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Optimal stopping of Brownian motion with broken drift

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Abstract

We solve an optimal stopping problem where the underlying diffusion is Brownian motion on **R** with a positive piecewise constant drift changing at zero. It is assumed that the drift μ_1 on the negative side is smaller than the drift μ_2 on the positive side. The main observation is that if $\mu_2 - \mu_1 > 1/2$, then there exist values of the discounting parameter for which it is not optimal to stop in the vicinity of zero where the drift changes. However, when the discounting gets larger, the stopping region becomes connected and contains zero. This is in contrast with results concerning optimal stopping of skew Brownian motion where the skew point is for all values of the discounting parameter in the continuation region in case the skew parameter is bigger than 1/2. In all cases, a practical implementation of these optimal stopping strategies, based on hitting times of boundaries of continuation regions, would require access to the highest-possible frequency data which is consistent with the continuous-time modeling framework.

JEL CLASSIFICATION

60J60, 60J65, 62L15

KEYWORDS

disconnected stopping region, excessive function, integral representation of excessive functions

1 | INTRODUCTION

1.1 | Motivation

Optimal stopping problems for one-dimensional diffusions are a much studied topic, and there exists a variety of methods for finding solutions. During the recent decade, some interest has been focused on cases where the underlying diffusion has exceptional points such as: sticky points; skew points; discontinuities in the infinitesimal variance; and/or drift. The main observation in the presence of sticky points is that the smooth fit does not in general hold (see, for instance, Crocce, 2014; Crocce & Mordecki, 2014; Salminen & Ta, 2015). If the diffusion has skew points, then it is possible that in the vicinity of such a point, it is not optimal to stop for any value of the discounting parameter, as was found by Alvarez E and Salminen (2017) and also by Presman (2012). Explicit solutions of optimal stopping problems when the underlying diffusion has discontinuous coefficients have not, to our best knowledge, been encountered in the literature. Many of the methods to solve optimal stopping problems do not, however, exclude such diffusions. This is, in particular, the case in the approach based on excessive functions, see, for example, Christensen and Irle (2011), Crocce and Mordecki (2014), Dayanik and Karatzas (2003), and Salminen (1985). Also, the approach via variational inequalities and free boundary problems, see Lamberton and Zervos (2013) and Rüschendorf and Urusov (2008), does not seem to require continuity of the diffusion coefficients. However, our main motivation for the present work is to study how the stopping set changes as a function of the discounting parameter when, in our case, the drift is piecewise constant and discontinuous at the origin.

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1.2 | Diffusions and the optimal stopping problem

Consider a nonterminating and regular one-dimensional (or linear) diffusion $X = \{X_t : t \geq 0\}$ in the sense of Itô and McKean (1974) (see also Borodin & Salminen, 2015). The state space of X is denoted by $\mathbf{I}$, an interval of the real line $\mathbf{R}$ with left endpoint $a = \inf \mathbf{I}$ and right endpoint $b = \sup \mathbf{I}$, where $-\infty \leq a < b \leq \infty$. The notations m and S are used for the speed measure and the scale function, respectively, of X. Moreover, for $r \geq 0$ let φ_r and ψ_r denote the decreasing and increasing, respectively, fundamental solutions of the generalized ODE (see Borodin & Salminen, 2015 II.10 p.18)

$$\frac{d}{dm}\frac{d}{dS}u = ru. \tag{1}$$

Let $\mathbb{P}_x$ stand for the probability measure associated with X when starting from x, and by $\mathbb{E}_x$ the corresponding mathematical expectation. Denote by $\mathcal{M}$ the set of all stopping times with respect to $\{\mathcal{F}_t : t \geq 0\}$, the usual augmentation of the natural filtration generated by X. Given a non-negative lower semicontinuous reward function $g : \mathbf{I} \to \mathbf{R}$ and a discount parameter $r \geq 0$, consider the optimal stopping problem consisting of finding a function V_r and a stopping time $\tau^* \in \mathcal{M}$, such that

$$V_r(x) = \mathbb{E}_x[e^{-r\tau^*} g(X_{\tau^*})] = \sup_{\tau \in \mathcal{M}} \mathbb{E}_x[e^{-r\tau} g(X_\tau)]. \tag{2}$$

The *value function* V_r and the *optimal stopping time* τ^* constitute the solution of the problem.

Regarding applications, in certain situations, such as in finance, decisions about optimal stopping, or similar decisions such as when to place a market order to control inventory in a limit-order book market making activity, it is important to keep track of information at a ultra-high frequency level, that is, practically in continuous time, to be able to make optimal decisions at essentially the exact moment when the edge of a continuation region is reached. In other words, high-frequency data are needed to make these optimal decisions at stopping times in the continuous-time framework. Nevertheless, it is well understood that the stochastic limit models corresponding to microstructure financial market models can be more complex than the ones considered here (see, e.g., Cont, 2011), and the corresponding stopping problems for these complex models are beyond the scope of the present work.

1.3 | References

After the classical works of McKean (1965), Merton (1973) and Taylor (1968), there has been, in recent times, an important effort to characterize optimal stopping problems with one-sided solutions, that is, such that the optimal stopping time is a threshold stopping time, usually of the form

$$\tau^* = \inf\{t \geq 0 : X(t) \geq x^*\},$$

for some critical x^*. In this situation, we say that the problem has a *one-sided* solution. For typical examples of one-sided solutions derived using the principle of smooth fit, we refer to papers on Russian options due to Shepp and Shiryaev (1993, 1994). Villeneuve (2007) gives sufficient conditions to have threshold optimal strategies, and Arkin (2015) gives necessary and sufficient conditions for Itô diffusions with C^2 payoffs functions to have one-sided solutions, whereas Arkin and Slastnikov (2017) and Crocce and Mordecki (2014) give also necessary and sufficient conditions in different and more general diffusion frameworks. For more general Markov processes, Christensen and Irle (2017), Christensen, Salminen, and Ta (2013), and Mordecki and Salminen (2007) propose verification results for one-sided solutions, but also for problems where the optimal stopping time is of the form

$$\tau^* = \inf\{t \geq 0 : X(t) \notin (x_*, x^*)\}.$$

In this second situation, it is said that the problem has a *two-sided* solution. For general reference of the theory of optimal stopping, see the books by Shiryaev (1978) and by Peskir and Shiryaev (2006).

1.4 | Present study

In this paper, we are interested to understand situations where the stopping region is disconnected due to the behavior of the underlying stochastic process and not due to the properties of the payoff function. Such a case has been found in Alvarez E and Salminen (2017) with skew Brownian motion as the underlying. The idea is to study the shape of the continuation set as a function of the discount parameter r. To depart, we consider a model where the solution of the optimal stopping problem is trivial in cases $r = 0$ (no stopping) and $r = \infty$ (immediate stopping), and describe the stopping set as a function of r when it increases from 0 to ∞. When isolated continuation intervals appear when r increases, we say that a *shape transition* occurs. Such intervals are called *bubbles*, for the definition, see Section 2. We demonstrate this phenomenon by analyzing a diffusion with a constant drift parameter changing when the process passes a given level.

The rest of the paper is organized as follows. In Section 2, we present some preliminary general results mainly on optimal stopping for diffusions. In Section 3, the Brownian motion with broken drift as the solution of a stochastic differential equation is introduced and its main characteristics are analyzed. In Section 4, we solve the optimal stopping

problem for the Brownian motion with broken drift with reward $x \mapsto (1+x)^+$. In particular, it is seen that if the discontinuity in the drift is large enough, a shape transition in the continuation region occurs.

2 | PRELIMINARY RESULTS

From the general theory of Markovian stopping problems, the optimal stopping time τ^* in (2), if such a time exists, can be characterized (see Theorem 3, Section 3.3 in Shiryaev, 1978) as the first entrance time into the stopping set

$$\mathbf{S}(r) = \{x \in \mathbf{I}:\ V_r(x) = g(x)\}, \tag{3}$$

where we have indicated the dependence on the discounting parameter $r \geq 0$. The complement of $\mathbf{S}(r)$, that is, $\mathbf{C}(r) = \mathbf{I} \backslash \mathbf{S}(r)$ is called the continuation set.

Proposition 1 *Let $0 \leq r_1 < r_2$ be two discounting parameters and consider the corresponding OSPs as given in (2). Then,*

$$V_{r_1}(x) \geq V_{r_2}(x) \quad \text{for all } x \in \mathbf{I}, \tag{4}$$

and

$$\mathbf{S}(r_1) \subseteq \mathbf{S}(r_2). \tag{5}$$

Proof. *Let τ be a fixed stopping time. Then,*

$$e^{-r_1 \tau} g(X_\tau) \geq e^{-r_2 \tau} g(X_\tau),$$

and taking the suprema yields (4). Using the characterization of the stopping set in (3), we obtain now that if $x \in S(r_1)$, then $x \in S(r_2)$ resulting to (5).

We recall next the smooth fit theorem (cf. Peskir, 2007; Salminen, 1985; Salminen & Ta, 2015; Samee, 2010). To fix ideas and to focus on the case studied below, the theorem is formulated here for a left boundary point of the stopping region $\mathbf{S}(r)$.

Theorem 1 *Let z be a left boundary point of $\mathbf{S}(r)$, that is, $[z,\ z+\varepsilon_1) \subset \mathbf{S}(r)$ and $(z-\varepsilon_2,\ z) \subset \mathbf{C}(r)$ for some positive ε_1 and ε_2. Assume that the reward function g and the fundamental solutions φ_r and ψ_r are differentiable at z. Then, the value function V_r in (2) is differentiable at z and it holds $V'(z) = g'(z)$.*

Since the value function V is greater than the reward g on the continuation set $\mathbf{C}(r)$, we introduce the following terminology.

Definition 1 A bounded open interval $(x_1, x_2) \subseteq \mathbf{C}(r)$ is called a bubble if $x_1, x_2 \in \mathbf{S}(r)$.

In Alvarez E and Salminen (2017), it is seen that for skew Brownian motion and a large class of reward functions, one can find a lower bound r_0 for the discounted parameter r such that for all $r \geq r_0$ there is a bubble (containing the skew point). This is, in particular, true for the reward function $g(x) = (1+x)^+$, that is, no matter how large is the discounting it is not optimal to stop at the skew point. In the present paper, we study the appearance and the disappearance of a bubble for a Brownian motion with piecewise constant positive drift discontinuous at the origin. We define this process—which we call a Brownian motion with broken drift—in the next section.

To make the presentation more self-contained, we display a result from Alvarez E and Salminen (2017), see Lemma 2 therein, which is used to verify that a candidate solution of OSP (2) is indeed the value function. This is essentially Corollary on p. 124 in Shiryaev (1978).

Proposition 2 *Let* A *be a nonempty Borel subset of* $\mathbf{I}$ *and*

$$H_A = \inf\{t \geq 0 : X_t \in A\}.$$

Assume that the function

$$\widehat{V}(x) := \mathbb{E}_x \left[e^{-rH_A} g(X_{H_A}) \right]$$

is r-excessive and dominates g. Then, $\widehat{V}$ coincides with the value function of OSP (2) and H_A is an optimal stopping time.

The following technical result is needed in the proof of Proposition 8.

Proposition 3 *Consider a family $\mathcal{H} = \{h_r : \mathbf{I} \to [0,\infty);\ r \in I\}$ such that for each $r \in I \subset \mathbf{R}$ the function h_r is r-excessive and $\mathcal{H}$ is dominated by a measurable function $\widehat{h}$, that is, $h_r(x) \leq \widehat{h}(x)$ for all $x \in \mathbf{I}$, $r \in I$. Assume also that $\mathbb{E}_x(\widehat{h}(X_t)) < \infty$ for all $t \geq 0$ and $x \in \mathbf{I}$. If for some r_0 the limit*

$$\lim_{r \to r_0} h_r(x) =: h_0(x),$$

exists for all $x \in I$, then the function h_0 is r_0-excessive.

Proof. Consider

$$\mathbb{E}_x \left[e^{-r_0 t} h_0(X_t) \right] = \mathbb{E}_x \left[\lim_{r \to r_0} e^{-rt} h_r(X_t) \right] = \lim_{r \to r_0} \mathbb{E}_x \left[e^{-rt} h_r(X_t) \right] \leq \lim_{r \to r_0} h_r(x) = h_0(x),$$

where in the second step, we use the dominated convergence theorem which is applicable since $e^{-rt} h_r(X_t) \leq \widehat{h}(X_t)$.

3 | BROWNIAN MOTION WITH BROKEN DRIFT

Consider a diffusion

$$X(t)=x+\int_0^t \mu(X(s))ds+W(t),$$

where for $0 \le \mu_1 < \mu_2$

$$\mu(x)=\begin{cases}\mu_1, & \text{for } x<0,\\ \mu_2, & \text{for } x\ge 0\end{cases},$$

and $\{W_t : t\ge 0\}$ is a standard Brownian motion. The speed measure of this diffusion is given by

$$m(dx)=\begin{cases}2e^{2\mu_1 x}dx, & \text{for } x<0,\\ 2e^{2\mu_2 x}dx, & \text{for } x>0,\end{cases}$$

while the scale function is

$$S(x)=\begin{cases}\frac{1}{2\mu_1}(1-e^{-2\mu_1 x}), & \text{for } x<0,\\ \frac{1}{2\mu_2}(1-e^{-2\mu_2 x}), & \text{for } x\ge 0.\end{cases}$$

We call the diffusion $\{X_t : t\ge 0\}$ a Brownian motion with broken drift, and remark that in the literature, one can also find a diffusion called the Brownian motion with alternating drift (also the bang–bang Brownian motion), see Borodin and Salminen (2015) p. 128 and references therein. Notice that the scale function is differentiable everywhere with the derivative

$$S'(x)=\begin{cases}e^{-2\mu_1 x}, & \text{for } x<0,\\ e^{-2\mu_2 x}, & \text{for } x\ge 0.\end{cases}$$

We find next the fundamental solutions for $r>0$. Introduce

$$\lambda_1^- = -\sqrt{\mu_1^2+2r}-\mu_1<0, \quad \lambda_1^+ = \sqrt{\mu_1^2+2r}-\mu_1>0,$$
$$\lambda_2^- = -\sqrt{\mu_2{}^2+2r}-\mu_2<0, \quad \lambda_2^+ = \sqrt{\mu_2{}^2+2r}-\mu_2>0.$$

The decreasing fundamental solution is

$$\varphi_r(x)=\begin{cases}A_1\exp(\lambda_1^- x)+A_2\exp(\lambda_1^+ x), & \text{for } x<0,\\ \exp(\lambda_2^- x), & \text{for } x\ge 0,\end{cases} \tag{6}$$

where the constants A_1 and A_2 are determined so that φ_r is differentiable at 0. Hence,

$$A_1=\frac{\lambda_1^+-\lambda_2^-}{\lambda_1^+-\lambda_1^-}=\frac{\lambda_1^+-\lambda_2^-}{2\sqrt{\mu_1^2+2r}}>0, \qquad A_2=\frac{\lambda_2^--\lambda_1^-}{\lambda_1^+-\lambda_1^-}=\frac{\lambda_2^--\lambda_1^-}{2\sqrt{\mu_1^2+2r}}<0.$$

Observe that $A_1+A_2=1$. Analogously, the increasing solution is

$$\psi_r(x)=\begin{cases}\exp(\lambda_1^+ x), & \text{for } x<0,\\ B_1\exp(\lambda_2^+ x)+B_2\exp(\lambda_2^- x), & \text{for } x\ge 0\end{cases} \tag{7}$$

with

$$B_1=\frac{\lambda_1^+-\lambda_2^-}{\lambda_2^+-\lambda_2^-}=\frac{\lambda_1^+-\lambda_2^-}{2\sqrt{\mu_2^2+2r}}>0, \quad B_2=\frac{\lambda_2^+-\lambda_1^+}{\lambda_2^+-\lambda_2^-}=\frac{\lambda_2^+-\lambda_1^+}{2\sqrt{\mu_2^2+2r}}<0.$$

The above stated properties $A_1>0$, $B_1>0$, and $A_2<0$ are easily seen from the explicit expressions for λ_i^+, λ_i^-, $i=1,2$. For $B_2<0$, notice that $\mu\mapsto\sqrt{\mu^2+2r}-\mu$ is decreasing when $\mu>0$. In Figure 1, we have visualized the functions φ_r and ψ_r when $r=3$, $\mu_1=1$, $\mu_2=10$.

4 | SOLUTION OF THE OPTIMAL STOPPING PROBLEM

We analyze the optimal stopping problem (2) for the broken drift diffusion introduced above and the reward function $g(x)=(1+x)^+$, that is,

$$\sup_{\tau\in\mathcal{M}} \mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\right]. \tag{8}$$

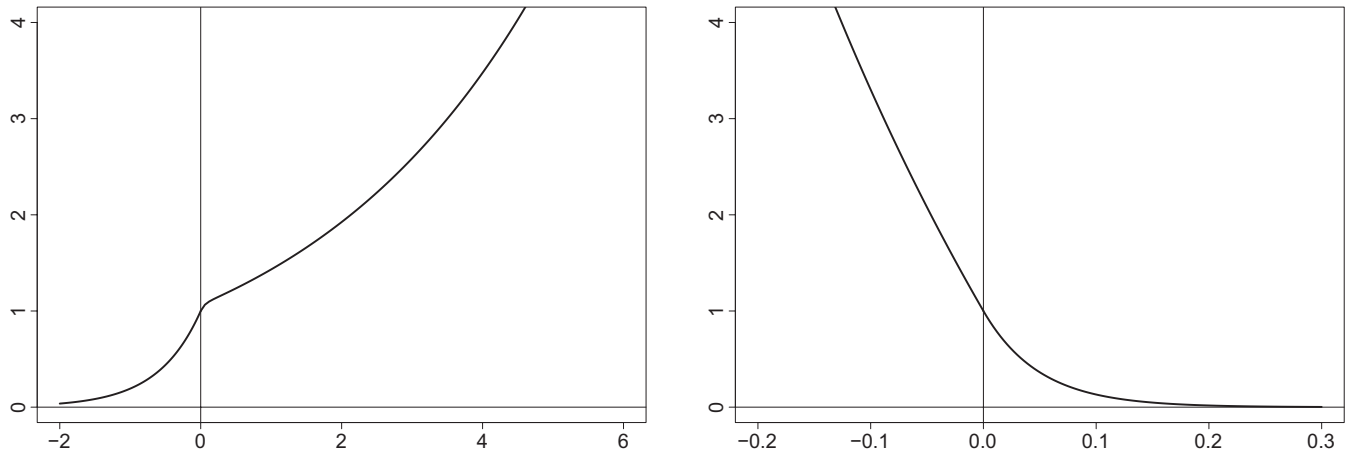


FIGURE 1 The functions ψ_r (left) and φ_r (right), for $r=3$, $\mu_1=1$, $\mu_2=10$

The main issue is to show that there are values of μ_1 and μ_2 such that, for some values on r, the continuation region is disconnected and contains 0, that is, the point where the drift changes. However, letting here r to increase makes the continuation region connected and then 0 is in the stopping set. For the exact statements, see Propositions 5 and 8. This situation described therein is different from the one studied in Alvarez E and Salminen (2017) where the skew point 0 is for all values of r and the skewness parameter $\beta > 1/2$ in the continuation set.

Clearly, it follows from the assumption $0 \le \mu_1 < \mu_2$ that $X(t) \to +\infty$ as $t \to +\infty$. Consequently, to make OSP (2) non-trivial we assume that $r > 0$. Because $g(x) = 0$ for $x \le -1$, it holds $(-\infty, -1) \subset \mathbf{C}(r)$. Notice also that the smooth fit theorem applies for all values of r, that is, the value function meets the reward smoothly at every boundary point between $\mathbf{C}(r)$ and $\mathbf{S}(r)$.

For the analysis to follow, we define (cf. Salminen, 1985)

$$G_-(x) := \left(\psi_r'(x)(x+1) - \psi_r(x)\right)/S'(x), \quad (9)$$

$$G_+(x) := \left(\varphi_r(x) - (1+x)\varphi_r'(x)\right)/S'(x), \quad (10)$$

and their derivatives with respect to the speed measure for $x > -1$ and $x \ne 0$ are given by

$$\begin{aligned} G_-'(x) &= m(x)\tfrac{d}{dm}G_-(x) = m(x)\psi_r(x)\left(r(1+x) - \tfrac{d}{dm}\tfrac{d}{dS}(1+x)\right) \\ &= m(x)\psi_r(x)\begin{cases} r(1+x)-\mu_1, & x<0, \\ r(1+x)-\mu_2, & x>0, \end{cases} \end{aligned} \quad (11)$$

and, similarly,

$$G_+'(x) = m(x)\varphi_r(x)\begin{cases} \mu_1 - r(1+x), & x<0, \\ \mu_2 - r(1+x), & x>0, \end{cases} \quad (12)$$

where we have used that the fact that φ_r and ψ_r solve the differential equation

$$\frac{d}{dm}\frac{d}{dS}u = ru.$$

The functions G_- and G_+ are used to check the excessivity of a proposed value function. An alternative way to check the excessivity is to evoke the generalization of the Itô formula developed in Peskir (2005).

Proposition 4 *In case $0 < r \le \mu_1 \le \mu_2$, the continuation region for OSP (8) is given by*

$$\mathbf{C}(r) = (-\infty, c),$$

where $c = c(r) > 0$ is the unique solution of the equation

$$\psi_r'(x)(x+1) - \psi_r(x) = 0. \quad (13)$$

Proof. We show first that Equation (13) has a unique positive solution. For this, consider for $x > -1$ the function G_ defined in (9). Since $S'(x) > 0$ for all x, the claim is equivalent to the statement that G_ has a unique positive zero. In fact, we prove a bit more; namely, that the function G_ attains the global minimum at $x_0 := (\mu_2 - r)/r > 0$, is negative and decreasing for $x \le x_0$, is increasing for $x > x_0$, and has, therefore, a unique zero. Analyzing G_-' as given in (11), it is straightforward to deduce, since $0 < r \le \mu_1 \le \mu_2$, the claimed properties of G_. Let

$$H_c := \inf\{t : X_t \ge c\},$$

where c is the unique solution of (13), and define

$$\widehat{V}(x) := \mathbb{E}_x\left[e^{-rH_c}g(X_{H_c})\right] = \begin{cases} \frac{\psi_r(x)}{\psi_r(c)}g(c), & x \le c, \\ g(x), & x \ge c. \end{cases} \quad (14)$$

If $\widehat{V}$ is an r-excessive majorant of g, it follows from Proposition 3 that $\widehat{V}$ is the value function of OSP (8). The excessivity can be checked with the method based on the representation theory of excessive functions (cf. Salminen, 1985 Section 3). This boils down to study for $x \ne -1$ the functions

$$I_V(x) := \left(\psi_r'(x)\widehat{V}(x) - \psi_r(x)\widehat{V}'(x)\right)/S'(x), \quad (15)$$

$$D_V(x) := \left(\varphi_r(x)\widehat{V}'(x) - \widehat{V}(x)\varphi_r'(x)\right)/S'(x). \quad (16)$$

Clearly, $I_V(x) = 0$ for $x \le c$ and increasing for $x > c$. Notice that $I_V = G_-$ on $[c, +\infty)$. Studying the derivative (with respect to the speed measure) of D_V, it is easily seen that D_V is positive and decreasing to 0 on $[c, +\infty)$. Consequently, I_V and D_V induce a (probability) measure which represents $\widehat{V}$ proving that $\widehat{V}$ is r-excessive. To prove that $\widehat{V}$ is a majorant of g, consider for $-1 < x < c$

$$\widehat{V}(x) \ge g(x) \Leftrightarrow \frac{\psi_r(x)}{g(x)} \ge \frac{\psi_r(c)}{g(c)}.$$

The right hand side of this equivalence holds since the derivative of $x \mapsto \psi_r(x)/g(x)$ is G_- which is negative for $-1 < x < c$, as is shown above.

In case $\mu_1 = \mu_2$, it is well known (see Salminen, 1985; Taylor, 1968) where the problem is solved using the representation theory of the excessive functions) that $\mathbf{S}(r) = [c, +\infty)$ with $c = c(r)$ as in Proposition 4, that is,

$$c = \frac{1}{\lambda_1^+} - 1.$$

Consequently, it is expected that if μ_2 is relatively close to μ_1, the stopping region is of this form for all values of

the discounting parameter r; in other words, there is no bubble. This is indeed the case, and the exact formulation is as follows.

Proposition 5 *In case* $0 \le \mu_1 \le \mu_2 \le \mu_1 + \frac{1}{2}$, *the continuation region for the OSP (8) is given by*

$$\mathbf{C}(r) = (-\infty, c),$$

where c = c(r) is the unique solution of Equation (13):

$$\psi_r'(x)(x+1) - \psi_r(x) = 0.$$

As r increases from 0 to $+\infty$, then $c(r)$ decreases monotonically from $+\infty$ to -1. In particular, $c(r) = 0$ for $r = \mu_1 + \frac{1}{2}$.

Proof. If $r \le \mu_1$ (and $\mu_1 > 0$), the statement is the same as in Proposition 4. We assume next that $r \ge \mu_2$. The proof in this case is very similar to the proof of Proposition 4. It can be proved that G_- attains the global minimum at $x_1 := (\mu_1 - r)/r < 0$, is negative and decreasing for $x \le x_1$, is increasing for $x > x_1$, and has, therefore, a unique zero. Consequently, this root can be taken to be an optimal stopping point $c = c(r)$ and the analogous function $\widehat{V}$ as in (14) can be proved to be the value of OSP (8). Finally, assume $\mu_1 < r < \mu_2$. In this case, G_- has a local maximum at 0, which is negative since

$$G_-(0) = \psi_r'(0) - \psi_r(0) = \lambda_1^+ - 1 < 0 \quad \Leftrightarrow \quad r < \mu_1 + 1/2.$$

Hence, Equation (13) has a unique positive root and the proof can be completed as in the previous cases.

Finally, we study the situation $0 \le \mu_1 < \mu_1 + \frac{1}{2} < \mu_2$. The main observation is that there exists a bounded interval of values of the discount r such that the continuation set has a bubble. The first result concerns the localization of a possible bubble and is perhaps intuitively obvious. Anyway, we present its proof since the result is needed when proving Proposition 8 below which characterizes the continuation (and the stopping) set in the present case.

Proposition 6 *Assume* $r > 0$ *is such that* (c_1(r), c_2(r)) *is a bubble. Then, it holds that* $0 \in [c_1(r), c_2(r))$, *and there exists at most one bubble.*

Proof. Let G_- and G_+ be given as in (9) and (10), respectively. Since g, φ_r, and ψ_r are differentiable everywhere, we may apply (4.7) Theorem in Salminen (1985) p. 95 to deduce that $c_1 = c_1(r)$ and $c_2 = c_2(r)$ satisfy

$$G_-(c_1) = G_-(c_2),$$
$$G_+(c_1) = G_+(c_2).$$

Moreover, G_- and G_+ are positive, nondecreasing, and nonincreasing, respectively, on $(c_1 - \varepsilon, c_1] \cup [c_2, c_2 + \varepsilon) \subseteq \mathbf{S}(r)$ for some $\varepsilon > 0$. Studying these explicit expressions of G_-' and G_+' given in (11) and (12), respectively, we conclude that necessarily $c_1 \le 0 < c_2$, as claimed.

Recall that the principle of smooth fit holds for our stopping problem. Hence, it is enlightening to investigate which "good" candidates satisfying the smooth fit principle cannot be value functions since they fail to be excessive. The following result shows that for $\mu_1 + \frac{1}{2} \le r < \mu_2$, there exist smooth fit (at 0) functions which are harmonic on $\mathbf{R}_-$ but which are not r-excessive.

Proposition 7 *For* $r \ge \mu_1 + \frac{1}{2}$, *there exist* A *and* B *such that the function*

$$F(x) := \begin{cases} A \exp(\lambda_1^+ x) + B \exp(\lambda_1^- x), & x \le 0, \\ 1 + x, & x \ge 0, \end{cases} \tag{17}$$

satisfies the principle of smooth fit at 0, that is, $F'(0-) = F'(0+) = 1$. The function F is r-harmonic (and positive) on $(-\infty, 0)$ but not r-excessive if $r < \mu_2$. For $r < \mu_1 + \frac{1}{2}$, the coefficient B is negative and the function $F(x) \to -\infty$ as $x \to -\infty$ (and the function is not r-excessive).

Proof. We study only the case $r = r_0 := \mu_1 + \frac{1}{2}$ and leave the details of the other cases to the reader. In this case, $\lambda_1^+ = \sqrt{\mu_1^2 + 2r} - \mu_1 = 1$, and, obviously,

$$F(x) := \begin{cases} \exp(x), & x \le 0, \\ 1 + x, & x \ge 0, \end{cases}$$

satisfies smooth fit at 0. Consequently, F is r_0-harmonic (and positive) on $(-\infty, 0)$ and it remains to prove that F is not r_0-excessive. For this, consider the representing function (this corresponds G_- in (9))

$$x \mapsto \left(\psi_{r_0}'(x) F(x) - \psi_{r_0}(x) F'(x) \right) / S'(x).$$

The claim is that this function is not nondecreasing. Indeed, take derivative with respect to the speed measure to obtain

$$\begin{aligned} \frac{d}{dm}\left((\psi_{r_0}'(x) F(x) - \psi_{r_0}(x) F'(x)) / S'(x) \right) &= F(x) \frac{d}{dm} \frac{d}{dS} \psi_{r_0}(x) - \psi_{r_0}(x) \frac{d}{dm} \frac{d}{dS} F(x) \\ &= \psi_{r_0}(x) \begin{cases} 0, & x < 0, \\ r_0(1+x) - \mu_2, & x > 0. \end{cases} \end{aligned}$$

Since $r_0 < \mu_2$, this derivative is negative, for example, for small positive x-values; therefore, F is not r_0-excessive.

Remark 1 From Proposition 5 and 8, we may conclude that if $r \geq \mu_2$ ($r \leq \mu_1 + 1/2$), then the problem is one-sided and the optimal stopping point is negative (positive). Notice that the smooth fit function F in (17) could be excessive for $r \geq \mu_2$ but since the optimal stopping point is negative F is not the smallest excessive majorant of the reward.

Next proposition can be seen as our main result concerning OSP (8). It is proved that if $r \in (\mu_1 + 1/2, \mu_2)$ but is "close to" μ_2, then $\mathbf{C}(r)$ has a bubble. However, the bubble disappears when r becomes bigger than μ_2 or tends to $\mu_1 + 1/2$. We give a complete description of $\mathbf{C}(r)$ although there is some overlap with Proposition 5.

Proposition 8 *In case* $0 \leq \mu_1 < \mu_1 + \frac{1}{2} < \mu_2$, *there exists* $r_0 \in (\mu_1 + 1/2, \mu_2)$ *with the following properties:*

1. If $r \in [r_0, \mu_2)$, the continuation region is given by

$$\mathbf{C}(r) = (-\infty, c_1) \cup (c_2, c_3),$$

where $c_i = c_i(r), i = 1,2,3$, are such that $c_3 > 0 \geq c_2 \geq c_1 > -1$. In particular, for $r = r_0$ it holds $c_1 = c_2 < 0$.

2. If $r \geq \mu_2$, the continuation region is given by

$$\mathbf{C}(r) = (-\infty, c_-),$$

where $c_- = c_-(r) < 0$ is the unique solution of (13). In particular, for $r = \mu_2$

$$c_-(\mu_2) = \frac{1}{\lambda_1^+} - 1 = \left(\sqrt{\mu_1^2 + 2\mu_2} + \mu_1 - 2\mu_2 \right) / 2\mu_2 < 0. \quad (18)$$

3. If $r < r_0$, the continuation region is given by

$$\mathbf{C}(r) = (-\infty, c_+),$$

where $c_+ = c_+(r) > 0$ is the unique solution of (13).

Proof. The proof of (2) is as the proof of Proposition 5 when $r \geq \mu_2$. Notice, however, that in the present case, $c(r) < 0$ for all $r \geq \mu_2$. We consider next (3) in case $r \leq \mu_1 + \frac{1}{2}$. Studying the derivative of G_- and the value of G_- at zero, it is seen, as in the proof of Proposition 4, that equation $G_-(x) = 0$ has for $r < \mu_1 + \frac{1}{2}$ one (and only one) root $\rho = \rho(r) > 0$. In case $r = \mu_1 + \frac{1}{2}$, there are two roots $\rho_1 = \rho_1(r) = 0$ and $\rho_2 = \rho_2(r) > 0$. Proceeding as in the proof of Proposition 4 it is seen that the stopping region is as claimed with $c_+ = \rho$ if $r < \mu_1 + \frac{1}{2}$ and $c_+ = \rho_2$ if $r = \mu_1 + \frac{1}{2}$. Assume now that there does not exist a bubble for any $r \in [\mu_1 + \frac{1}{2}, \mu_2]$. Then, for all $r \in [\mu_1 + \frac{1}{2}, \mu_2]$, we can find $c = c(r)$ such that $\mathbf{S}(r) = [c, +\infty)$. Knowing that $c(r) > 0$ for $r = \mu_1 + \frac{1}{2}$ and $c(r) < 0$ for $r = \mu_2$, we remark first there does not exists r such that $c(r) = 0$. Indeed, by Theorem 1 the value should satisfy the smooth fit principle at 0 but from Proposition 7, we know that such functions are not r-excessive. Next, using $\mathbf{S}(r_1) \subseteq \mathbf{S}(r_2)$ for $r_1 < r_2$ (cf. Proposition 1) it is seen that $r \mapsto c(r)$ is nonincreasing, and has, hence, left and right limits. Consequently, there exists a unique point $\hat{r}$ such that

$$\hat{c}_+ := \lim_{r \uparrow \hat{r}} c(r) > 0 \quad \text{and} \quad \hat{c}_- := \lim_{r \downarrow \hat{r}} c(r) < 0.$$

Under the assumption that there is no bubble, the value function is of the form given in (14), that is,

$$V_r(x) = \begin{cases} \psi_r(x) \frac{1 + c(r)}{\psi_r(c(r))}, & x \leq c(r), \\ 1 + x, & x \geq c(r). \end{cases} \quad (19)$$
$$= \mathbb{E}_x \left(e^{-rH_c} (1 + X_{H_c}) \right),$$

where $H_c := \inf\{y : X_t \geq c(r)\}$. Letting in (19) $r \uparrow \hat{r}$ yields by Proposition 3 an $\hat{r}$-excessive function which by Proposition 2 is the value of the corresponding OSP (8). Notice that as a dominating function $\widehat{h}$, we may take an r-excessive with r small enough. Similarly, letting $r \downarrow \hat{r}$ yields an $\hat{r}$-excessive function which should also be the value of the same OSP. However, the functions are clearly different, and since the value is unique we have reached a contradiction showing that there exists at least one bubble. Evoking Proposition 6 completes the proof.

Remark 2 The fact that there is a bubble when $r < \mu_2$ but "close" to μ_2 would also follow if we can prove that G_- has a unique negative zero. Notice that G_- in this case is not monotone around 0. This would then imply the existence of a bubble if we can verify that the local minimum on $(0, +\infty)$ is positive. However, we have not been able to show this. Numerical calculations with some parameter values give evidence that the local minimum on $(0, +\infty)$ is indeed positive.

Remark 3 For $r < r_0$ we have the value function (cf. (14))

$$V_r(x) = \begin{cases} \psi_r(x) \frac{1 + c_+}{\psi_r(c_+)}, & x \leq c_+, \\ 1 + x, & x \geq c_+. \end{cases} \quad (20)$$

Since $c_+ > 0$, it holds $V_r(0) > g(0)$, and, hence, $(1 + c_+)/\psi_r(c_+) > 1$. Consequently,

$$V_r'(0) = \lambda_1^+ \frac{1+c_+}{\psi_r(c_+)} > 1,$$

because also $\lambda_1^+ > 1$ for $r > \mu_1 + \frac{1}{2}$. Moreover, $V_r'(x) \to 0$ as $x \to -\infty$, and there exists a unique point $a = a(r)$ such that $V_r'(a) = 1$. From Proposition 8, we know that the bubble appears as r increases and takes the value r_0. Therefore, we may describe the value function V_{r_0} to be of the form in (20) satisfying the smooth fit at $c_+(r_0) > 0$ and also at another point $a(r_0) < 0$ which is a tangent point with the reward.

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