# OPTIMAL STOPPING FOR A COMPOUND POISSON PROCESS WITH EXPONENTIAL JUMPS

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Abstract. In this paper we give the closed form solution of some optimal stopping problems for processes derived from a compound Poisson process with exponential jumps. Within the possible applications, the results can be interpreted as pricing perpetual American Options under pure jump information.

## 0.Introduction.

Let be given on a probability space $(\Omega, \mathcal{F}, P)$ a Poisson process $N = (N_t)_{t\geq 0}$ with intensity $\lambda > 0$ and a sequence of independent nonnegative random variables $Y = (Y_k)_{k\geq 1}$, with identical distribution $F$.

According to Feller, ([3]), the compound Poisson process $X = (X_t)_{t\geq 0}$, is given by

$$X_t = \sum_{k=1}^{N_t} Y_k \qquad t \geq 0 \tag{0.1}$$

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by $X$, that is the minimal filtration such that $X$ is $\mathbb{F}$-adapted and satisfies the "usual" Dellacherie conditions (see Jacod and Shiryaev, [4]). $\tau$ is a stopping time (relative to $\mathbb{F}$) if $\tau\colon \Omega \to [0, +\infty]$ and $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \leq 0$. Denote by $\mathcal{M}$ the class of all stopping times.

The aim of this paper is to give closed form solution to the optimal stopping problem $\sup_{\tau\in\mathcal{M}} E(f(S_\tau))$, that is, we find a pair $(\tau^*, s(x))$, such that

$$s(x) = \sup_{\tau\in\mathcal{M}} E(f(S_\tau)) = E(f(S_{\tau^*})) \tag{0.2}$$

where $\tau^*$ is the optimal stopping time and $s(x)$ is the cost function in the following four cases. The function $f$ is given either by

$$f(x) = (x - K)^+,$$

or

$$f(x) = (K - x)^+,$$

with $K$ a real constant, and the process $S$ by

$$S_t = x + X_t - at, \qquad x \in \mathbb{R}, \tag{0.3}$$

or

$$S_t = x \exp\{-at + X_t\}, \qquad x > 0. \tag{0.4}$$

## 1. Main Results.

**Theorem 1.** *Let $S$ be given by (0.3), with $F(x) = 1 - e^{-\alpha x}$, and $0 < \lambda < a\alpha$. Denote*

$$x_0 = \frac{\lambda}{\alpha(a\alpha - \lambda)} + K, \qquad \rho = \alpha - \frac{\lambda}{a}, \tag{1.1}$$

*and define*

$$\tau^* = \inf\{t \geq 0 : S_t \geq x_0\},$$
$$s(x) = \begin{cases} x - K, & \text{if } x \leq x_0, \\ (x_0 - K)\exp\{\rho(x - x_0)\}, & \text{if } x < x_0. \end{cases}$$

*Then the pair $(\tau^*, s(x))$ is the solution to the problem (0.2) with $f(x) = (x - K)^+$.*

**Theorem 2.** *Let $S$ be given by (0.3). For arbitrary $F$ denote $\mu = \int_0^{+\infty} x dF(x)$, that can take the value $\mu = +\infty$. Suppose that $0 < a < \mu\lambda$. Denote by $\rho > 0$ the unique root of the equation*

$$\int_0^{+\infty} e^{-\rho y} dF(y) = 1 - \frac{a}{\lambda}\rho, \tag{1.2}$$

*and*

$$x_0 = K - 1/\rho.$$

*Define*

$$\tau^* = \inf\{t \geq 0 : S_t \leq x_0\},$$
$$s(x) = \begin{cases} K - x, & \text{if } x \leq x_0, \\ (K - x_0)\exp\{-\rho(x - x_0)\}, & \text{if } x > x_0. \end{cases}$$

*Then the pair $(\tau^*, s(x))$ is the solution to the problem (0.2), with $f(x) = (K - x)^+$.*

*Remark*: For $X$ given by (0.1) with exponential jumps, that is when $F(x) = 1 - e^{-\alpha x}$ we have $\mu = \frac{1}{\alpha}$ and $\rho = \frac{\lambda}{a} - \alpha$.

**Theorem 3.** *Let $S$ given by (0.4). Assume that $F(x) = 1 - e^{-\alpha x}$ with $\alpha > 1 + \frac{\lambda}{a}$. Denote*

$$x_0 = K\frac{a - \lambda/\alpha}{a - \lambda/(\alpha - 1)}, \qquad \rho = \alpha - \frac{\lambda}{a}. \tag{1.3}$$

*Define*

$$\tau^* = \inf\{t \geq 0 : S_t \geq x_0\},$$
$$s(x) = \begin{cases} x - K, & \text{if } x \geq x_0, \\ (x_0 - K)\left(\frac{x}{x_0}\right)^\rho, & \text{if } 0 \leq x < x_0. \end{cases}$$

*Then the pair $(\tau^*, s(x))$ is the solution to the problem (0.2), with $f(x) = (x - K)^+$.*

**Theorem 4.** *Let $S$ be given by (0.4). For arbitrary $F$ denote $\mu = \int_0^{+\infty} x dF(x)$, that can take the value $\mu = +\infty$. Suppose that $0 < a < \lambda\mu$. Denote by $\rho > 0$ the unique root of the equation (1.2), and*

$$x_0 = K\frac{\rho}{\rho+1}.$$

*Define*

$$\tau^* = \inf\{t \geq 0 : S_t \leq x_0\},$$
$$s(x) = \begin{cases} K - x, & \text{if } 0 < x \leq x_0, \\ (K - x_0)\left(\frac{x}{x_0}\right)^{-\rho}, & \text{if } x > x_0. \end{cases}$$

*Then the pair $(\tau^*, s(x))$ is the solution to the problem (0.2), with $f(x) = (K-x)^+$.*

*Remark*: For $X$ given by (0.1) with exponential jumps, that is when $F(x) = 1 - e^{-\alpha x}$ we have $\mu = \frac{1}{\alpha}$ and $\rho = \frac{\lambda}{a} - \alpha$, as in Theorem 2.

## 2. Pricing American Options under Poisson-Exponential information.

Beside the probabilistic interest we are motivated by the following interpretation: Suppose that $S$ models the evolution of the price of a risky asset in a financial market. Then, the proposed problems can be seen as the pricing of a perpetual American Option, as described for instance in Karatzas, [5] or Shiryev et al., [8].

For $S$ given by (0.3) we are in the linear case, having an analogous of the Bachelier model, with a compound Poisson process (eventualy compensated) instead of a Wiener process.

When $S$ is given by (0.4), we have American call and put Options, in an analogous of the Black and Scholes model (see [8]). Note that in the call case, we obtain a solution that is not differentiable at the optimal level $x_0$. This is due to the fact that the process is not stopped at $x_0$, because it jumps above this critical level, and the "smooth fitting" principle in its classical form, as described in [7] does not apply.

Other applications, for instance to investment under uncertainity can be found in Dixit and Pindyck, [1].

## 3.Proofs.

We begin with an auxiliary result of independent interest (compare with [6]).

**Lemma 3.1.** *Let $X$ be a compound Poisson process defined by (0.1). Let $S = (S_t)_{t\geq 0}$ be the solution of the stochastic differential equation*

$$dS_t = (aS_{t-} + b)dt + (cS_{t-} + e)dX_t, \qquad S_0 = x,$$

*with $a, b, c$ and $e$ real constants. $C^*$, the continuation region, is an open interval of the real line of the form $(x_0, +\infty)$, $(-\infty, x_0)$ or $(0, x_0)$ for some $x_0$ in $\mathbb{R}$. Let $s$ and $g$ denote Borel real functions, with $s$ continuous for $x \in R$ and differentiable for $x \neq x_0$.*

*Denote by $L$ the infinitesimal generator of the Markov process $S$, that is*

$$(Lf)(x) = \lambda \int_0^{+\infty} [f(x + cxy + ey) - f(x)]dF(y) + (ax + b)f'(x) \tag{3.1}$$

*for $f$ in the domain of $L$.*

*Let*

$$\tau^* = \inf\{t \geq 0 \colon S_t \notin C^*\}.$$

*Assume that the following five conditions hold:*

(1) $(Ls)(x) = 0 \qquad \forall x \in C^*.$

(2) $(Ls)(x) \leq 0 \qquad \forall x \neq x_0.$

(3) $0 \leq g(x) \leq s(x) \qquad \forall x \in \mathbb{R}.$

(4) $s(S_{\tau^* \wedge T \wedge t}) \leq Z$ *$P$-a.s for all $T \in \mathcal{M}$ and for all $t \in \mathbb{R}^+$, with $Z$ an integrable random variable, that is $E|Z| < +\infty$.*

(5) $s(S_{\tau^*}) = g(S_{\tau^*})$ *$P$-a.s.*

*Then the pair $(\tau^*, s)$ is the solution for the optimal stopping problem for the process $(g(S_t))_{t \geq 0}$:*

$$s(x) = \sup_{\tau \in \mathcal{M}} E(g(S_\tau)) = E(g(S_{\tau^*})).$$

*Proof.* For each $\omega \in \Omega$ we can write

$$\begin{aligned} s(S_t) - s(x) = \int_0^t s'(S_{s-})(aS_{s-} + b)ds + \sum_{0 \leq s \leq t} \Delta s(S_s) = \int_0^t \big\{ s'(S_{s-})(aS_{s-} + b) + \\ \lambda \int_0^{+\infty} [s(S_{s-} + cS_{s-}y + ey) - s(S_{s-})]dF(y) \big\} ds \\ + \sum_{0 \leq s \leq t} \Delta s(S_s) - \lambda \int_0^t [s(S_{s-} + cS_{s-}y + ey) - s(S_{s-})]ds \\ = \int_0^t (Ls)(S_{s-})ds + M_t. \end{aligned}$$

with $M = (M_t)_{t \geq 0}$ a local martingale (a compensated sum of jumps, [4]). In view of (1) we obtain that for any $T \in \mathcal{M}$

$$\int_0^{\tau^* \wedge T \wedge t} (Ls)(S_{s-})ds = 0, \tag{3.2}$$

and (2) gives

$$\int_0^{\sigma} (Ls)(S_{s-})ds \leq 0 \quad \text{for any} \quad \sigma \in \mathcal{M}. \tag{3.3}$$

Now in view of (2)

$$M_t = s(S_t) - s(x) - \int_0^t (Ls)(S_{s-})ds \geq -s(x) \tag{3.4}$$

and as $M = (M_t)_{t\geq 0}$ is a local martingale (3.4) gives that $M$ is a supermartingale by Fatou's Lemma, and in consequence

$$E(M_\sigma) \leq E(M_0) = 0 \quad \text{for any} \quad \sigma \in \mathcal{M}. \tag{3.5}$$

On the other side, by (4) and (3.2)

$$M_{\tau^* \wedge t \wedge T_n} \leq s(S_{\tau^* \wedge t \wedge T_n}) \leq Z$$

for a localizing sequence $(T_n)$ for $M$, that gives by dominated convergence

$$E(M_{\tau^* \wedge t}) = 0. \tag{3.6}$$

In conclusion (3.2) and (3.6) give

$$s(x) = E(s(S_{\tau^* \wedge t})), \tag{3.7}$$

and (3.3), (3.5) and (3) gives

$$s(x) \geq E(s(S_\sigma)) \geq E(g(S_\sigma)) \quad \text{for any} \quad \sigma \in \mathcal{M}.$$

By (4), (5) and (3.7) we obtain

$$s(x) = E(s(S_{\tau^*})) = E(g(S_{\tau^*})),$$

and the Lemma is proved.

The proof of the Theorems reduces now to the verification of the five conditions in Lemma 3.1.

*Proof of Theorem 1.* First observe that

$$(Lf)(x) = \lambda \int_0^{+\infty} [f(x+y) - f(x)]dF(y) - af'(x). \tag{3.8}$$

We now verify conditions (1) to (5) in Lemma 3.1.

(1) For $x < x_0$, by substitution

$$as'(x) + \lambda s(x) - \lambda \int_0^{+\infty} [f(x+y) - f(x)]\alpha \exp\{-\alpha y\}dy =$$
$$(x_0 - K)\exp\{\rho(x - x_0)\}\big(a\rho + \lambda + \frac{\lambda\alpha}{\rho - \alpha}\big)$$
$$- \lambda\alpha \exp\{\alpha(x - x_0)\}\big((x_0 - K)(\frac{1}{\alpha} + \frac{1}{\rho - \alpha}) + \frac{1}{\alpha^2}\big) = 0$$

Because

$$a\rho + \lambda + \frac{\lambda\alpha}{\rho - \alpha} = (x_0 - K)\big(\frac{1}{\alpha} + \frac{1}{\rho - \alpha}\big) + \frac{1}{\alpha^2} = 0$$

as can be verified by substitution of the values of $\rho$ and $x_0$ in (1.1).

(2) If $x > x_0$ we have

$$Ls(x) = \frac{\lambda}{\alpha} - a < 0.$$

(3) $s(x) > 0$ by definition, and $s(x) \geq x - K$ can be verified directly if $x \geq x_0$, and considering the tangent line to the convex function $s(x)$, at the point $x_0$ for $x < x_0$,

$$(x_0 - K)\exp\{\rho(x - x_0)\} \geq (x_0 - K) + \frac{\lambda}{a\alpha}(x - x_0) \geq x - K \qquad \text{if} \quad x < x_0.$$

because $\frac{\lambda}{a\alpha} < 1$.

(4) Take $Z = s(S_{\tau^*}) + x_0$. $s(S_{\tau^* \wedge T \wedge t}) \leq Z$ follows from the fact that on the set $\{\tau^* < +\infty\}$ we have $S_{\tau^* \wedge T \wedge t} \leq S_{\tau^*}$ and the function $s$ is increasing. On the set $\{\tau^* = +\infty\}$ we have $s(S_{\tau^* \wedge T \wedge t}) \leq x_0$.

Observe now that $s(S_{\tau^*}) = (S_{\tau^*} - K)^+ \leq \sup_{0 \leq t < +\infty} S_t = x + \sup_{0 \leq t < +\infty}(X_t - at)$ From [2] we know that

$$P\big(\sup_{0 \leq t < +\infty}(X_t - at) \geq u\big) = \frac{\lambda\mu}{a}\exp\{-\rho u\} \tag{3.9}$$

so we can compute

$$E\big(\sup_{0 \leq t < +\infty}(X_t - at)\big) = \frac{\lambda}{\alpha(a\alpha - \lambda)}$$

and (4) is proved.

(5) is direct, concluding the proof of Theorem 1.

*Proof of Theorem 2.* As in Theorem 1, we have to verify conditions (1) to (5) in Lemma 3.1. In this case, as in Theorem 1, $L$ is given by (3.8).

(1) If $x > x_0$, $s(x) = \frac{1}{\rho}\exp\{-\rho(x - x_0)\}$,

$$(Ls)(x) = \exp\{-\rho(x + y - x_0)\}\frac{\lambda}{\rho}\Big(\int_0^{+\infty}\exp\{-\rho y\}dF(y) - 1 + \frac{a\rho}{\lambda}\Big) = 0$$

by (1.2).

(2)[1] If $x < x_0$

$$\begin{aligned}(Ls)(x) &= -as'(x) + \lambda\int_0^{+\infty}(s(x + y) - s(x))dF(y) \\ &\leq -as'(x_0) + \lambda\int_0^{+\infty}(s(x_0 + y) - s(x_0))dF(y) = 0,\end{aligned}$$

[1]We add some missing arguments in the published version.

because $s'(x) = s'(x_0) = -1$, and

$$s(x+y) - s(x) \leq s(x_0+y) - s(x_0)$$

for each $y \geq 0$.

(3) $s(x) > 0$ is inmediate and $s(x) \geq K - x$ follows from the fact, that $s(x)$ is convex and the fucnction $\frac{1}{\rho}\exp\{-\rho(x-x_0)\}$ has tangent $y = K - x$ at the point $x = x_0$ (here applies the smooth pasting principle, because the function $f(x)$ is decreasing and we have jumps up).

(4) In this case we have $s(S_{\tau^*}) \leq x_0$.

(5) is also inmediate concluding the proof.

*Proof of Theorem 3.* Assume first that $F$ is arbitrary.

$$S_t = x\exp\{-at + X_t\} = x\exp\{-at\}\prod_{k=1}^{N_t} e^{Y_k}$$

that can be written as

$$S_t = x\mathcal{E}\big(-at + \sum_{k=1}^{N_t}(e^{Y_k} - 1)\big),$$

where $\mathcal{E}$ is the stochastic or Doléans-Dade exponential, so $S_t$ verifies the stochastic differential equation

$$dS_t = S_{t-}d(-at + \sum_{k=1}^{N_t}(e^{Y_k} - 1), \qquad S_0 = x,$$

and then, it has an infinitesimal generator given by

$$(Lf)(x) = \lambda\int_0^{+\infty}[f(xe^y) - f(x)]dF(y) - axf'(x) \tag{3.10}$$

If $F(x) = 1 - e^{-\alpha x}$, we transform this expresion into

$$(Lf)(x) = \lambda\int_0^{+\infty}[f(x(1+y)) - f(x)]\alpha(1+y)^{-(\alpha+1)}dy - axf'(x)$$

We now verify conditions (1) to (5) in Lemma 3.1.

(1) For $x < x_0$, by substitution

$$\begin{aligned} axs'(x) + \lambda s(x) - \lambda\int_0^{+\infty} s(x+xy)\alpha(1+y)^{-(\alpha+1)}dy = \\ (x_0 - K)\rho\big(\frac{x}{x_0}\big)^{\rho}[a + \frac{\lambda}{\rho-\alpha}] \\ + \big(\frac{x}{x_0}\big)^{\alpha}\big((\frac{\alpha}{\rho-\alpha} + \frac{\alpha}{\alpha-1})x_0 - (\frac{\alpha}{\rho-\alpha} + 1)K\big) = 0 \end{aligned}$$

Because

$$a + \frac{\lambda}{\rho - \alpha} = \big(\frac{\alpha}{\rho - \alpha} + \frac{\alpha}{\alpha - 1}\big)x_0 - \big(\frac{\alpha}{\rho - \alpha} + 1\big)K = 0$$

as can be verified by substitution of the values of $\rho$ and $x_0$ in (1.3).

(2) If $x > x_0$ we have

$$Ls(x) = x\big(\frac{\lambda}{\alpha - 1} - a\big) < 0.$$

(3) $s(x) > 0$ by definition, and $s(x) \geq x - K$ is direct when $x \geq x_0$. When $x < x_0$, it can be verified considering the tangent line to the convex function $s(x)$, at the point $x_0$, giving

$$s(x) \geq (x_0 - K) + \frac{\lambda}{a(\alpha - 1)}(x - x_0) \geq (x - K) \qquad \text{if} \quad 0 \leq x \leq x_0,$$

that gives the result because $\frac{\lambda}{a(\alpha-1)} < 1$.

(4) Take $Z = s(S_{\tau^*}) + x_0$. $s(S_{\tau^* \wedge T \wedge t}) \leq Z$ follows from the fact that on the set $\{\tau^* < +\infty\}$ we have $S_{\tau^* \wedge T \wedge t} \leq S_{\tau^*}$ and the function $s$ is increasing. On the set $\{\tau^* = +\infty\}$ we have $S_{\tau^* \wedge T \wedge t} \leq x_0$.

Now, as in the proof of Theorem 1, taking into account (3.9), and $\rho > 1$ we verify $E(Z) < +\infty$.

$$E\big(\sup_{0 \leq t < +\infty} \exp(X_t - at)\big) = \frac{\lambda}{a\alpha}\frac{1}{\rho - 1} < +\infty$$

and (4) is proved.

(5) is direct, concluding the proof of Theorem 3.

*Proof of Theorem 4.* As in Theorem 1, we have to verify conditions (1) to (5) in Lemma 3.1.

In this case, as in Theorem 3, $L$ is given by (3.10).

(1) If $x > x_0$, $s(x) = (K - x_0)\big(\frac{x}{x_0}\big)^{-\rho}$

$$\begin{aligned}(Ls)(x) &= \lambda \int_0^{+\infty} \big[(K - x_0)\big(\frac{xe^y}{x_0}\big)^{-\rho} - \big(\frac{x}{x_0}\big)^{-\rho}\big] dF(y) \\ &= (K - x_0)\big(\frac{x}{x_0}\big) - \rho[\lambda(E(e^{-\rho y}) - 1) - a\rho] = 0,\end{aligned}$$

by the definition of $\rho$.

(2)[2] If $0 < x < x_0$

$$\begin{aligned}(Ls)(x) &= -axs'(x) + \lambda \int_0^{+\infty} (s(xe^y) - s(x)) dF(y) \\ &\leq x\Big(a + \lambda \int_0^{+\infty} \frac{s(x_0 e^y) - s(x_0)}{x_0} dF(y)\Big) = 0\end{aligned}$$

[2]Idem pg. 6

because $s'(x) = -1$ and, by convexity for all $y \geq 0$

$$\frac{s(xe^y) - s(x)}{x} \leq \frac{s(x_0e^y) - s(x_0)}{x_0}.$$

(3) $s(x) > 0$ is inmediate and $s(x) \geq K - x$ follows from the fact, that $s(x)$ is convex and the function $(K - x_0)\left(\frac{x}{x_0}\right)^{-\rho}$ has tangent $y = K - x$ at the point $x = x_0$ (here applies the smooth pasting principle, because the function $f(x)$ is decreasing and we have jumps up).

(4) In this case we have $s(S_{\tau^*}) \leq x_0$.

(5) is also inmediate concluding the proof.

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