Zero Black-Derman-Toy Interest Rate Model

Grzegorz Krzyżanowski, Ernesto Mordecki, and Andrés Sosa

Grzegorz Krzyżanowski
is on the faculty of Pure and Applied Mathematics in the Hugo Steinhaus Center at the Wroclaw University of Science and Technology in Wroclaw, Poland.
grzegorz.krzyzanowski@pwr.edu.pl

Ernesto Mordecki
is on the Facultad de Ciencias in the Centro de Matemática, at the Universidad de la República in Uruguay.
mordecki@cmat.edu.uy

Andrés Sosa
is on the Facultad de Ciencias Económicas y de Administración in the Instituto de Estadística at the Universidad de la República Uruguay.
andres.sosa@fcea.edu.uy

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KEY FINDINGS

- The modification of the classical Black-Derman-Toy model reflecting the Zero Interest Rate Policy is proposed.
- The empirical analysis corresponding to the period 2002-2017 is presented.
- The novel model returns more accurate option prices in pre crisis period than its classical equivalent.

ABSTRACT

We propose a modification of the classical Black-Derman-Toy (BDT) interest rate tree model, which includes the possibility of a jump with a small probability at each step to a practically zero interest rate. The corresponding BDT algorithms are consequently modified to calibrate the tree containing zero interest rate scenarios. This modification is motivated by the recent 2007–2008 crisis in the United States and it quantifies the risk of future crises in bond prices and derivatives. The proposed model can be useful to price derivatives. A comparison of option prices and implied volatilities on US Treasury bonds computed with both the proposed and the classical tree model is provided in six different scenarios along the different periods comprising the years 2002–2017.

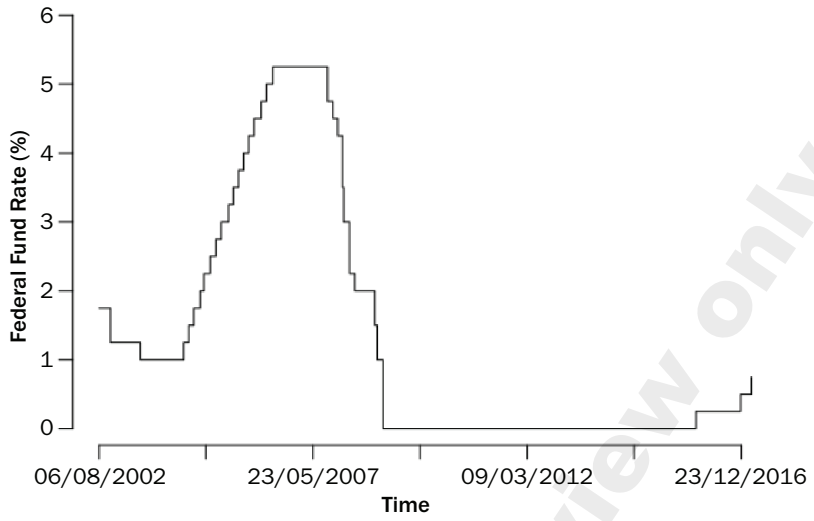
The Federal Funds Rate (i.e., the interest rate at which depository institutions lend reserve balances to other depository institutions overnight on an uncollateralized basis) is an important benchmark in financial markets. This interest rate affects monetary and financial conditions which influence certain aspects of the general economy in the United States, such as employment, growth, inflation, and term structure interest rates.

Following the 2007–2008 financial crisis in United States, the Federal Reserve reduced the Fed Funds Rate by 425 basis points to practically zero (targeting interest rates in the interval 0–0.25%) in one year. This decision was preserved for nine years and was called the Zero Interest Rate Policy (ZIRP policy). Exhibit 1 shows the evolution of the Federal Funds Rate between the years 2002 and 2017.

Motivated by this phenomena and inspired by Lewis's (2016) ZIRP models in continuous time, and by using the default models of Duffie and Singleton (1999), we propose a modification of the classical Black-Derman-Toy (BDT) model of interest rates (Black, Derman, and Toy 1990).

EXHIBIT 1

Federal Fund Rate (2002–2017)



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Different Approaches to Model the ZIRP

Recently, several approaches to model the ZIRP have appeared. Lewis (2016) makes two proposals, which he summarizes as: (i) slowly-reflecting boundaries, also known as sticky boundaries; and (ii) jump-returns from a boundary. The first model consists in the utilization of a resource used in diffusion, considered as Markovian processes, consisting of the introduction of sticky points. The sticky point retains the process for a longer time than the other points. To produce this phenomenon, in the continuous time model, an atomic point is introduced in the speed measure of diffusion (Borodin and Salminen 2002). The second model consists of the introduction of a delayed start of the process. This delay time is modeled by an exponential random variable. The process stays at the $x = 0$ level until this exponential time. It then jumps to an independent state, from which it continues its dynamics as a diffusion. The bond prices for these models are given in (Lewis 2016).

An alternative approach was proposed by Tian and Zhang (2018). These authors depart from the classical CIR process (Cox, Ingersoll, and Ross 1985), and add one skew point at a certain relatively small level of the interest rate. The skew phenomenon in diffusion models represents a permeable barrier. When the process reaches the skew point, the probability of upwards and downwards movements is modified according to a certain probability. In this way, if the probability of downwards continuation is higher that 1/2, as the CIR process never reaches zero, then the proposed process remains below the skew point for a longer time than the CIR process. The skew diffusion can be constructed by departing from the excursion theory for diffusions, and in many other ways (Lejay 2006). The discrete analogue of this model is a binary random walk with symmetric probabilities at all states except for one—skew point. At this point, there is a higher probability of going downwards. This produces a process that stays longer below the critical threshold than the original. It also can be seen that the weak limit of this process, properly normalized, goes to a skew diffusion (Lejay 2006). In the paper (Tian and Zhang 2018), based on stochastic calculus arguments, the authors give bond prices for this model.

Another approach to model the ZIRP phenomenon was introduced by Eberlein et al. (2018). This proposal is in the context of Lévy modeling of Libor rates, and the modification allows negative interest rates. This model is especially suited for

calibration in the presence of extremely low rates, it is presented in the framework of the semimartingale theory, and includes derivatives pricing, particularly caplets. As an application, European caplets market prices are used to calibrate the proposed model with the help of Normal inverse Gaussian Lévy processes.

Martin (2018) made an alternative proposal, which considers that the financial crisis changes the modeling perspective of the term structure. The main reason is that there are differences between interest rates that were previously linked. Therefore, the proposal is to use several interest rate curves in the same model, which reflect the different types of risk observed in the fixed income markets. The paradigm of valuation that the authors use is based on intensity models. The dynamics of the term structure is given by the exponential affine factor model. The hazard rate incorporates the risk observed in the interbank sector that affects the corresponding interest rate. The author states that the approach is important for long-term assets, such as swaps and swaptions.

Our Proposal

In view of the need of adequate models of the ZIRP, we propose to depart from the Black-Derman-Toy (BDT) binary tree model, incorporating into its dynamics the possibility of a downwards jump with a small probability at each time step to a practically zero interest rate value. Additionally, we assume that once the process reaches the zero interest rate zone, it remains there with high probability. This proposal mimics the intensity approach in default bond models proposed by Duffie and Singleton (1999), by jumping to near zero according to a geometric random variable with a small rate. In addition, the sticky phenomenon described by Lewis (2016), as the interest rate process, once this jump is realized, stays with high probability in this close to zero zone. In practical terms, the initial BDT binary tree model is modified to a mixed binary ternary tree model to find consistent interest rates with the market term structure. The new model is called the ZBDT model (Zero Black-Derman-Toy interest rate model).

The rest of this article is structured as follows. In the next section, we introduce the main ideas of the classical BDT model with an emphasis on calibration, with the aim of introducing the ZBDT model in the following section, together with its respective calibration equations. The section after that has empirical content. It contains a detailed account of critical financial events during the period of study (2002–2017) in the United States, which provides information about interest rates with their respective volatilities, and we choose six representative different scenarios to compare the results given by the BDT and the ZBDT models. In the final section, we conclude with a brief discussion of the results and comments on some possible future work.

THE BLACK-DERMAN-TOY MODEL

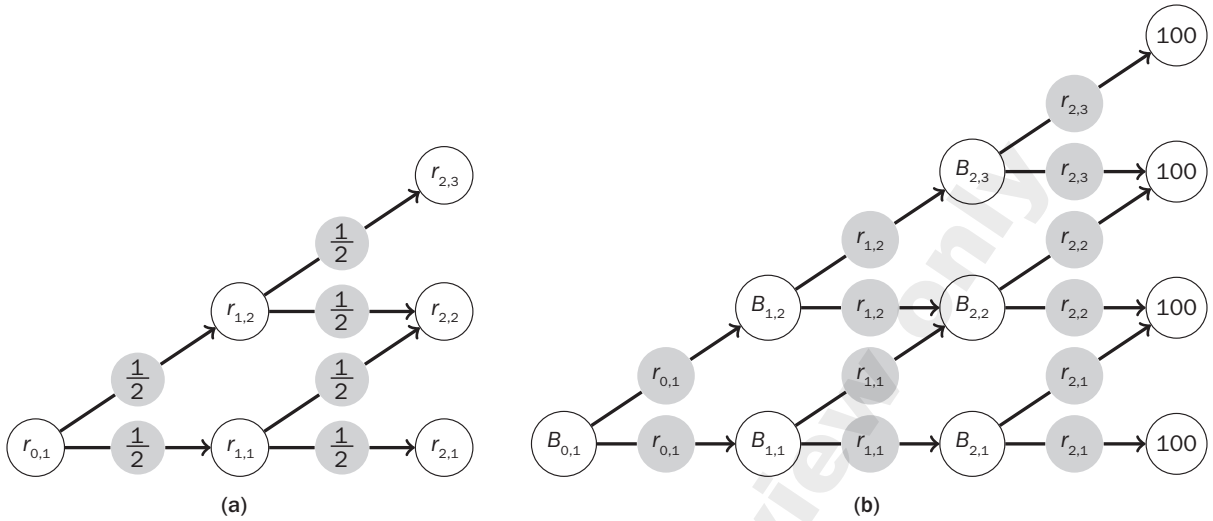
The Black-Derman-Toy model (Black et al. 1990), (Bjerksund and Stensland 1996) is one of the most popular and celebrated models in fixed income interest rate theory. It consists in a binary tree with equiprobable transitions, which makes it simple and flexible to use. More precisely, the model departs from the current interest rate curve, from where the yields for different maturities are extracted, and it uses a series of consecutive historical interest rate curves during a certain time interval to compute this yield volatility. The model assumes that the volatility only depends on time and not on the value of the interest rate. A calibration procedure is implemented to obtain the interest rates acting during the respective time intervals defined in the model.

The model assumes that the future interest rates evolve randomly in a binomial tree with two scenarios at each node, labeled, respectively, by u (for "up") and

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EXHIBIT 2

The BDT Interest Rate Tree (a) and the Corresponding Tree of Zc-Bond (b) with $T = 3$ and $FV = 100$



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d (for "down"), with the particularity that an *u* followed by a *d* take us to the same value as a *d* followed by an *u*. In this way, after *n* periods, we have $n + 1$ possible states for our stochastic process modeling the interest rate. With the aim of simplifying the presentation, we consider that one period is equivalent to one year. Moreover, in the whole article we focus on the zero-coupon bonds (zc-bonds). The corresponding modification to shorter periods or using bonds with coupons is straightforward. In Exhibit 2 (b), we present the tree corresponding to the prices of a zc-bond with expiration in $n = 3$ years, where we denote by B_{ij} the zc-bond price corresponding to the period *i* and state *j* for the same values of *i* and *j*. Here and in the whole article, we assume that the face value (FV) of the bond equals 100: $B_{nj} = 100$ for $j = 1, \ldots, n + 1$.

The evolution of this bond is associated with a tree with the interest rates that apply to each period, as shown in Exhibit 2 (a). In the BDT model, the probability of each *u* or *d* scenario at each node is 1/2, the evolutions are independent, and the values of the interest rates are obtained through calibration.

Calibration of the BDT Model

In a model with *n* time periods, we calibrate a tree of order *n* departing from the following data: the yields on zc-bonds $y(k)$, $k = 1, \ldots, n$, corresponding to the respective periods $[0, k]$ (the first *k* periods), and the yield volatilities for the same bonds $\beta(k)$, $k = 2, \ldots, n$, under the same convention.

The interest rates of the tree are $\{r_{i,j}: i = 0, \ldots, n - 1; j = 1, \ldots, i + 1\}$, and correspond to each time period in the up and down scenarios, giving $n(n + 1)/2$ unknowns to be calibrated.

The first step uses only $y(1)$ and concludes that $r_{0,1} = y(1)$:

$$B_{1,1} = B_{1,2} = 100,$$
$$B_{0,1} = \frac{100}{1 + y(1)} = \frac{1}{2}\frac{1}{1 + r_{0,1}}\left(B_{1,1} + B_{1,2}\right).$$

When $n > 1$, we introduce the yields y_u (up) and y_d (down) one year from now, corresponding to bond prices B_u and B_d. The relevant relations that these quantities satisfy are

$$B_u = \frac{1}{(1+y_u)^{n-1}}, \quad B_d = \frac{1}{(1+y_d)^{n-1}}.$$

Variance Equation at a Node

Consider a tree with n steps. We introduce a random variable Y that takes two values:

$$Y = \begin{cases} y_u, & \text{with probability } 1/2, \\ y_d, & \text{with probability } 1/2. \end{cases}$$

Then, $\log Y$ has a variance $\mathbf{var}\log Y = \beta^2(n)$, if and only if $y_u = y_d e^{2\beta(n)}$, equivalent to

$$\beta(n) = \frac{1}{2}\log\frac{y_u}{y_d}, \tag{1}$$

as follows from the following computation:

$$\mathbf{var}\log Y = \frac{1}{2}\log^2 y_u + \frac{1}{2}\log^2 y_d - \left(\frac{1}{2}(\log y_u + \log y_d)\right)^2 = \left(\frac{1}{2}(\log y_u - \log y_d)\right)^2$$
$$= \left(\frac{1}{2}\log\frac{y_u}{y_d}\right)^2 = \left(\frac{1}{2}\log\left(\frac{r_{i+1,j}\exp(2\sigma(i+1))}{r_{i+1,j}}\right)\right)^2 = \beta(n)^2.$$

The BDT model assumes that the variance of the log-interest rate with fixed time is constant for all nodes. The respective interest rates at each node at time $n-1$ are represented by an auxiliar random variable $R_{n-2,j}$.

$$R_{n-2,j} = \begin{cases} r_{n-1,j+1}, & \text{with probability } 1/2, \\ r_{n-1,j}, & \text{with probability } 1/2, \end{cases}$$

for $j = 1, \ldots, n-1$. The variance of this random variable is assumed to be constant for all nodes at the same time period, and satisfies

$$\sigma(n) = \frac{1}{2}\log\frac{r_{n-1,j+1}}{r_{n-1,j}}, \quad j = 1,\ldots,n-1. \tag{2}$$

In the second step of the calibration, the new data are $y(2)$ and $\beta(2)$. The unknowns are $r_{1,1}$, $r_{1,2}$ and $\sigma(2)$. In this case $\sigma(2) = \beta(2)$, because the local variation of the interest rate for one year coincides with the global variation. Accordingly, $y_u = r_{1,2}$ and $y_d = r_{1,1}$. The bond prices satisfy

$$B_{0,1} = \frac{100}{(1+y(2))^2} = \frac{1}{2}\frac{1}{(1+r_{0,1})}\left(B_u + B_d\right),$$
$$B_{2,j} = 100, j = 1,2,3,$$
$$B_u = \frac{100}{(1+y_u)}, \quad B_d = \frac{100}{(1+y_d)},$$
$$\beta(2) = \frac{1}{2}\log\frac{y_u}{y_d}.$$

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For general n the new data are $y(n)$ and $\beta(n)$. The unknowns are $r_{n-1,j}$ for $j = 1, \ldots, n$ and $\sigma(n)$. The value $\sigma(n)^2$ is the variance of the interest rate at each node (see Equation 1). The bond prices satisfy

$$\frac{100}{(1+y(n))^n} = \frac{1}{2}\frac{1}{(1+r_{0,1})}\left(B_u + B_d\right),$$
$$B_{n,j} = 100,\ j = 1,\ldots,(n+1),$$
$$B_{i,j} = \frac{1}{2}\frac{1}{(1+r_{i,j})}\left(B_{i+1,j} + B_{i+1,j+1}\right),\ i = 0,\ldots,(n-1),\ j = 1,\ldots,i+1,$$
$$B_u = \frac{100}{(1+y_u)^{n-1}}, \quad B_d = \frac{100}{(1+y_d)^{n-1}},$$
$$\beta(n) = \frac{1}{2}\log\frac{y_u}{y_d},$$
$$\sigma(n) = \frac{1}{2}\log\frac{r_{n-1,j}}{r_{n-1,j-1}},\ \ j = 2,\ldots,n.$$

THE ZBDT MODEL

Our modification of the classical BDT interest rate tree model adds to the dynamics the possibility of a downwards jump with a small probability at each time step to a practically zero interest rate, where, after its arrival, the process remains with high probability. More precisely, in the new model, the nodes labeled (i, j) with $j \geq 2$ have the same characteristics as in the BDT model (up and down probabilities 1/2, and jump to values to be calibrated). In addition, the nodes of the form $(1, j)$ add a third possible downwards jump with a small probability p and the other two possible jumps have probability $\hat{p} = (1-p)/2$. If this downward jump is realized, then the process enters the so-called ZIRP zone, meaning that interest rate becomes a small value x_0 (close to the target of the policy). When the process is in the ZIRP zone, it remains there with a high probability $(1 - q)$ and exits with probability q. Finally, to calibrate the tree, following the same convention as in the classical BDT model, we further impose that the variance at each node for the same time period remains the same (to be determined by calibration, denoted below by $\sigma(n)$ for the period n). With the previous $r_{i,j}$ and $B_{i,j}$ corresponding to the BDT model, the ZBDT model adds the (unknown) bond prices $B_{i,0}$ for $i = 1, \ldots, n - 1$ and $B_{0,n} = 100$. The corresponding interest rates $r_{i,0}$ for $1, \ldots, i + 1$ are fixed to x_0. In Exhibit 3 (b), we present the tree corresponding to the prices of a zc-bond with expiration in $n = 3$ years. Note that p, q, x_0 have a clear interpretation as a probability of the crisis (in the basic period of time, which in this article is 1 year), a conditional probability of economic recovery from the financial crisis (in the basic period of time) and an assumed value of the interest rate in the ZIRP zone respectively.

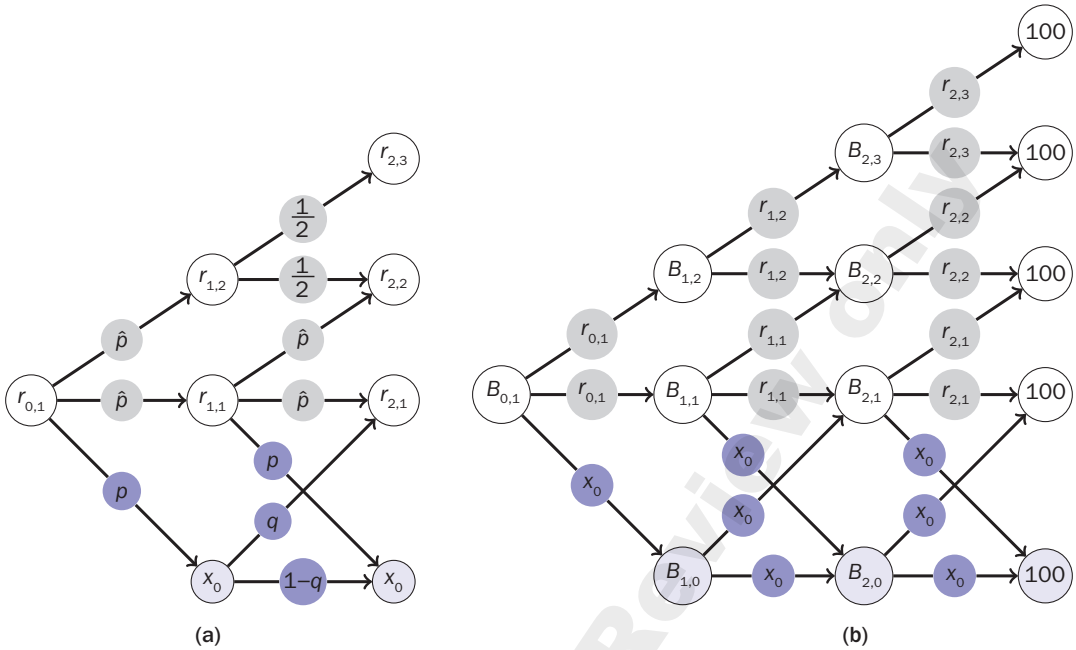
Calibration of the ZBDT Model

For the calibration, we use the same data as in the BDT model. The strategy is modified to cope with the new unknowns but follows the same ideas. The first step uses only $y(1)$ and we conclude that $r_{01} = y(1)$. The equations are

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EXHIBIT 3

The ZBDT Interest Rate Tree (a) and the Corresponding Zc-Bonds (b) with $T = 3$ and $FV = 100$



$$B_{1,0} = B_{1,1} = B_{1,2} = 100,$$

$$B_{0,1} = \frac{100}{1+y(1)} = \frac{1}{1+r_{0,1}}\left(\frac{1-p}{2}\left(B_{1,2}+B_{1,1}\right)+pB_{1,0}\right).$$

Variance Equation at a Node

In the present situation, the random variable y takes three values:

$$Y = \begin{cases} y_u, & \text{with probability } \hat{p}, \\ y_d, & \text{with probability } \hat{p}, \\ y_0, & \text{with probability } p. \end{cases}$$

Then, $\log Y$ has the same variance as the random variable

$$\log\frac{Y}{y_0} = \begin{cases} \log\dfrac{y_u}{y_0}, & \text{with probability } \hat{p}, \\ \log\dfrac{y_d}{y_0}, & \text{with probability } \hat{p}, \\ 0, & \text{with probability } p. \end{cases}$$

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The mean of $\log (Y/y_0)$ is

$$\frac{1-p}{2}\left(\log\frac{y_u}{y_0}+\log\frac{y_d}{y_0}\right),$$

then

$$\mathbf{var}\log\frac{Y}{y_0}=\frac{1-p}{2}\left(\left(\log\frac{y_u}{y_0}\right)^2+\left(\log\frac{y_d}{y_0}\right)^2\right)-\left(\frac{1-p}{2}\left(\log\frac{y_u}{y_0}+\log\frac{y_d}{y_0}\right)\right)^2.$$

Introducing the notation

$$\ell_u=\log\frac{y_u}{y_0},\quad \ell_d=\log\frac{y_d}{y_0}, \tag{3}$$

we obtain

$$\mathbf{var}\log y=\frac{1-p^2}{4}(\ell_u^2+\ell_d^2)-\frac{(1-p)^2}{2}\ell_u\ell_d. \tag{4}$$

Considering now the interest rates, if the node $n-1$, j has two edges (i.e., $j = 2, \ldots, n$), then the variance at the node satisfies Equation 2, (the same as in the BDT case). If the node has three edges (when $j = 1$), then the variance satisfies Equation 4.

In the second step of calibration, the new data are $y(2)$ and $\beta(2)$. The unknowns are $r_{1,1}$, $r_{1,2}$ and $\sigma(2)$. In this case $r_{1,1} = y_d$, $r_{1,2} = y_u$, $y_0 = x_0$ and $\sigma(2) = \beta(2)$ (because in this case the local variation of the interest rate for one year coincides with the global variation). Accordingly, $y_d = r_{1,1}$ and $y_u = r_{1,2}$.

$$B_{0,1}=\frac{1}{(1+y(2))^2}=\frac{1}{1+r_{0,1}}\left(\frac{1-p}{2}\left(B_u+B_d\right)+pB_0\right),$$
$$B_{2,j}=100,\quad j=1,2,3,$$
$$B_u=\frac{100}{(1+y_u)},\quad B_d=\frac{100}{(1+y_d)},\quad B_0=\frac{100}{(1+y_0)},$$
$$\beta(2)^2=\frac{1-p^2}{4}\left(\ell_u^2+\ell_d^2\right)-\frac{(1-p)^2}{2}\ell_u\ell_d.$$

with ℓ_u and ℓ_d given in Equation 3.

For general n, the new data are $y(n)$ and $\beta(n)$. The unknowns are $r_{n-1,j}$ for $j = 1, \ldots, n$, and $\sigma(n)$. The calibration equations are:

$$B_{0,1}=\frac{1}{(1+y(n))^n}=\frac{1}{1+r_{0,1}}\left(\frac{1-p}{2}B_u+\frac{1-p}{2}B_d+pB_0\right),$$
$$B_{n,j}=100,\ j=0,\ldots,(n+1),$$
$$B_{i,j}=\frac{1}{2}\frac{1}{1+r_{i,j}}\left(B_{i+1,j+1}+B_{i+1,j}\right),\ i=1,\ldots,(n-1),\ j=2,\ldots,i+1,$$

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$$B_{i,1} = \frac{1}{1+r_{i,j}}\left(\frac{1-p}{2}B_{i+1,2} + \frac{1-p}{2}B_{i+1,1} + pB_{i+1,0}\right), i = 1,\ldots,n-1,$$

$$B_{i,0} = \frac{1}{1+x_0}\left(qB_{i+1,1} + (1-q)B_{i+1,0}\right), i = 1,\ldots,n-1,$$

$$B_u = \frac{100}{(1+y_u)^{n-1}}, \quad B_d = \frac{100}{(1+y_d)^{n-1}}, \quad B_0 = \frac{100}{(1+y_0)^{n-1}},$$

$$\ell_u = \log\frac{y_u}{y_0}, \quad \ell_d = \log\frac{y_d}{y_0},$$

$$\beta(n)^2 = \frac{1-p^2}{4}\left(\ell_u^2 + \ell_d^2\right) - \frac{(1-p)^2}{2}\ell_u\ell_d,$$

$$\sigma(n) = \frac{1}{2}\log\frac{r_{n-1,j+1}}{r_{n-1,j}}, \; j = 2,\ldots,n,$$

$$\ell_1 = \log\frac{r_{n-1,1}}{x_0}, \qquad \ell_2 = \log\frac{r_{n-1,2}}{x_0},$$

$$\sigma(n)^2 = \frac{1-p^2}{4}\left(\ell_1^2 + \ell_2^2\right) - \frac{(1-p)^2}{2}\ell_1\ell_2.$$

EMPIRICAL ANALYSIS OF DIFFERENT SCENARIOS WITH US TREASURY BONDS DATA

The main motivation of our work is to analyze the new features observed in bond prices as a consequence of the ZIRP implemented by the US Government in 2008. In Exhibit 4, we give an account of the main events related to the US economy during the period of the study.

Interest Rates, Yields, and Volatilities 2002–2017

We present the yields and its volatilities used to calibrate the ZBDT model (including interest rates and bond prices) with the aim of computing bond option prices. The daily interest rates correspond to the period from August 6, 2002 to April 28, 2017, and were obtained from the Federal Reserve Board of the United States. The data are denoted by y(*t*, *k*), where *t* denotes the day and *k* the corresponding six maturities used in this study (*k* = 1/2,1,2,3,4,5 in years). In the previous sections, *t* was omitted because the analysis was performed for a fixed time. To compute the volatility β (*t*, *k*) corresponding to these values, we use the equations

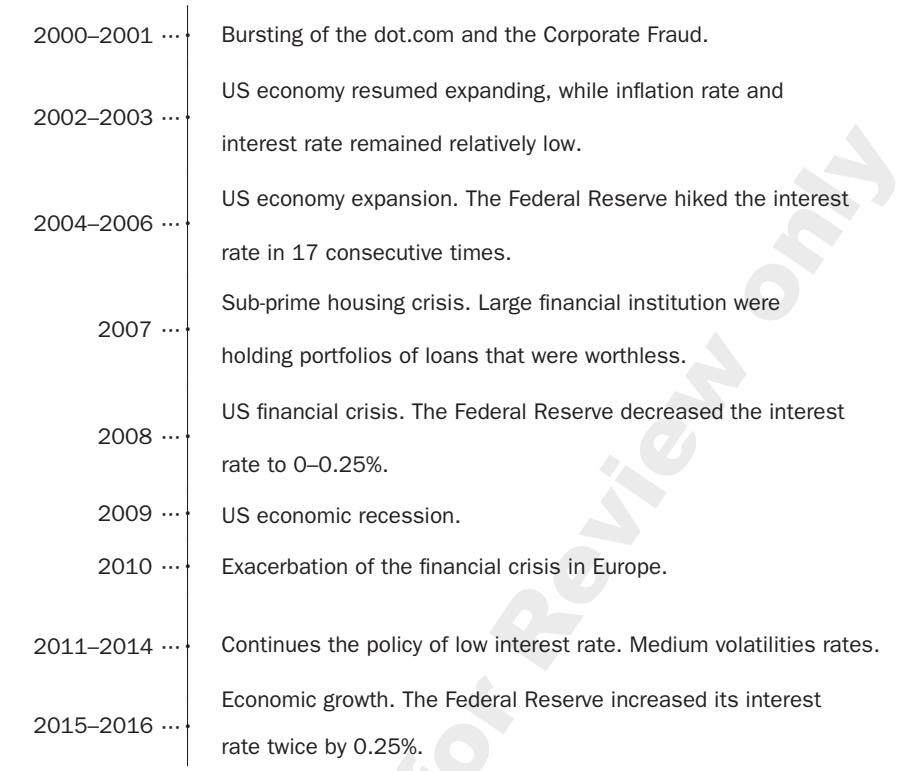
$$\overline{\ell}(t,k) = \frac{1}{252}\sum_{i=0}^{251}\log\frac{y(t-i,k)}{y(t-i-1,k)},$$

$$\beta^2(t,k) = \sum_{i=0}^{251}\left(\log\frac{y(t-i,k)}{y(t-i-1,k)} - \overline{\ell}(t,k)\right)^2,$$

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EXHIBIT 4

Timeline of Relevant Financial Events 2000–2016



where the factor 252 corresponds to the number of business days of one year; that is, the window chosen to compute the volatility. The obtained data is presented in Exhibit 5.

Six Observed Typical Scenarios

We choose six different days, corresponding to six different periods, expecting to analyze the impact of the downwards jump in the interest rate included in the ZBDT model. To select each of these days, the interest rates depicted in Exhibit 5 were taken into account.

In each of the six chosen scenarios, we calibrate the BDT and ZBDT models, presenting the corresponding interest rates and bond prices. These numerical results can be seen in Exhibits 6, 8, 10, 12 14 and 16 in the Appendix.

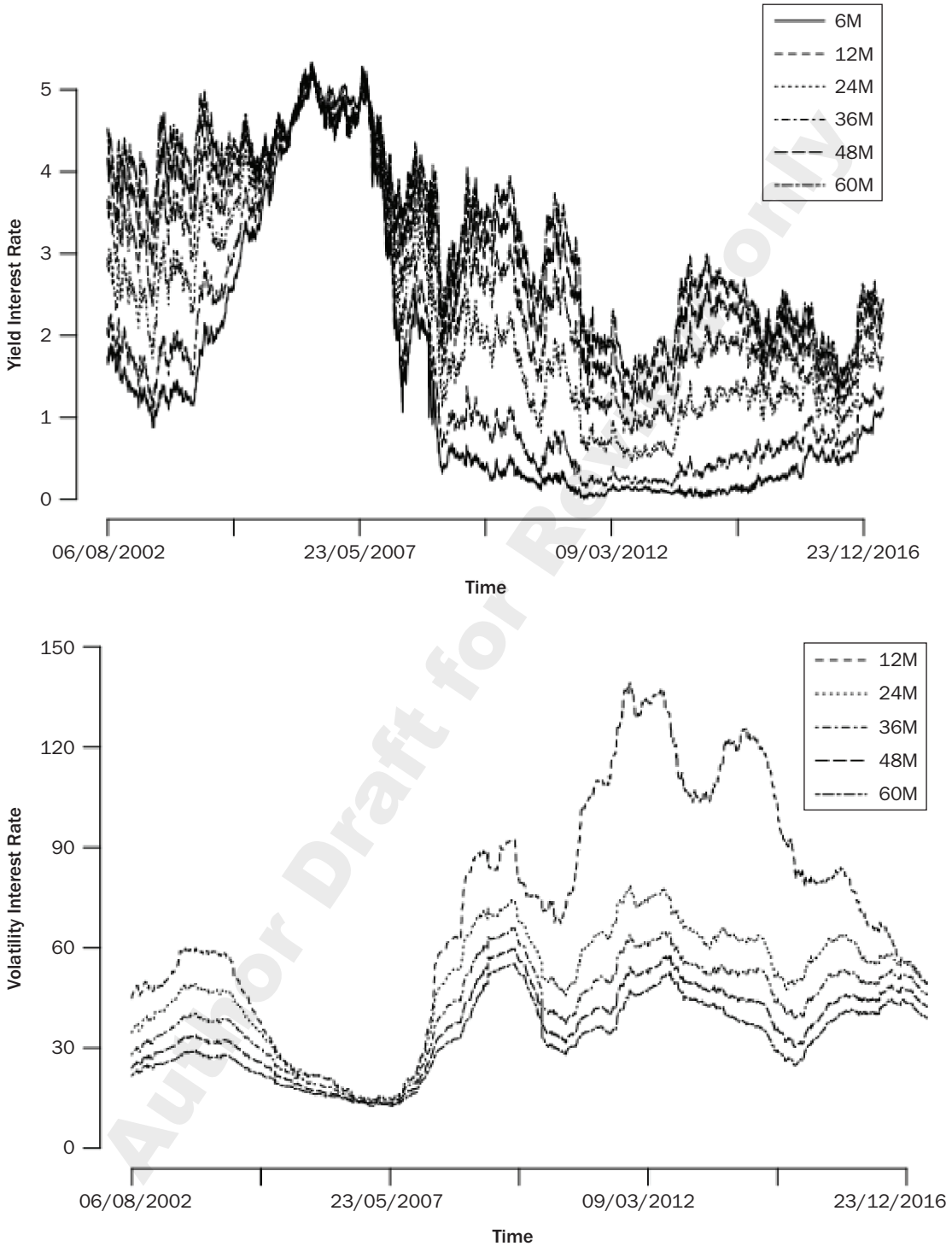
With this information, we compute the vanilla call option prices along the strikes of bond prices ranging from 80 to 100, obtaining the respective implied volatility. To compute the implied volatility at time t of an option written on a zc-bond that expires at time T, with strike K and maturity S ($t < S < T$), we use Black's formula (see Black 1976), which states

$$C = B(t,T)\Phi(d_1) - KB(t,S)\Phi(d_2),$$

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EXHIBIT 5

Yield Rates and Yield Volatilities for Different Maturities (2002–2017)



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where

$$d_{1,2} = \frac{\log\left(\frac{B(t,T)}{KB(t,S)}\right)}{\sigma\sqrt{S-t}} \pm \frac{\sigma\sqrt{S-t}}{2},$$

and Φ is the cumulative normal distribution function. Note that regardless of the model under consideration, we assume that the implied volatility is equal to 0 if the corresponding option is worthless. For more details see (McDonald 2006). In our empirical exercise, we consider a zc-bond with expiration in five years ($T = 5$) and European call options written at $t = 0$ with exercise time two years ($S = 2$). For the ZBDT model, we assume the parameters $x_0 = 0.25\%$, $p = 0.02$, $q = 0.07$. The results are presented in Exhibits 7, 9, 11, 13, 15 and 17 in the Appendix. A primary conclusion is that, in contrast to the BDT model, the ZBDT allows us to price options with strikes close to the face value of the bond, which corresponds to low interest rate periods. This gives more accurate option prices in the precrisis period.

CONCLUSIONS

In the present work, we propose a novel and practical approach to model the possibility of a drop in the interest rate structure of sovereign bonds. This modification is motivated by the recent 2008–2009 crisis in United States.

Our approach is inspired by Lewis's (2016) ZIRP models in continuous time, and also in Duffie and Singleton's (1999) default framework of bond pricing models. Our proposal consists in adding a new branch at each period to the classical Black-Derman-Toy tree model that takes into account the small probability of this drop event to happen. We name this the ZBDT model, "Z" standing for (close to) zero interest rate. To the best of our knowledge, our model is the first discrete space–time model proposed for the ZIRP, and it shares the motivation of including this phenomenon as previously considered in continuous time models through sticky diffusion (such as in Lewis 2016) and skew diffusions (Tian and Zhang 2018).

This article includes the development of the corresponding modified calibration scheme (that, naturally, happens to be more complex than the classical BDT calibration and uses the same information) to obtain the interest rate tree and corresponding bond prices. With this information, we valuate European option prices provided by both models. The comparison between the two models is carried out although the implied volatility analysis is provided by the Black option pricing formula. Our proposal opens the possibility of correcting option prices in different scenarios, especially under the risk of future interest rates. The analysis of the implied volatility curves provided by the US bond market is a tool that can reveal in which situations this drop probability is not negligible. Our main conclusion is that the ZIRP model allows to price options with high strikes. All observed implied volatilities are higher in the ZBDT model than in the BDT model. This gives more accurate option prices in the precrisis period.

Further research includes the consideration of other usual derivatives in the bond market, methods of calibration of the parameters of the proposed model (such as the probability of drop and the probability of staying in the ZIRP zone) and the more complex task of proposing a continuous time model analog.

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APPENDIX

Scenario I (May 23, 2003): Expanding economy, normal term structure.

EXHIBIT 6

BDT Interest Rates (top-left), BDT Bond Prices (top-right), ZBDT Interest Rates (bottom-left), and ZBDT Bond Prices (bottom-right) in Scenario I

				9.32				91.47	100
			8.34	6.56			85.52	93.84	100
		6.52	5.30	4.62		82.36	89.95	95.58	100
	3.76	3.50	3.36	3.26	82.30	88.42	93.08	96.85	100
1.36	1.54	1.87	2.13	2.29	84.53	89.05	92.44	95.27	97.76	100
									100
				21.24				82.48	100
			14.57	8.81			76.10	91.90	100
		8.91	5.51	3.65		75.92	89.27	96.48	100
	4.21	2.92	2.09	1.52	79.49	89.76	95.50	98.51	100
1.36	1.13	0.97	0.80	0.63	84.53	91.81	95.90	98.16	99.37	100
	0.25	0.25	0.25	0.25		98.76	99.17	99.48	99.75	100

EXHIBIT 7

Call Option Prices and Implied Volatility for Both Models in Scenario I

Strikes	BDT	v	ZBDT	v
80	7.6328	1.5372	8.5965	1.5691
81	6.6716	1.5058	7.8715	1.5479
82	5.7103	1.4714	7.1465	1.5254
83	4.9003	1.4399	6.4214	1.5014
84	4.1768	1.409	5.6964	1.4757
85	3.4533	1.3744	4.9714	1.4477
86	2.7298	1.3343	4.2463	1.4170
87	2.0063	1.2862	3.5213	1.3826
88	1.2827	1.2236	2.7963	1.3430
89	0.8363	1.1714	2.0713	1.2955
90	0.5934	1.1343	1.4609	1.2458
91	0.3505	1.0825	1.2150	1.2226
92	0.1076	0.9855	0.9691	1.1955
93	0.00	0.00	0.7235	1.1624
94	0.00	0.00	0.4774	1.119
95	0.00	0.00	0.2315	1.0522
96	0.00	0.00	0.009	0.8459
97	0.00	0.00	0.0062	0.8300
98	0.00	0.00	0.0033	0.8049
99	0.00	0.00	0.0005	0.7364

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Scenario II (August 07, 2006): Flat term structure curves.

EXHIBIT 8

BDT Interest Rates (top-left), BDT Bond Prices (top-right), ZBDT Interest Rates (bottom-left), and ZBDT Bond Prices (bottom-right) in Scenario II

				8.41				92.24	100
			7.12	6.40			86.92	93.98	100
		6.37	5.51	4.88		83.04	89.72	95.35	100
	5.66	4.77	4.26	3.71		80.33	86.71	91.96	96.42	100
4.97	3.96	3.57	3.30	2.83	78.66	84.82	89.65	93.74	97.25	100
										100
				17.79				84.89	100
			12.23	8.58			78.86	92.09	100
		9.06	6.08	4.14		76.80	88.67	96.02	100
	6.70	4.32	3.02	2.00		77.06	87.65	94.19	98.04	100
4.97	3.00	2.13	1.54	0.98	78.66	88.02	93.63	97.04	99.03	100
	0.25	0.25	0.25	0.25		98.58	99.10	99.47	99.75	100

EXHIBIT 9

Call Option Prices and Implied Volatility for Both Models in Scenario II

Strikes	BDT	v	ZBDT	v
80	5.9450	1.4960	6.6568	1.5239
81	5.0361	1.4600	5.9706	1.4995
82	4.1271	1.4196	5.2844	1.4734
83	3.2181	1.3731	4.5982	1.4449
84	2.5267	1.3320	3.9120	1.4135
85	1.8431	1.2828	3.2258	1.3782
86	1.1595	1.2184	2.5396	1.3374
87	0.6079	1.1417	1.8534	1.2881
88	0.3788	1.0940	1.3277	1.2412
89	0.1497	1.0135	1.0945	1.2171
90	0.00	0.00	0.8614	1.1887
91	0.00	0.00	0.6283	1.1535
92	0.00	0.00	0.3951	1.1061
93	0.00	0.00	0.1620	1.0272
94	0.00	0.00	0.0139	0.8702
95	0.00	0.00	0.0111	0.8607
96	0.00	0.00	0.0085	0.8487
97	0.00	0.00	0.0057	0.8322
98	0.00	0.00	0.0030	0.8059
99	0.00	0.00	0.0003	0.7224

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Scenario III (November 14, 2007): Start of financial crisis.

EXHIBIT 10

BDT Interest Rates (top-left), BDT Bond Prices (top-right), ZBDT Interest Rates (bottom-left), and ZBDT Bond Prices (bottom-right) in Scenario III

				7.91					92.67	100
			7.03	6.08				87.33	94.27	100
		6.06	5.24	4.68			83.68	90.17	95.53	100
	4.73	4.23	3.91	3.60		81.77	87.59	92.42	96.53	100
3.56	3.06	2.95	2.91	2.77	81.22	86.46	90.62	94.17	97.31	100
										100
				16.86					85.57	100
			12.18	8.18				79.34	92.44	100
		8.62	5.72	3.97			77.59	89.21	96.19	100
	5.58	3.75	2.69	1.92		78.69	88.58	94.60	98.11	100
3.56	2.26	1.68	1.29	0.95	81.22	89.48	94.39	97.33	99.06	100
	0.25	0.25	0.25	0.25		98.63	99.11	99.47	99.75	100

EXHIBIT 11

Call Option Prices and Implied Volatility for Both Models in Scenario III

Strikes	BDT	v	ZBDT	v
80	6.8635	1.5215	7.4143	1.5414
81	5.9340	1.4884	6.7131	1.5185
82	5.0046	1.4519	6.0118	1.4940
83	4.0751	1.4110	5.3105	1.4677
84	3.2198	1.3676	4.6092	1.4390
85	2.5208	1.3265	3.9079	1.4073
86	1.8218	1.2765	3.2066	1.3715
87	1.1229	1.2103	2.5053	1.3300
88	0.6141	1.1396	1.8041	1.2794
89	0.3799	1.0014	1.2969	1.2339
90	0.1456	1.0090	1.0589	1.2089
91	0.00	0.00	0.8209	1.1792
92	0.00	0.00	0.5829	1.1419
93	0.00	0.00	0.3449	1.0897
94	0.00	0.00	0.1069	0.9926
95	0.00	0.00	0.0114	0.8587
96	0.00	0.00	0.0086	0.8467
97	0.00	0.00	0.0059	0.8304
98	0.00	0.00	0.0031	0.8044
99	0.00	0.00	0.0003	0.7252

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Scenario IV (August 08, 2008): US crisis.

EXHIBIT 12

BDT Interest Rates (top-left), BDT Bond Prices (top-right), ZBDT Interest Rates (bottom-left), and ZBDT Bond Prices (bottom-right) in Scenario IV

				10.87				90.20	100
			9.70	7.10			83.67	93.37	100
		8.04	5.67	4.64		80.10	89.40	95.57	100
	5.43	3.84	3.32	3.03	79.69	87.93	93.22	97.06	100
2.47	1.86	1.85	1.94	1.98	82.16	88.70	92.76	95.70	98.06	100

				25.29				79.82	100
			16.84	9.42			73.27	91.39	100
		10.72	5.80	3.51		73.21	88.85	96.61	100
	5.95	3.17	2.00	1.39	76.77	89.46	95.75	98.71	100
2.47	1.40	0.95	0.69	0.49	82.16	91.56	96.18	98.43	99.51	100
	0.25	0.25	0.25	0.25		98.79	99.19	99.49	99.75	100

EXHIBIT 13

Call Option Prices and Implied Volatility for Both Models in Scenario IV

Strikes	BDT	v	ZBDT	v
80	6.8145	1.5171	8.3751	1.5714
81	6.0812	1.4919	7.6631	1.5500
82	5.3708	1.4655	6.9511	1.5272
83	4.6603	1.4369	6.2391	1.5030
84	3.9499	1.4051	5.5271	1.4769
85	3.2395	1.3693	4.8151	1.4486
86	2.5291	1.3277	4.1031	1.4175
87	1.8187	1.2770	3.3911	1.3825
88	1.1398	1.2132	2.6791	1.3421
89	0.9003	1.1853	1.9670	1.2935
90	0.6608	1.1508	1.5078	1.2558
91	0.4213	1.1048	1.2653	1.2334
92	0.1818	1.0299	1.0227	1.2074
93	0.00	0.00	0.7802	1.1760
94	0.00	0.00	0.5376	1.1359
95	0.00	0.00	0.2951	1.0780
96	0.00	0.00	0.0525	0.9463
97	0.00	0.00	0.0062	0.8323
98	0.00	0.00	0.0033	0.8075
99	0.00	0.00	0.0005	0.7417

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Scenario V (August 03, 2010): European crisis.

EXHIBIT 14

BDT Interest Rates (top-left), BDT Bond Prices (top-right), ZBDT Interest Rates (bottom-left), and ZBDT Bond Prices (bottom-right) in Scenario V

				9.65					91.20	100
			8.04	6.33				85.73	94.05	100
		5.90	4.84	4.15			83.27	90.65	96.01	100
	2.89	2.98	2.91	2.72		84.02	89.63	93.95	97.35	100
0.51	1.13	1.50	1.75	1.79	86.87	90.60	93.62	96.12	98.24	100
										100
				21.45					82.34	100
			13.76	8.22				76.80	92.40	100
		7.92	4.93	3.15			77.38	90.22	96.94	100
	3.27	2.46	1.77	1.21		81.55	90.96	96.17	98.81	100
0.51	0.84	0.77	0.64	0.47	86.87	93.03	96.62	98.54	99.54	100
	0.25	0.25	0.25	0.25		98.82	99.20	99.49	99.75	100

EXHIBIT 15

Call Option Prices and Implied Volatility for Both Models in Scenario V

Strikes	BDT	v	ZBDT	v
80	8.8374	1.5667	9.4675	1.5859
81	7.8620	1.5378	8.7327	1.5659
82	6.8866	1.5067	7.9978	1.5448
83	5.9112	1.4730	7.2629	1.5225
84	5.1113	1.4430	6.5280	1.4987
85	4.3777	1.4128	5.7931	1.4731
86	3.6440	1.3791	5.0582	1.4454
87	2.9104	1.3405	4.3233	1.4149
88	2.1767	1.2945	3.5884	1.3808
89	1.4431	1.2358	2.8535	1.3415
90	0.8912	1.1755	2.1186	1.2945
91	0.6453	1.1402	1.4042	1.2362
92	0.3993	1.0922	1.1555	1.2121
93	0.1533	1.0101	0.9068	1.1836
94	0.00	0.00	0.6581	1.1483
95	0.00	0.00	0.4095	1.1003
96	0.00	0.00	0.1608	1.0191
97	0.00	0.00	0.0063	0.8279
98	0.00	0.00	0.0034	0.8033
99	0.00	0.00	0.0006	0.7389

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Scenario VI (May 20, 2015): End of US-crisis.

EXHIBIT 16

BDT Interest Rates (top-left), BDT Bond Prices (top-right), ZBDT Interest Rates (bottom-left), and ZBDT Bond Prices (bottom-right) in Scenario VI

				7.63				92.91	100
			6.52	4.78			88.41	95.44	100
		5.34	3.72	2.99		86.02	92.81	97.10	100
	2.97	2.35	2.13	1.87	86.46	92.04	95.60	98.17	100
0.61	0.92	1.04	1.21	1.17	89.14	92.90	95.47	97.32	98.84	100
									100
				16.66				85.72	100
			11.02	6.01			81.09	94.33	100
		6.91	3.70	2.17		81.26	92.67	97.88	100
	3.21	1.91	1.24	0.78		84.53	93.22	97.34	99.22	100
0.61	0.70	0.53	0.42	0.28	89.14	94.80	97.68	99.06	99.72	100
	0.25	0.25	0.25	0.25		98.91	99.23	99.50	99.75	100

EXHIBIT 17

Call Option Prices and Implied Volatility for Both Models in Scenario VI

Strikes	BDT	v	ZBDT	v
80	11.1312	1.6207	11.1312	1.6207
81	10.1562	1.5959	10.1562	1.5959
82	9.1811	1.5697	9.3581	1.5750
83	8.2061	1.5419	8.6233	1.5551
84	7.2310	1.5120	7.8886	1.5341
85	6.2560	1.4796	7.1538	1.5118
86	5.2809	1.4439	6.4190	1.4880
87	4.5431	1.4146	5.6842	1.4625
88	3.8094	1.3821	4.9494	1.4347
89	3.0756	1.3450	4.2417	1.4040
90	2.3419	1.3013	3.4799	1.3695
91	1.6081	1.2467	2.7451	1.3297
92	0.8744	1.1694	2.0103	1.2818
93	0.6085	1.1302	1.2756	1.2188
94	0.3623	1.0795	0.9197	1.1793
95	0.1161	0.9860	0.6710	1.1447
96	0.00	0.00	0.4222	1.0981
97	0.00	0.00	0.1734	1.0205
98	0.00	0.00	0.0035	0.7996
99	0.00	0.00	0.0007	0.7397

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