\ell \geq k$ and define a and r such that $\ell = a(k-1) + r$ with $1 \leq r \leq k-1$. Then, for $B > 0$, we have*:

$$\mathbf{P}\big(N_u \geq \ell, |X^{(k)}|_\infty \leq B\big) \leq aC(2B)^{h+1}\prod_{i=0}^{h}\left(\frac{|I|}{2a}\right)^{k-i}\frac{1}{(k-i)!},$$

where C is the bound in (H3).

When (H4) *does not hold, the inequality is true for ℓ sufficiently large only.*

Proof. We divide I into a equal subintervals $I_1, \dots I_a$, s.t. at least one of them has k zeros. After this, we use an union bound:

$$\begin{aligned}
&\mathbf{P}\big(N_u \geq \ell, |X^{(k)}|_\infty \leq B\big) \\
&\quad = \mathbf{P}\big(N_u \geq a(k-1)+r, |X^{(k)}|_\infty \leq B\big) \\
&\quad \leq \mathbf{P}\left(\bigcup_{j=1}^{a} \{N_u(I_j) \geq k\}, |X^{(k)}|_\infty \leq B\right) \\
&\quad \leq \sum_{j=1}^{a} \mathbf{P}\left(|X(\bar{I}_j) - u| \leq \frac{B}{k!}\left(\frac{|I|}{2a}\right)^k, \bigcap_{i=1}^{h} |X^{(i)}(\bar{I}_j)| \leq \frac{B}{(k-i)!}\left(\frac{|I|}{2a}\right)^{k-i}\right) \\
&\quad \leq aC \prod_{i=0}^{h} \frac{2B}{(k-i)!}\left(\frac{|I|}{2a}\right)^{k-i},
\end{aligned}$$

obtaining the result. Of course, when $h = 0$, the intersection in the equation above does not appear. □

Proof of Theorem 1. Assume (H4) for the moment. From Lemma 2, we have

$$\mathbf{E}\big(N_u^p\big) \leq (k-1)^p + \sum_{\ell=k}^{\infty} p\ell^{p-1}\mathbf{P}(N_u \geq \ell). \tag{2.3}$$

In order to bound the addends in the r.h.s. of (2.3), for given ℓ consider a and r as in Lemma 3, and $\alpha > 0$ to be defined later. We use the following bound:

$$\mathbf{P}(N_u \geq \ell) \leq \mathbf{P}\big(N_u \geq \ell, |X^{(k)}|_\infty \leq a^\alpha\big) + \mathbf{P}\big(|X^{(k)}|_\infty > a^\alpha\big).$$

Using now Lemma 3 for the first addend and Chebyshev's inequality for the second one, we obtain

$$\begin{aligned}
\mathbf{E}\big(N_u^p\big) &\leq (k-1)^p + p(k-1)^{p-1} \sum_{a=1}^{\infty} (a+1)^{p-1}\mathbf{P}(N_u \geq \ell) \\
&\leq (k-1)^p + p(k-1)^{p-1}\left(C2^{h+1}\left(\frac{|I|}{2}\right)^{(h+1)(k-h/2)}\right. \\
&\quad \left. \times \prod_{i=0}^{h} \frac{1}{(k-i)!} \sum_{a=1}^{\infty} (a+1)^{p-1} a^{(h+1)(\alpha+h/2-k)+1} + D_m \sum_{a=1}^{\infty} (a+1)^{p-1} a^{-m\alpha}\right).
\end{aligned}$$

The conditions for the simultaneous convergence of the two series above are

$$\frac{p}{m} < \alpha < k - \frac{h}{2} - \frac{1+p}{1+h}.$$

Under condition (2.1), there always exists an adequate α, concluding the proof of Theorem 1 when (H4) holds.

When (H4) does not hold, the calculation above is true only for ℓ (or a) sufficiently large, and this does not change the condition of convergence of the series. □

3 Examples

We present below four examples. The first is a simple one, that shows the power and limitations of Theorem 1. The other three give new results on moments of the number of crossings: the first one is the classical case of Gaussian processes; the second one considers Chi-square processes derived from Gaussian random variables; and the third one considers regularized diffusions as considered in Nualart and Wschebor (1991).

Example 1 (Conditional sine-cosine process). We begin with a toy example. Consider a random variable ω such that $\mathbf{E}(|\omega|^M) < \infty$ and $\mathbf{E}(|\omega|^{M+1}) = \infty$. This variable is the random frequency of a process $X = \{X(t) : 0 \le t \le 2\pi\}$ that we construct as

$$X(t) = \xi_1 \sin \omega t + \xi_2 \cos \omega t,$$

where ξ_1 and ξ_2 are two standard normal independent random variables, also independent from ω. Conditionally to the value of ω, the process X is a Gaussian sine-cosine process. It is clear that the density of $X(t)$ is standard normal for each t. Nevertheless, it should be observed that $X(t)$ is, in general, not a Gaussian process. The number N_0 of roots of $X(t)$ on $[0, 2\pi]$ coincides with the number of roots of the process

$$\tilde{X}(t) = \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}} \sin \omega t + \frac{\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}} \cos \omega t = \cos(\omega t - \theta),$$

where $\theta = \arctan(\xi_1/\xi_2)$ is defined a.s. Therefore, N_0 satisfies $|N_0 - 2\omega| \le 2$. This means that for the considered process, the maximal finite moment of N_0 is M. To apply Theorem 1, we compute

$$|X^{(k)}|_\infty \le \omega^k \sqrt{\xi_1^2 + \xi_2^2}.$$

Note that $\sqrt{\xi_1^2 + \xi_2^2}$ follows a Rayleigh's distribution. Then, based on the independence of the three random variables above, we have

$$\mathbf{E}(|X^{(k)}|_\infty^m) \le H_m \mathbf{E}(\omega^{mk}),$$

where H_m is the finite moment of the Rayleigh distribution. So, we can apply Theorem 1 with $h = 0$, $k = M$ and $m = 1$, obtaining that the moments of order $p \le (M-1)/2$ are finite. This result should be compared with the true result which is M.

Example 2 (Stationary Gaussian process). Consider a stationary Gaussian process with spectral measure supported in a set with an accumulation point. By Ex. 3.4 in Azaïs and Wschebor (2009) we know that for all h the random vector $X(t), X'(t), \ldots, X^{(h)}(t)$ admits a joint density. By the Borell–Sudakov–Tsirelson theorem (see, for example, Th. 2.8 in Azaïs and Wschebor (2009)), for all m:

$$\mathbf{E}(|X^{(k)}|_\infty^m) \le +\infty.$$

It is then possible to take $h = k$ and m arbitrarily large. Thus, Formula (2.1) gives a finite moment of order

$$p \le \frac{k(k+1)}{2} - 2.$$

To visualize it, see Table 1. This result improves the previous one obtained in Nualart and Wschebor (1991) that is recovered in the case $h = 0$ and $m = 1$.

Table 1 *If X is C^k, following Theorem 1 we obtain that $\mathbf{E}(N_u^p)$ is finite for values of k and p above*

k	2	3	4
p	1	≤ 4	≤ 8

Example 3 (Chi-square process). Consider a Chi-square process $Y = \{Y(t) : t \in \mathbb{R}\}$ given by

$$Y(t) = \big|X(t)\big|^2 = \sum_{i=1}^{n} X_i(t)^2,$$

where $X = \{X(t) : t \in \mathbb{R}\}$ with $X(t) = (X_1(t), \ldots, X_n(t))$ is an $\mathbb{R}^n$ valued random process whose coordinates are n independent copies of a stationary Gaussian process with $C^k(\mathbb{R})$ paths and variance one. First, observe that the case $n = 1$ admits an ad-hoc treatment. If $n = 1$ for $u = 0$ we have $N_0^Y[0, T] = N_0^X[0, T]$, and for $u > 0$ we have $N_u^Y[0, T] = N_{\sqrt{u}}^X[0, T] + N_{-\sqrt{u}}^X[0, T]$. For $n \geq 2$, the level $u = 0$ is not interesting, as for this level the process is differentiable and nonnegative and has a bounded density, so Bulinskaya Lemma (Prop. 1.20 in Azaïs and Wschebor (2009)) gives that the number of crossings is zero a.s. Consequently, we consider $n \geq 2$ and $u > 0$. Observe that, excluding the uninteresting case where $\lambda_2 = \mathbf{Var}X'(t) = 0$, the joint density of the random variables $X(t)$, $X'(t)$ is bounded, as they conform a pair of independent Gaussian random variables.

It is not difficult to check the hypothesis of Theorem 1 in two different situations: $n = 2$ and $n \geq 3$. It is direct to see that Y has C^k paths and that $|Y|_\infty$ has moments of every order when considering a compact interval $I \subset \mathbb{R}$. Observe now that $Y(0)$ has a $\chi^2(n)$ density that is bounded for $n \geq 2$. We can then apply Theorem 1 with the given k, arbitrary m, and $h = 0$. Based on (2.1), we obtain the finiteness of the moments of the crossings, for $u > 0$, of order

$$p = k - 2,$$

which is relevant only in the case $k \geq 3$. A more refined analysis can be carried out. Regarding the derivative, we have

$$Y'(t) = 2\sum_{i=1}^{n} X_i(t)X_i'(t).$$

We see that conditional to $X(t)$, the random variable $Y'(t)$ has a Gaussian distribution with variance $4\lambda_2 Y(t)$. Its conditional density $p_{Y'|X=x}$ is bounded by

$$(\text{Const})\big(Y(t)\big)^{-1/2},$$

where (Const) denotes a meaningless constant whose value may change from line to line.

Let now B_1 and B_2 be two Borel sets of $\mathbb{R}$ with respective measures $|B_1|$ and $|B_2|$. Writing Y and Y' for $Y(t)$ and $Y'(t)$

$$\begin{aligned}
\mathbf{P}\big(Y' \in B_1, Y \in B_2\big) &= \int_{\mathbb{R}^n} dx \int_{B_1} p_{Y'|X=x}(y)\mathbf{1}_{\{Y\in B_2\}} p_X(x)\, dy \\
&\leq (\text{Const}) \int_{\mathbb{R}^n} |B_1| Y^{-1/2} \mathbf{1}_{\{Y\in B_2\}} p_X(x)\, dx \\
&= (\text{Const})|B_1| \int_{B_2} y^{-1/2} p_Y(y)\, dy
\end{aligned}$$

$$= (\text{Const})|B_1| \int_{B_2} y^{n/2-3/2} e^{-y/2}\, dy, \tag{3.1}$$

where $p_X(\cdot)$ and $p_Y(\cdot)$ are the density functions of X and Y, respectively. Note that we used the explicit expression of the density of a $\chi^2(n)$ distribution. When $n \geq 3$, the integrand in (3.1) is bounded yielding that

$$\mathbf{P}\big(Y' \in B_1, Y \in B_2\big) \leq (\text{Const})|B_1||B_2|.$$

This proves that the process satisfies the hypotheses of Theorem 1 with $h = 1$. The conclusion is that for $n \geq 3$, the number of crossings of Y has a finite moment of order

$$p = 2k - 3.$$

Example 4 (Regularized processes). In this example, we consider the number of crossings with a level u of a regularized diffusion. We depart from a diffusion $X = \{X(t) : t \geq 0\}$ defined as the solution of the stochastic differential equation

$$X(t) = x_0 + \int_0^t b\big(s, X(s)\big)\, ds + \int_0^t \sigma\big(s, X(s)\big)\, dW(s), \quad t \geq 0.$$

Here, $W = \{W(s) : s \geq 0\}$ is a standard Brownian motion, $x_0 \in \mathbb{R}$, and $b, \sigma : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are Lipschitz w.r.t. the second variable, that is, for $T > 0$ there exists $K_T > 0$ such that for $x, y \in \mathbb{R}$ and $0 \leq s \leq T$ it holds that

$$\big|b(s, x) - b(s, y)\big| + \big|\sigma(s, x) - \sigma(s, y)\big| \leq K_T |x - y|.$$

Assume also the linear growth condition

$$\big|b(s, x)\big| + \big|\sigma(s, x)\big| \leq K_T\big(1 + |x|\big).$$

Then, the above equation has a unique strong solution that satisfies

$$\sup_{0 \leq t \leq T} \mathbf{E}\big(X(t)^2\big) \leq H_T < \infty.$$

for a constant H_T (see Theorem 7.1 in Borodin (2017)). The regularized diffusion is defined by

$$X_\Psi(t) = (\Psi * X)(t),$$

where $\Psi : \mathbb{R} \to [0, \infty)$ is a C^∞-function with support contained in $[-1, 1]$ that integrates to one. Thus, X_Ψ is obtained from X by path-wise convolution with Ψ. We consider two cases:

Case I. We assume that $b = 0$ and $\sigma : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is strictly positive and C^3.

Case II. The diffusion coefficient σ is strictly positive and C^3 as in Case I, and there exist constants B_T, c_T and C_T such that

$$\big|b(s, x)\big| \leq B_T, \qquad 0 < c_T \leq \sigma(s, x) \leq C_T$$

for all $0 \leq s \leq T$ and $x \in \mathbb{R}$.

We start with Case I. With respect to the hypotheses of Theorem 1, we have that X_Ψ is C^∞ since it inherits the regularity of Ψ. Furthermore, for a large enough (depending on Ψ) the random variable $X_\Psi(t)$ has a uniformly bounded density on an interval $[a, T]$ for all finite T, see Lemma 3.1 in Nualart and Wschebor (1991). Besides, a direct computation gives

$$\big|X_\Psi^{(h)}\big|_\infty \leq c \cdot |X|_\infty,$$

with $c = \int |\Psi^{(h)}(u)|\,du$ and the infinity norm is taken on the interval $[a, T]$. Hence, it suffices to bound the moments of $|X|_\infty$. Now, the Burkholder–Davis–Gundy inequality (Th. IV-48 in Protter (2004)) gives

$$\mathbf{E}\big(|X|_\infty^2\big) \le C_1 \mathbf{E}\big([X]_T\big),$$

with $[X]_T$ the quadratic variation of X on $[0, T]$. Now, by Th. II-29 in Protter (2004), we have

$$\begin{aligned}[X]_T &= \left[\int_0^T \sigma\big(s, X(s)\big)\,dW(s)\right]_T = \int_0^T \sigma\big(s, X(s)\big)^2\,ds \\ &\le K_T^2 \int_0^T \big(1 + |X(s)|\big)^2\,ds \le 2K_T^2 \int_0^T \big(1 + |X(s)|^2\big)\,ds.\end{aligned}$$

Taking expectations, we have

$$\mathbf{E}\big([X]_T\big) \le 2K_T^2 \int_0^T \big(1 + \mathbf{E}\big(X(s)\big)^2\big)\,ds \le 2K_T^2 T(1 + H_T) < \infty.$$

Hence, by Theorem 1 with $h = 0$, $m = 2$ and arbitrary k, we obtain that

$$\mathbf{E}\big(N_u^p\big) < \infty,$$

for all p, as obtained in Nualart and Wschebor (1991).

For Case II, we apply Girsanov's theorem (Borodin (2017), Th. 10.1–10.2). We then have two SDE

$$\begin{aligned}dX_0(t) &= \sigma\big(t, X_0(t)\big)\,dW(t), \\ dX(t) &= b\big(t, X(t)\big)\,dt + \sigma\big(t, X(t)\big)\,dW(t).\end{aligned}$$

Consider the process density:

$$\rho(t) = \exp\left(\int_0^t \frac{b(s, X(s))}{\sigma(s, X(s))}\,dW(s) - \frac{1}{2}\int_0^t \left(\frac{b(s, X(s))}{\sigma(s, X(s))}\right)^2 ds\right). \tag{3.2}$$

Then, Girsanov's theorem states that for an arbitrary function on the trajectories of the process $F\colon C([0, T], \mathbb{R}) \to \mathbb{R}$ we have $\mathbf{E}(F(X)) = \mathbf{E}(\rho(T)F(X_0))$. If F is the p-power of the number of crossings on $[0, T]$, we have

$$\mathbf{E}\big(N_u(I, X)^p\big) = \mathbf{E}\big(\rho(T)N_u(I, X_0)^p\big) \le \mathbf{E}\big(\rho(T)^2\big)^{1/2}\mathbf{E}\big(N_u(I, X_0)^{2p}\big)^{1/2},$$

applying the Hölder's inequality (X and X_0 are defined above). To bound the first expectation in the r.h.s. consider the square of the density process in (3.2). Then, applying the bounds corresponding to Case II, we obtain

$$\rho(t)^2 \le \exp\left(2\int_0^t \frac{b(s, X(s))}{\sigma(s, X(s))}\,dW(s) - 2\int_0^t \left(\frac{b(s, X(s))}{\sigma(s, X(s))}\right)^2 ds\right)\exp\left(\left(\frac{B_T}{c_T}\right)^2 T\right).$$

Since the first exponential on the r.h.s. above is a martingale, we get

$$\mathbf{E}\big(\rho(T)^2\big) \le \exp\left(\left(\frac{B_T}{c_T}\right)^2 T\right) < \infty.$$

As, for all p, $\mathbf{E}(N_u(I, X_0)^{2p})$ is finite by Case I, the finiteness of $\mathbf{E}(N_u(I, X)^p)$ follows.

4 Shot noise processes

We now apply our main result in a particular relevant case, the shot noise process. This process was introduced by Rice (1944) and is currently under intense research (see references below). The crossings for shot noise processes where first considered in Orsingher and Battaglia (1982), under exponential kernel functions.

Definition 2. The stationary shot noise process $X = \{X(t) : t \in \mathbb{R}^d\}$ is defined by

$$X(t) = \sum_i \beta_i g(t - \tau_i), \tag{4.1}$$

where the *impulses* (β_i) conform a sequence of integrable i.i.d. random variables, (τ_i) is a Poisson field on $\mathbb{R}^d$ with constant intensity λ, the impulses are independent from the Poisson field, and $g : \mathbb{R}^d \to \mathbb{R}$ is some integrable function called the *kernel* function.

The facts that β_1 is an integrable random variable and that $g \in L^1(\mathbb{R}^d)$ ensures the a.s. convergence of the series in (4.1). A key point to apply Theorem 1 is the verification of condition (H4) with $h = 0$, which requires the boundedness of a density. This is a delicate issue as was previously noticed in Biermé and Desolneux (2012b), which is the main reference of this section from where we borrow the presentation and notations (see also Biermé and Desolneux (2011)). We consider here $d = 1$, the case $d > 1$ will be considered in Section 5.1 and the case of shot noise random fields defined on the sphere will be treated in Section 6.1. We begin by specializing Theorem 1 to the present situation.

Corollary 1. *Consider a stationary shot noise process* (4.1) *satisfying condition* (H1) *for some* $k \geq 1$, (H2) *for some* $m \geq 1$ *and* k *in* (H1), *and* (H3) *for* $h = 0$. *Then*,

$$\mathbf{E}(N_u^p) < \infty$$

for

$$p < \frac{m}{m+1}(k-1). \tag{4.2}$$

The rest of the section is devoted to obtain sufficient conditions to verify this corollary. The differentiability of the sample paths follows directly from the differentiability of the kernel g:

$$X^{(k)}(t) = \sum_i \beta_i g^{(k)}(t - \tau_i),$$

provided that the k-th derivative of g is integrable, that is, $g^{(k)} \in L^1(\mathbb{R})$.

Boundedness of moments (H2). We now give conditions on the impulse and the kernel in order to verify (H2) for given k and m. For simplicity of exposition, we assume that $I = [-1, 1]$ and the general case can be treated in the same way. We consider the partition of the real line $I_1 = I = [-1, 1]$ and $I_n = [-n, -n+1) \cup (n-1, n]$ $(n \geq 2)$. In this way, we can write

$$X^{(k)}(t) = \sum_{n=1}^{\infty} \sum_{i:\tau_i \in I_n} \beta_i g^{(k)}(t - \tau_i).$$

Proposition 1. *Consider the shot noise process in* (4.1) *with impulse such that* $\mathbf{E}(|\beta_1|^m) < \infty$ *for given* $m \geq 1$. *Define*

$$d_{k,n} = \sup_{t \in I_1, s \in I_n} \left| g^{(k)}(t - s) \right|,$$

for $k \geq 1$, and assume that

$$D_k = \sum_{n=1}^{\infty} d_{k,n} < \infty. \tag{4.3}$$

Then, the condition (H2) *holds true for m and k as above, that is,*

$$\mathbf{E}\big(|X^{(k)}|_{\infty}^m\big) < \infty.$$

Proof. We have

$$\begin{aligned}
|X^{(k)}|_{\infty} &= \max_{-1\leq t\leq 1} |X^{(k)}(t)| \leq \max_{-1\leq t\leq 1} \sum_{n=1}^{\infty} \sum_{i:\tau_i\in I_n} |\beta_i| |g^{(k)}(t-\tau_i)| \\
&\leq \sum_{n=1}^{\infty} \sum_{i:\tau_i\in I_n} |\beta_i| \max_{-1\leq t\leq 1} |g^{(k)}(t-\tau_i)| \leq \sum_{n=1}^{\infty} \sum_{i:\tau_i\in I_n} |\beta_i| \sup_{-1\leq t\leq 1, s\in I_n} |g^{(k)}(t-s)| \\
&= \sum_{n=1}^{\infty} d_{k,n} Z_n,
\end{aligned}$$

where $Z_n := \sum_{\tau_i\in I_n} |\beta_i|$. As the sets I_n are disjoint and have the same length, the random variables Z_n are i.i.d. Each one has a compound Poisson distribution. As $\mathbf{E}(|\beta_1|^m) < \infty$, we obtain $\mathbf{E}(Z_1^m) < \infty$. We now use Jensen's inequality, based on the convergence in (4.3), to obtain the following bound:

$$\left(\sum_{n=1}^{\infty} d_{k,n} Z_n\right)^m = D_k^m \left(\sum_{n=1}^{\infty} \frac{d_{k,n}}{D_k} Z_n\right)^m \leq D_k^{m-1} \sum_{n=1}^{\infty} d_{k,n} Z_n^m.$$

Now, as the random variables Z_n^m are i.i.d. and have finite moments,

$$\mathbf{E}\big(|X^{(k)}|_{\infty}^m\big) \leq D_k^{m-1} \sum_{n=1}^{\infty} d_{k,n} \mathbf{E}(Z_n^m) = D_k^m \mathbf{E}(Z_1^m) < \infty,$$

it concludes that condition (H2) holds true for k and m. □

Remark 4. Given a kernel $g^{(k)} : \mathbb{R} \to \mathbb{R}$ we define by $G^{(k)} : [0, \infty) \to [0, \infty)$ the monotone hull of $\max(|g^{(k)}(t)|, |g^{(k)}(-t)|)$, $t \geq 0$, as the smallest non-increasing function that dominates $|g^{(k)}(t)|$ and $|g^{(k)}(-t)|$ for $t \geq 0$. More precisely,

$$G^{(k)}(t) = \sup_{s>t} \big(|g^{(k)}(s)|, |g^{(k)}(-s)|\big).$$

Then, condition (4.3) is implied by the integrability of $G^{(k)}$. Then, when the kernel $g^{(k)}$ decreases monotonously for large positive and large negative values, condition (4.3) follows automatically from the integrability of the kernel $g^{(k)}$.

Boundedness of the density (H3). In Biermé and Desolneux (2011) Section 3.2, it is shown that when $\beta_1 = 1$ a.s., in the two following particular situations the stationary shot noise process has a bounded density:

(a) The kernel is $g(t) = e^{-t} \mathbf{1}_{\{t\geq 0\}}$, and the intensity $\lambda > 1$.
(b) The kernel satisfies $g(t) = t^{-\alpha}$ for $t \geq A$ for some $A > 0$ and $\alpha > 1/2$.

We present below a generalization of the results of Biermé and Desolneux (2012b) that constitutes one of the contributions of the present paper.

Proposition 2. *Consider a shot noise process* (4.1) *with a differentiable kernel. Denote by* T *an exponential random variable with parameter* λ. *Assume that either*

(A) *The impulse* β_1 *has a density bounded by* B *and either* $\mathbf{E}(1/|g(-T)|) < \infty$ *or* $\mathbf{E}(1/|g(T)|) < \infty$.

(B1) *Assume that* $\mathbf{E}(1/|\beta_1|) < \infty$. *Assume further that* $g(x)$ *has a strictly negative derivative in* $(0, +\infty)$, *and the function*

$$g_*^+(t) := \inf_{0\le s\le t} |g'(s)|, \quad (t \ge 0),$$

satisfies $\mathbf{E}(1/g_*^+(T)) < \infty$.

(B2) *The same as* (B1) *replacing* $g(x)$ *by* $g(-x)$ *and* $g_*^+(t)$ *by*

$$g_*^-(t) := \inf_{0\le s\le t} |g'(-s)|, \quad (t \ge 0),$$

Then $X(0)$ *has a bounded density.*

Note that Proposition 2 implies (H3) with $h = 0$. The proof requires the following simple result, that has a direct proof.

Lemma 4. *Consider two independent random variables* X *and* Y, *where* X *has a density bounded by* B.

(a) *Then, the sum* $X + Y$ *has a density bounded by* B.

(b) *If* $\mathbf{E}(1/|Y|) < \infty$, *the product* XY *has a density bounded by* $B\mathbf{E}(1/|Y|)$.

Proof of Proposition 2. Consider (A), assuming that $\mathbf{E}(1/|g(-T)|) < \infty$ (the other case is analogous). Based on Lemma 4 and the decomposition

$$X(0) = \sum_{i\,:\,\tau_i>0} \beta_i g(-\tau_i) + \sum_{i\,:\,\tau_i\le 0} \beta_i g(-\tau_i) =: X^+(0) + X^-(0),$$

as $X^+(0)$ and $X^-(0)$ are independent, it is enough to show that $X^+(0)$ has a bounded density. Define by T_1 and T_2 the first two positive occurrences of the Poisson process. Condition on T_2 and apply (b) in Lemma 4 to obtain that $\beta_1 g(-T_1)$ has a conditional density bounded by $B\mathbf{E}(1/|g(-T_1)||T_2)$. Applying now (a) in Lemma 4, we obtain that the sum $X^+(0)$ has a conditional density, denote it by $p_{X^+(0)|T_2}(x)$ with the same bound. Finally, integrating

$$p_{X^+(0)}(x) = \mathbf{E}\big(p_{X^+(0)|T_2}(x)\big) \le \mathbf{E}\big(B\mathbf{E}\big(1/|g(-T_1)||T_2\big)\big) = B\mathbf{E}\big(1/|g(-T_1)|\big),$$

concluding the proof in this case.

Let us consider now case (B1), the proof in the case (B2) is similar. We see first that, conditional on T_2, the random variable $g(-T_1)$ has a bounded density. In fact, by the change-of-variable formula

$$p_{g(-T_1)|T_2}(u) = \frac{1}{|g'(g^{-1}(u))|}\frac{1}{T_2}\mathbf{1}_{\{g^{-1}(u)\le T_2\}} \le \frac{1}{T_2 g_*^+(T_2)}.$$

By (b) in Lemma 4 the product $\beta_1 g(T_1)$ has a conditional density bounded by

$$\frac{1}{T_2 g_*^+(T_2)}\mathbf{E}\big(1/|\beta_1|\big). \tag{4.4}$$

Then, by conditional independence and (a) in Lemma 4, the sum $X^+(0)$ has a conditional density with the same bound. Finally, as the density of T_2 is $\lambda^2 t_2 e^{-\lambda t_2}$, integrating the bound

(4.4), we obtain that

$$p_{X^+(0)}(u) = \mathbf{E}\big(p_{X^+(0)|T_2}(u)\big) \le \int_0^\infty \frac{1}{t_2 g_*^+(t_2)} \mathbf{E}\big(1/|\beta_1|\big)\lambda^2 t_2 e^{-\lambda t_2}\, dt_2$$
$$= \lambda \mathbf{E}\big(1/|\beta_1|\big)\mathbf{E}\big(1/g_*^+(T)\big).$$

This concludes the proof of the proposition giving the respective bounds. □

Corollary 2. *Consider a shot noise process with a $C^\infty(\mathbb{R})$ kernel g such that for every $k \ge 1$*

$$g^{(k)}(t) \simeq c_k(t)e^{\alpha t} \quad \text{as } t \to -\infty \quad \text{and} \quad g^{(k)}(t) \simeq C_k(t)e^{-\alpha t} \quad \text{as } t \to +\infty, \tag{4.5}$$

where $\simeq$ means equivalence and c_k, C_k are polynomials of an arbitrary degree. Assume furthermore that $\lambda > \alpha$ and $\mathbf{E}(|\beta_1|) < \infty$. If in addition either

(i) *β_1 has bounded density or*
(ii) *$\mathbf{E}(1/|\beta_1|) < \infty$ and g, or $g(-x)$, has a strictly negative derivative on $(0, +\infty)$,*

then

$$\mathbf{E}\big(N_u^p\big) < \infty, \quad \text{for any } p \ge 1.$$

Proof. We apply Corollary 1 with $m = 1$ and arbitrarily large k. In view of (4.2), we derive the result for arbitrary p. The kernel is differentiable for any k. The relation (4.5) ensures condition (4.3), as it gives the integrability of the derivatives of any order of the kernel. It remains to see that the density of $X(0)$ is bounded, and this follows in case (i) from the fact that $\lambda > \alpha$ giving $\mathbf{E}(1/|g(-T_1)|) < \infty$, and the boundedness of the density follows by (A) in Proposition 2. In case of (ii), we apply (B) in Proposition 2. In this way, we conclude the proof. □

5 Random fields from $\mathbb{R}^d$ to $\mathbb{R}$

In this section, we consider a real valued random field defined in $\mathbb{R}^d$. Given a level u and a subset of the domain, the object of study is the level set of the trajectories corresponding to that level with the parameter restricted to the subset.

Definition 3. Consider a real valued random field $X = \{X(t) : t \in \mathbb{R}^d\}$ and define, for a given $u \in \mathbb{R}$, the level set C_u restricted to $\mathbb{D}_a$ (the closed ball with radius a centered at the origin) by the formula

$$C_u = C_u(\mathbb{D}_a) := \big\{t \in \mathbb{D}_a : X(t) = u\big\}.$$

Remark 5. Observe that under regularity conditions, the level set C_u is almost surely a manifold of co-dimension one (with or without boundary). In fact, Theorem 1 in Belyaev (1966) states that the set $\{t \in \mathbb{D}_a : X(t) = u, \nabla X(t) \text{ is singular}\}$ is a.s. empty and, provided the a.s. absence of critical points, the Implicit Function theorem gives a local chart a.s. (see also Wschebor (1985) for a proof).

The aim of this section is to generalize Theorem 1 into the present framework. To this end, we need to measure the object of study, that is, the level sets, and this is done through the $(d-1)$-dimensional integral geometric measure of a set following Morgan (2016).

Definition 4. Given a Borel set $B \subset \mathbb{D}_a$ we define its integral geometric measure by

$$\mathcal{I}_{d-1}(B) := c_{d-1}(a) \int_{v \in S^{d-1}} \int_{y \in v^\perp \cap \mathbb{D}_a} \#\{B \cap \ell_{v,y}\}\, dv_a^\perp(y)\, dS^{d-1}(v), \tag{5.1}$$

where the norming constant is half the volume of the sphere

$$c_{d-1}(a) = \frac{1}{2}\mathcal{H}_{d-1}(S_a^{d-1}) = \frac{\pi^{d/2}}{\Gamma(d/2)} a^{d-1}. \tag{5.2}$$

Above, $\mathcal{H}_{d-1}$ denotes the Hausdorff measure, dS^{d-1} is the uniform probability on the sphere S^{d-1}, $dv_a^\perp$ is the uniform probability on $v^\perp \cap \mathbb{D}_a$, and $\ell_{v,y}$ is the affine linear space $\{y + tv : t \in \mathbb{R}\}$.

Remark 6. When B is a rectifiable set in $\mathbb{D}_a$ the Crofton's formula (Morgan (2016), Eq. 3.16, p. 31) implies that its integral geometric measure coincides with its Hausdorff measure. If in addition, $B \subset \mathbb{D}_a$ is a codimension one smooth submanifold $\mathcal{H}_{d-1}(B)$ coincides with the (induced) Riemannian measure.

Once the definitions have been established, we are interested in the finiteness of the moments of the random variable $\mathcal{I}_{d-1}(C_u)$. In view of (5.1), to obtain the finiteness of the moments, the idea is to give conditions on X that ensure that its restriction to an arbitrary line in $\mathbb{R}^d$ (i.e., the integrand in (5.1)) verifies the hypotheses of Theorem 1 for some values of m, h and k, as stated in the following result.

Theorem 2. *Consider a real valued random field $X = \{X(t) : t \in \mathbb{D}_a\}$. Assume that X satisfies*

(H1′) *The sample paths of X are $C^k(\mathbb{D}_a)$, $k \geq 2$.*

(H2′) *For some $m = 1, 2, \ldots$ there exists a constant D_m such that*

$$\max_{|v|=1} \mathbf{E}\left(\left| \frac{\partial^k X}{\partial v^k} \right|_\infty^m \right) \leq D_m,$$

where $\frac{\partial^k}{\partial v^k}$ denotes the k-th directional derivative w.r.t. the vector v.

(H3′) *For some $0 \leq h \leq k$ there exists a constant $C > 0$ such that the joint density of*

$$X(t), \frac{\partial X(t)}{\partial v}, \ldots, \frac{\partial^h X(t)}{\partial v^h}$$

is bounded by C uniformly in $t \in \mathbb{D}_a$, $v \in S^{d-1}$, and $(u_1, \ldots, u_{h+1}) \in \mathbb{R}^{h+1}$.

Then, for $p \geq 1$ satisfying (2.1), the p-th moment of the measure of the level set $\mathcal{I}_{d-1}(C_u)$ is finite and is bounded by

$$\big(c_{d-1}(a)\big)^p \big((k-1)^p + D_m \cdot E_{\alpha,k,p} + C|2a|^{(h+1)(k-h/2)} \cdot D_{\alpha,k,h,p}\big),$$

where α and the coefficients $E_{\alpha,k,p}$ and $D_{\alpha,k,h,p}$ are as in Theorem 1.

Proof of Theorem 2. We first apply Jensen's inequality in (5.1):

$$\begin{aligned}\big(\mathcal{I}_{d-1}(C_u)\big)^p &= \left(c_{d-1}(a) \int_{v \in S^{d-1}} \int_{y \in v^\perp \cap \mathbb{D}_a} \#\{C_u \cap \ell_{v,y}\}\, dS^{d-1}(v)\, dv_a^\perp(y) \right)^p \\ &\leq \big(c_{d-1}(a)\big)^p \int_{v \in S^{d-1}} \int_{y \in v^\perp \cap \mathbb{D}_a} \big(\#\{C_u \cap \ell_{v,y}\}\big)^p\, dS^{d-1}(v)\, dv_a^\perp(y).\end{aligned}$$

Now, take expectation and apply Tonelli's theorem,

$$\mathbf{E}(\mathcal{I}_{d-1}(C_u))^p \le (c_{d-1}(a))^p \int_{v\in S^{d-1}} \int_{y\in v^\perp\cap\mathbb{D}_a} \mathbf{E}(\#\{C_u\cap \ell_{v,y}\})^p \, dS^{d-1}(v)\, dv_a^\perp(y).$$

Let us apply Theorem 1 to the expectation inside the integral. It is clear that the bound is maximal when the interval is maximal, and it corresponds to $y=0$. So,

$$\mathbf{E}(\mathcal{I}_{d-1}(C_u))^p \le (c_{d-1}(a))^p\big((k-1)^p + D_m\cdot E_{\alpha,k,p} + C|2a|^{(h+1)(k-h/2)}\cdot D_{\alpha,k,h,p}\big).$$

The expression in the r.h.s. above is finite due to Theorem 1. □

An easy consequence is the following result.

Corollary 3. *Assume that the random field in Theorem 2 is stationary Gaussian with a spectral density. Then, the moment of order p of the integral geometric measure of C_u is finite with $p=\frac{k(k+1)}{2}-2$.*

Proof. The only point to check is that, if we limit the parameter set to a line, the obtained Gaussian stationary process has a density so it satisfy the conditions of Example 2. □

Let us mention that the case $p=2$ and the case of arbitrary p were considered in Azaïs and León (2020) and in Armentano et al. (2019), respectively.

5.1 Shot noise random field in $\mathbb{R}^d$

Consider the shot noise random field $X\colon \mathbb{R}^d\to\mathbb{R}$ in Definition 2, where we now consider the case $d>1$. We are interested in the finiteness of high-order moments of the measure of level sets, restricted to the closed centered ball $\mathbb{D}_a$, of this differentiable random field. For the computation of the expectation of the measure of excursion sets of shot noise random fields with trajectories of bounded variation (possibly discontinuous), see Biermé and Desolneux (2016).

The following result is a direct application of Theorem 2.

Corollary 4. *Consider a shot noise random field* (4.1) *satisfying condition* (H1′) *for some* $k\ge 1$, (H2′) *for some* $m\ge 1$ *and* k *above, and* (H3′) *for* $h=0$. *Then,*

$$\mathbf{E}(\mathcal{I}_{d-1}(C_u(\mathbb{D}_a))^p) < \infty \tag{5.3}$$

for

$$p < \frac{m}{m+1}(k-1).$$

In the following paragraphs, we obtain sufficient conditions to verify this corollary. The differentiability of the trajectories follows directly from the differentiability of the kernel g, see Biermé and Desolneux (2016). Consider a multi-index $\alpha=(\alpha_1,\dots,\alpha_d)$ of non-negative integers such that $|\alpha|=\sum_i\alpha_i=k$ and the usual notation for the partial derivatives. Then

$$\frac{\partial^k X(t)}{\partial t^\alpha} = \sum_i \beta_i \frac{\partial^k g(t)}{\partial t^\alpha}(t-\tau_i),$$

is a shot noise provided that the α-th derivative of g is integrable.

Boundedness of moments (H2′). For simplicity of exposition, we take $a=1$, that is, the unitary ball $\mathbb{D}_1$. Define the partition of $\mathbb{R}^d$ given by $A_1=\mathbb{D}_1$ and

$$\{A_n=\mathbb{D}_{r_n}\setminus\mathbb{D}_{r_{n-1}}\colon n=2,3,\dots\},$$

where the sequence (r_n) has $r_1 = 1$ and is such that the volume of each set A_n equals the volume of A_1. In this way, we can write

$$\frac{\partial^k X}{\partial v^k}(t) = \sum_{n=1}^{\infty} \sum_{i:\tau_i \in A_n} \beta_i \frac{\partial^k g}{\partial v^k}(t - \tau_i).$$

We now present a result useful to verify the moment condition (H2′). The proof follows the same lines as that of Proposition 1 in dimension one, and is omitted.

Proposition 3. *Consider a shot noise with impulse s.t.* $\mathbf{E}(|\beta_1|^m) < \infty$ *for given* m, *and kernel s.t.*

$$D'_k = \sum_{n=1}^{\infty} d'_{k,n} < \infty, \quad \text{for given } k,$$

with

$$d'_{k,n} = \sup_{|v|=1} \sup_{t \in A_1, s \in A_n} \left| \frac{\partial^k g}{\partial v^k}(t - s) \right|.$$

Then, the condition (H2′) *holds true for* m *and* k *as above, that is,*

$$\sup_{|v|=1} \mathbf{E}\left(\left| \frac{\partial^k X}{\partial v^k} \right|_{\infty}^m \right) < \infty.$$

Boundedness of the density (H3′). Introduce the volume of the d-dimensional ball of radius r by

$$V(r) = k_d r^d, \quad k_d = \pi^{d/2} / \Gamma(1 + d/2).$$

Proposition 4. *Consider a shot noise random field* (4.1) *with a differentiable kernel. Denote by* T_1 *the occurrence of the Poisson field closest to the origin with the Euclidean distance* (*a.s. defined*). *Assume that either*

(A) *The impulse* β_1 *has a density bounded by* B *and* $\mathbf{E}(1/|g(-T_1)|) < \infty$.
(B) $\mathbf{E}(1/|\beta_1|) < \infty$ *and the function*

$$G_*(r) := \sup_{u \in \mathbb{R}} \int_{t \in g^{-1}(u) \cap B_r} \frac{1}{|\nabla g(t)|} \, d\mathcal{I}_{d-1}(t)$$

satisfies $\mathbf{E}(G_*(|T_1|)) < \infty$.

Then $X(0)$ *has a bounded density.*

The following auxiliary result is needed in the proof under (B). Its proof is elementary and thus omitted.

Lemma 5. *Denote by* (T_n) *the occurrence of the Poisson field ordered by the Euclidean distance to the origin. Then, the random variables* $|T_1|$ *and* $|T_2|$ *have the following densities*:

$$p_{|T_1|}(x) = d\lambda k_d e^{-\lambda k_d x^d} x^{d-1}, \qquad p_{|T_2|}(x) = d\lambda^2 k_d^2 e^{-\lambda k_d x^d} x^{2d-1}.$$

Proof of Proposition 4. This proof follows the same lines as those of the proof of Proposition 2. To prove case (A), we write

$$X(0) = \beta_1 g(-T_1) + \sum_{i=2}^{\infty} \beta_i g(-T_i).$$

Applying (b) in Lemma 4, we obtain that $\beta_1 g(-T_1)$ has a conditional density bounded by $B\mathbf{E}(1/|g(-T_1)||T_2)$. Applying now (a) in Lemma 4, by conditional independence we obtain that the sum $X(0)$ has a conditional density: $p_{X(0)|T_2}(x)$ with the same bound. Finally, integrating

$$p_{X(0)}(x) = \mathbf{E}\big(p_{X(0)|T_2}(x)\big) \le \mathbf{E}\big(B\mathbf{E}(1/|g(-T_1)||T_2)\big) = B\mathbf{E}\big(1/|g(-T_1)|\big),$$

concluding the proof in this case.

Let us consider now case (B). Applying the co-area formula we see that, conditional on T_2, the random variable $g(-T_1)$ has a bounded density:

$$p_{g(-T_1)|T_2}(u) = \int_{t\in g^{-1}(u)\cap \mathbb{D}_{|T_2|}} \frac{1}{|\nabla g(t)|}\frac{1}{k_d|T_2|^d}\mathcal{I}_{d-1}(dt) \le \frac{G_*(|T_2|)}{k_d|T_2|^d}.$$

By (b) in Lemma 4, the product $\beta_1 g(-T_1)$ has a conditional density bounded by

$$\frac{G_*(|T_2|)}{k_d|T_2|^d}\mathbf{E}(1/|\beta_1|). \tag{5.4}$$

Then, by conditional independence and (a) in Lemma 4, the sum $X(0)$ has a conditional density with the same bound (5.4). Integrating the bound (5.4) with the density of $|T_2|$ in Lemma 5, we obtain that

$$\begin{aligned} p_{X(0)}(x) = \mathbf{E}\big(p_{X(0)|T_2}(x)\big) &\le \frac{\mathbf{E}(1/|\beta_1|)}{k_d}\int_0^\infty \frac{G_*(r)}{r^d} e^{-\lambda k_d r^d}\lambda^2\, dk_d^2 r^{2d-1}\, dr \\ &= dk_d\lambda^2\mathbf{E}(1/|\beta_1|)\int_0^\infty G_*(r)e^{-\lambda k_d r^d} r^{d-1}\, dr = \lambda\mathbf{E}(1/|\beta_1|)\mathbf{E}(G_*(|T_1|)), \end{aligned}$$

which is finite because of our hypotheses, concluding the proof. □

Example 5. Consider a shot noise random field with a radial kernel of the form $g(t) = e^{-|t|^{2q}}$ $(q = 1, 2, \dots)$, and impulse s.t. $\mathbf{E}(|\beta_1|) < \infty$. Assume further that

(i) Either $2q < d$, or $2q = d$ and $\lambda k_d > 1$,
(ii) The impulse β_1 has a bounded density or $\mathbf{E}(1/|\beta_1|) < \infty$.

Let us check the finiteness of the moments (5.3). It is straightforward to verify (H1′) and (H2′). To see (H3′) assume first that β_1 has a bounded density. Then, as g is radial, we have

$$\mathbf{E}\left(\frac{1}{g(-T_1)}\right) = \mathbf{E}\left(\frac{1}{g(|T_1|)}\right) = \int_0^\infty \frac{1}{g(r)} p_{|T_1|}(r)\, dr < \infty, \tag{5.5}$$

due to the form of the kernel, Lemma 5, and condition (i).

Assume now that $\mathbf{E}(1/|\beta_1|) < \infty$. The sup in the function $G_*(r)$ in (B) in Proposition 4 is attained when $u = e^{-r^{2q}}$, giving $G_*(r) = e^{r^{2q}} r^{d-2q}$. A computation similar to (5.5) gives $\mathbf{E}(G_*(|T_1|)) < \infty$. In conclusion, the hypotheses of Corollary 4 are valid with arbitrary k, obtaining that the shot noise random field (4.1) verifies (5.3) with arbitrary p.

Example 6. Suppose that

- The kernel $g(\cdot)$ is of class $\mathcal{C}^5$ with compact support included in, say, $(-\eta/2, \eta/2)$.
- The impluse β has a finite moment of order 4.
- $\beta_1 g(T)$ has a bounded density, where T has a uniform distribution over $(-\eta/2, \eta/2)$.

Then

$$\mathbf{E}\big(\mathcal{I}_{d-1}\big(C_u(\mathbb{D}_a)\big)^3\big) < \infty.$$

We have a central limit theorem. For $u \neq 0$

$$\frac{\mathcal{I}_{d-1}(C_u(\mathbb{D}_a)) - \mathbf{E}(\mathcal{I}_{d-1}(C_u(\mathbb{D}_a)))}{\sqrt{\mathbf{Var}(\mathcal{I}_{d-1}(C_u(\mathbb{D}_a)))}} \to N(0,1) \quad \text{as } a \to \infty. \tag{5.6}$$

Clearly the random field $X(\cdot)$ is η-dependent is the sense that if $A_1, \ldots, A_m$ are m Borel sets with pairwise distance greater than η then $\{X(t); t \in A_1\} \ldots \{X(t); t \in A_m\}$ are independent.

As the paths are $\mathcal{C}^5$, using Proposition 3, we obtain

$$\sup_{|v|=1} \mathbf{E}\left(\left| \frac{\partial^k X}{\partial v^k} \right|_\infty^4 \right) < \infty,$$

and using the results of Lemma 6 below, the distribution of $X(t)$ is the sum of one atom at zero and a defective probability with bounded density. This implies by Theorem 2 that, for $u \neq 0$, $\mathcal{I}_{d-1}(C_u(\mathbb{D}_a))$ admits a finite moments of order 3. The proof is concluded using the central limit theorem for η dependent random fields (see Leonenko (1975)).

6 Random fields from S^d to $\mathbb{R}$

We now move to random fields defined on a sphere. Without loss of generality, we can assume that the sphere is $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$. Given $t \in S^d$ and a tangent vector $v \in T_t S^d$, we denote by $\frac{\partial^k X(t)}{\partial v^k}$ the k-th order derivative of X along the sphere. In this case, we define the integral geometric measure following Santaló (1976) or Howard (1993).

Definition 5. Consider a Borel subset $B \subset S^d$. The integral geometric measure of B is defined by

$$\mathcal{I}_{d-1}(B) = \beta_{2,d+1} \int_{E \in G_{2,d+1}} \#(B \cap E)\, dG_{2,d+1}(E), \tag{6.1}$$

where the norming constant is half the volume of the sphere of radius one,

$$\beta_{2,d+1} = \frac{\pi^{d/2}}{\Gamma(d/2)}.$$

Furthermore, $G_{2,d+1}$ denotes the Grassmanian of 2-dimensional subspaces of $\mathbb{R}^{d+1}$, and the integral is with respect to the uniform distribution on this space.

Remark 7. (1) Observe that $dG_{2,d+1}$ is the unique probability measure that is invariant under the action of the orthogonal group (see, for example, Knapp (1996), Theorem 8.36). (2) Observe that Crofton's formula holds true in this case. (3) Note that $E \cap S^d$ is a great circle for every $E \in G_{2,d+1}$.

With similar arguments as in the previous section, we obtain the following result.

Theorem 3. *Consider a real valued random field $X = \{X(t) : t \in S^d\}$. Assume that*:

(H1″) *X is $C^k(S^d)$ for a given k.*
(H2″) *There exist $m \in \mathbb{N}$ and a constant D_m such that*

$$\mathbf{E}\left(\max_{t \in S^d, v \in T_t S^d, |v|=1} \left| \frac{\partial^k X(t)}{\partial v^k} \right|^m \right) \leq D_m.$$

(H3″) *Let $0 \le h \le k$. Assume that there exists a constant $C > 0$ such that the joint density of*

$$X(t), \frac{\partial X(t)}{\partial v}, \dots, \frac{\partial^h X(t)}{\partial v^h}$$

is bounded by C uniformly in $t \in S^d$ and $v \in T_t S^d$, $|v| = 1$, and for $(u_1, \dots, u_{h+1}) \in \mathbb{R}^{h+1}$.

Then, for $p = 1, 2, \dots$, satisfying (2.1) the pth moment of the measure of the level curve $\mathcal{I}_{d-1}(C_u)$ is finite and bounded by

$$\beta_{2,d+1}^p\big((k-1)^p + D_m \cdot E_{\alpha,k,p} + C|2\pi|^{(h+1)(k-h/2)} \cdot D_{\alpha,k,h,p}\big),$$

where α and the coefficients $E_{\alpha,k,p}$, $D_{\alpha,k,h,p}$ are given in Theorem 1.

Proof. From Remark 7 and (6.1) applied to $\mathcal{M} = C_u$, we observe that

$$\#(C_u \cap E) = \#\big\{t \in E \cap S^d \colon X(t) = u\big\}$$

is the number of crossings of a process defined on a great circle of radius 1 in such a way that Theorem 1 can be applied. Now, the proof follows the same lines as proof of Theorem 2. □

6.1 Shot noise random field in S^d

We now define the shot noise random field on a sphere.

Definition 6. Let $\mathcal{P}$ be the standard Poisson field on the unit sphere S^d of $\mathbb{R}^{d+1}$ realized as a sequence $T_1, T_2, \dots, T_N$ of uniformly distributed random points on S^d where N is a Poisson random variable, all variables being independent. Let $g \colon [0, \pi) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function. The shot noise process is defined by

$$X(t) = \sum_{i=1}^{N} \beta_i g\big(\operatorname{dist}^2(t, T_i)\big), \quad t \in S^d, \tag{6.2}$$

where dist is the geodesic distance, and the β_i's are i.i.d. with distribution F.

We assume the following conditions:

(H4) $\beta_1 g(\operatorname{dist}^2(t, T))$ admits a bounded density, where T has a uniform distribution on the sphere.
(H5) The distribution F has moments of any order.

Note that (H4) is rather weak, it is met, for example, if β_1 has a bounded density and g is strictly positive on $[0, \pi]$ (apply Lemma 4). We have the following result.

Proposition 5. *Consider a shot noise random field defined by (6.2) and assume* (H4)–(H5). *Then, for every level $u \neq 0$, for every compact set W and for every integer p*

$$\mathbf{E}\big(\mathcal{I}_{d-1}^p(C_u \cap W)\big) < \infty.$$

The proof follows from Theorem 3.

Differentiability (H1″) *and Boundedness of moments* (H2″). We now check (H1″) and (H2″) for arbitrary k and m.

Let v be a norm 1 vector orthogonal to $t \in S^d$. Since the number of realizations of $\mathcal{P}$ is almost surely finite, for any k, the derivative of X along the sphere at $t \in S^d$ in the direction of v is given by

$$\frac{\partial^k X(t)}{\partial v^k} = \sum_{i=1}^{N} \beta_i \frac{d^k}{dz^k} g\big(\operatorname{dist}^2\big(\gamma_{t,v}(z), T_i\big)\big)\Big|_{z=0},$$

where $\gamma_{t,v}(z)$ is a $\mathcal{C}^\infty$ arc-length parametrization of a great circle through t in the direction of v such that $\gamma_{t,v}(0) = t$.

Then, X is $\mathcal{C}^k$ for all k, and by compactness of the sphere S^d we obtain

$$\frac{\partial^k X(t)}{\partial v^k} \leq (\text{Const}) \sum_{i=1}^{N} \beta_i.$$

The condition (H5) implies that the compound Poisson distribution of $\sum_{i=1}^{N} \beta_i$ admits moments of every order giving the desired result.

Boundedness of the density (H3″) *with* $h = 0$. For simplicity, we consider the hypothesis (H3″) with $h = 0$ and study the marginal density of $X(t)$.

Lemma 6. *For every $t \in S^d$ the distribution of $X(t)$ is the sum of one atom at zero and a defective probability with bounded density.*

Proof. We consider the distribution of $X(t)$ conditional to $\mathcal{P}(S^d) = k$ in the case $k > 0$. Under that condition it is well known that the T_i's, $i = 1 \ldots, k$, are i.i.d. with uniform distribution on S^d, so we write

$$X(t) = \beta_1 g\big(\text{dist}^2(t, T_1)\big) + \sum_{i=2}^{k} \beta_i g\big(\text{dist}^2(t, T_i)\big).$$

The terms in this sum are independent and, because of (H4), the first term has a bounded density. By convolution it is the same for the conditional distribution of $X(t)$ which admits a density bounded by the same constant. Since this bounds does not depend on k, it is also a bound for the density of $X(t)$ conditional to $\mathcal{P}(S^d) > 0$. Obviously, when $\mathcal{P}(S^d) = 0$, $X(t) = 0$ which gives the atom at zero. □

7 Critical points of a random field

In order to illustrate the application of our tools, in this section, we restrict ourselves to the simple case of a two dimensional parameter random field with C^3 paths. In this way, we obtain results on the moments of the number of critical points.

Let $X(\cdot)$ be a random field defined on $T := [-1, 1]^2 \to \mathbb{R}$ and denote its gradient by $X' = (X'_1, X'_2)$.

Definition 7. The number of critical points of X in T is defined by

$$\text{Crit}(T) := \#\big\{t \in T : X'(t) = 0\big\}.$$

Theorem 4. *Let $X(\cdot)$ be a random field defined on $T := [-1, 1]^2 \to \mathbb{R}$ such that*

1. *X has C^3 paths.*
2. *For all integer m, $\sup_{|v|=1} \mathbf{E}(\|\frac{\partial^3 X}{\partial v^3}\|_\infty^m) = C_m < \infty$.*
3. *The density of $X'(t) = (X'_1(t), X'_2(t))$ exists for $t \in T$ and is continuous and uniformly bounded by K_1 when $t \in T$.*
4. *We have*

$$\sup_{i=1,2; v \in S^1, t \in T; u \in \mathbb{R}^2} \mathbf{E}\big(\big|X''_{iv}(t)\big| \big| X'(t) = u\big) p_{X'(t)}(u) \leq K_2,$$

Then, for all real $p \in (0, 2)$

$$\mathbf{E}\big(\big(\text{Crit}(T)\big)^p\big) < \infty.$$

Note that these conditions are easily checked for stationary Gaussian processes. The proof follows the same lines as those of Theorems 1–2. We first prove some auxiliary lemmas to obtain a bound similar to the one obtained in in Lemma 3.

Lemma 7. *Under the conditions in Theorem 4, let I be a segment of T, let $h > 0$, then*

$$\mathbf{P}\Big(\min_{t\in I}\big\|X'(t)\big\|_\infty \le h\Big) \le K_1 h\big(h + 8K_2|I|\big), \tag{7.1}$$

where $|I|$ is the length of the segment.

Proof. We use a parameterization $\eta : [0, |I|] \to [-1, 1]^2$ for I by its arc length, and set $\bar{X}(t) = (\bar{X}_1, \bar{X}_2) = X(\eta(t))$, accordingly $\bar{X}_i'(t) = X_i'(\eta(t))$, for $i = 1, 2$. By the third hypothesis, we have

$$\mathbf{P}\big(\big\|\bar{X}'(0)\big\|_\infty \le h\big) \le K_1 \cdot h^2.$$

On the other hand, set

$$\mu := \min_{t\in[0,|I|]}\big\|\bar{X}'(t)\big\|_\infty.$$

Then,

$$\mathbf{P}(\mu \le h) \le \mathbf{P}\big(\big\|\bar{X}'(0)\big\|_\infty \le h\big) + \mathbf{P}\big(\exists t \in [0, |I|] : \big\|\bar{X}'(t)\big\|_\infty = h\big).$$

The event $\{\|\bar{X}'(t)\|_\infty = h\}$ can be decomposed into four sub-cases that are all equivalent, so we consider only one which is

$$\big\{\bar{X}_1'(t) = h; \big|\bar{X}_2'(t)\big| \le h\big\}.$$

By the Kac Lemma (Azaïs and Wschebor (2009), Lemma 3.1) and Fatou's lemmas we obtain the upper bound part of the KRF, setting $M = \#\{t \in I : \bar{X}_1'(t) = h, |\bar{X}_2'(t)| \le h\}$, we have

$$\begin{aligned}&\mathbf{P}\big(\exists t \in [0, |I|]\colon \bar{X}_1'(t) = h, \big|\bar{X}_2'(t)\big| \le h\big)\\ &\quad\le \mathbf{E}(M) \le \int_0^{|I|}\int_{-h}^{h} \mathbf{E}\big(\big|\bar{X}''_{1,v}(t)\big|\big|\bar{X}'(t) = h, x_2'\big)p_{\bar{X}'(t)}\big(h, x_2'\big)\,dx_2'\,dt.\end{aligned}$$

Here, $v \in S^1$ has the direction of I. Under the third and fourth hypotheses

$$\mathbf{P}\big(\exists t \in I : \bar{X}_1'(t) = h, \big|\bar{X}_2'(t)\big| \le h\big) \le 2|I|hK_1.K_2. \qquad \square$$

Lemma 8. *Under the conditions of Theorem 4, let $\ell = a^2 + r$ with $0 < r \le (a + 1)^2 - a^2$, then*

$$\mathbf{P}\big(\mathrm{Crit}(T) > \ell; \big\|X^{(3)}(\cdot)\big\|_\infty \le B\big) \le (\mathrm{Const})Ba^{-3}.$$

Proof. We partition T into a^2 sub-squares of size $2/a$, one of which, S say, contains at least two critical points C_1 and C_2. Consider the two central axes A_1 and A_2 of S of length $\frac{2\sqrt{2}}{a}$. The line passing through C_1, C_2 crosses at least one axis. We consider the case in which it intersects A_1, the other one is strictly equivalent (see Figure 1). Let P be the intersection of L with A_1. Applying the tools of Lemma 1 along L to X_1' and X_2' (we use the first hypothesis), we get

$$\big|X_i'(P)\big| \le (\mathrm{Const}) \cdot a^{-2}B, \quad i = 1, 2.$$

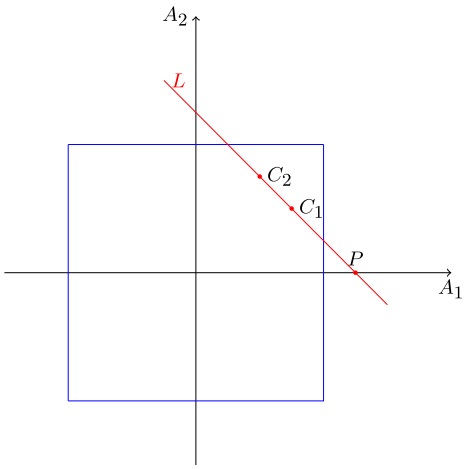


Figure 1 *The square S and the two axes.*

This implies that

$$m := \inf_{t\in A_1} \|X'(t)\|_\infty \le (\text{Const})a^{-2}B.$$

Applying Lemma 7 and recalling that $|A_1| = (\text{Const})/a$, the probability of this event is bounded by $(\text{Const}) \cdot a^{-3}B$. □

Proof of Theorem 4. We write C for $\text{Crit}(T)$. Using Lemma 3 and setting $\ell = a^2 + r$, $0 < r \le (a+1)^2 - a^2$, we have

$$\mathbf{E}(C^p) = p\mathbf{P}\{C \ge 1\} + p\sum_{\ell=2}^{\infty} \ell^{p-1}\mathbf{P}\{C \ge \ell\}$$

$$\le p + p\sum_{a=1}^{\infty}(a+1)^{2(p-1)}\mathbf{P}\{C \ge \ell\}.$$

We split $\mathbf{P}\{C \ge \ell\}$ as in the proof of Theorem 1 using Lemma 8 with $B = a^\alpha$, $\alpha > 0$, and Chebyshev's inequality (we use the second hypothesis), we get

$$\mathbf{E}(C^p) \le p + \text{Const} \cdot p\sum_{a=1}^{\infty}(a+1)^{2(p-1)}\left[a^{\alpha-3} + a^{-m\alpha}\right].$$

Thus, for convergence we need:

$$5 - 2p - \alpha > 1, \qquad 2 - 2p + m\alpha > 1.$$

Namely,

$$\frac{1-2p}{m} < \alpha < 4 - 2p.$$

Taking $m \to \infty$, we can take $\alpha \to 0^+$ and deduce that the p-th moment of C is finite for $p < 2$. □

Remark 8. Modulo boundary problems, the number of connected components of a nodal set is bounded by the number of critical points in the whole domain. The above result could be used to bound the moments of the number of nodal components, with some extra work.

8 Non Gaussian Kac–Rice formula

Gaussian KRFs are valid under weak and simple conditions and a comprehensive reference is the book Azaïs and Wschebor (2009) that treats all the relevant dimensions, that is, random fields X from $\mathbb{R}^D$ to $\mathbb{R}^d$ with $d \leq D$. These formulas give the expectation or the higher moments of the $\mathcal{H}_{D-d}$ Hausdorff measure of

$$C_u(H) = \{t \in H : X(t) = u\},$$

that is the level set restricted to a compact set $H \subset \mathbb{R}^D$. The proofs use basically the change-of-variable formula (or its generalization: the co-area formula) and have nothing to do with Gaussianity, its generalization to non-Gaussian cases encounters difficulties in defining properly the quantities involved in the formulas.

For instance, in the simplest case, the KRF for the expectation when $D = d = 1$ formally reads for a compact interval I:

$$\mathbf{E}\big(N_u(I)\big) = \int_I \mathbf{E}\big(|X'(t)| \big| X(t) = u\big) p_{X(t)}(u)\, dt. \tag{8.1}$$

In the non-Gaussian case, the conditional expectation is defined only for almost every level u. As a consequence, the punctual values of the r.h.s. of (8.1) are not defined unless some kind of continuity is established. This is why the non-Gaussian KRF requires complicated conditions. See Marcus (1977) and Azaïs and Wschebor (2009) for the case $D = d = 1$; for the case $D = d > 1$ the only reference is Adler and Taylor (2007). To our knowledge, in the non-Gaussian case, with the exception of the complicated treatment in Wschebor (1985), there exists no proof of the KRF for the case $D > d$.

Often, the process or the random field has C^k paths with k "large" and the conditions can be drastically simplified. This is the object of this section. Note that in its full generality, the statement of the KRF cannot be stated in the classical form.

Our first main result follows.

Theorem 5. *Let us consider a real-valued process $X = \{X(t), t \in I\}$, where I is a compact interval of $\mathbb{R}$. Assume that there exist $u \in \mathbb{R}$ and $\epsilon > 0$ such that*:

(a) *The sample paths of X are $C^1(I)$.*
(b) $\sup_{|v-u|<\epsilon} \mathbf{E}(N_v(I)^2) < \infty$.
(c) *The density of $X(t)$ at point v is uniformly bounded for $t \in I$ and for $|v - u| < \epsilon$.*

Then,

$$\mathbf{E}\big(N_u(I)\big) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_I \mathbf{E}\big(|X'(t)| \mathbf{1}_{|X(t)-u|\leq\delta}\big)\, dt. \tag{8.2}$$

Suppose in addition that $X'(t)$ has a finite expectation for every $t \in I$. Define for $|v - u| < \epsilon$

$$R(v) := \int_I \mathbf{E}\big(|X'(t)| \big| X(t) = v\big) p_{X(t)}(v)\, dt.$$

Then, (8.2) *can be rewritten as*

$$\mathbf{E}\big(N_u(I)\big) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_{u-\delta}^{u+\delta} R(v)\, dv. \tag{8.3}$$

Remark 9. In many situations it is easy to prove that the value given by the r.h.s. of (8.2) is equal to $R(u)$.

Proof. By (b) there exists a finite K such that

$$\mathbf{E}(N_v^2) \leq K,$$

for all $|v-u| < \epsilon$. Our conditions imply that, with probability 1, the process $X(t)$ cannot take the value u at the two extremities of I. In addition by the Bulinskaya lemma (Prop. 1.20 in Azaïs and Wschebor (2009)) there are, with probability 1, no extremes at the level u. Thus, the Kac lemma (Lemma 3.1 in Azaïs and Wschebor (2009)) yields

$$N_u = \lim_{\delta \to 0} N_u^\delta, \quad \text{almost surely,}$$

where N_u^δ is the Kac's counter defined by

$$N_u^\delta := \frac{1}{2\delta} \int_I |X'(t)| \mathbf{1}_{\{|X(t)-u| \leq \delta\}} \, dt.$$

By the area formula (Prop 6.1 in Azaïs and Wschebor (2009))

$$N_u^\delta = \frac{1}{2\delta} \int_{u-\delta}^{u+\delta} N_v \, dv.$$

This, associated to the Jensen inequality, yields

$$\mathbf{E}((N_u^\delta)^2) \leq K.$$

This implies in turn that the family N_u^δ is uniformly integrable. As a consequence, N_u^δ converges also in L^1, yielding

$$\mathbf{E}(N_u) = \lim_{\delta \to 0} \mathbf{E}(N_u^\delta).$$

Making explicit the r.h.s. above we get directly (8.2). Under the integrability of $X'(t)$, the conditional expectation $\mathbf{E}(|X'(t)||X(t))$ is well defined giving (8.3). □

Remark 10. Analogous to the situation in Theorem 5, the finiteness of the second moment of the level set can be obtained using Theorem 2.

Example 7 (The case $D = d = 1$). Suppose that $X(t)$ satisfies the conditions of Theorem 1 with and k, m, h such that

$$p = 2 < \left(k - \frac{h}{2} - \frac{1}{1+h}\right)\left(\frac{1}{m} + \frac{1}{1+h}\right)^{-1}.$$

In most of the cases, it is very easy to see that the process also satisfies all the conditions of Theorem 5, although this fact is not an exact consequence. This is the case for all the examples considered in Section 3. Of course, for Gaussian processes and χ^2 processes, the validity of the KRF has been known for a long time.

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