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On optimal stopping of multidimensional diffusions

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Abstract

This paper develops an approach for solving perpetual discounted optimal stopping problems for multidimensional diffusions, with special emphasis on the d-dimensional Wiener process. We first obtain some verification theorems for diffusions, based on the Green kernel representation of the value function. Specializing to the multidimensional Wiener process, we apply the Martin boundary theory to obtain a set of tractable integral equations involving only harmonic functions that characterize the stopping region of the problem in the bounded case. The approach is illustrated through the optimal stopping problem of a d-dimensional Wiener process with a positive definite quadratic form reward function.

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1. Introduction

1.1. Background

The theory of optimal stopping in continuous time is a well established and active area of research in stochastic processes. Initiated in the field of sequential sampling in statistics, it receives a second important impulse with the emergence of mathematical finance and the problem of pricing American type contracts. For the interested reader we refer to the monographs by Shiryaev [36] and Peskir and Shiryaev [31], which seem to correspond respectively to these two periods of the topic. It can be observed that, although the general theory of optimal stopping is developed for processes with general state space, very few explicit solutions seem to be known in concrete examples for underlying multidimensional Markov processes, which constitute the subject of the present paper. There is also a relevant distinction between finite horizon and perpetual problems: in the present work we focus our attention on perpetual problems.

When considering the dimensionality of an optimal stopping problem, the first question to be answered is whether the problem has a representation in terms of a one dimensional problem. Important cases of this situation are for instance the optimal stopping of Bessel processes with integer index, that corresponds to the stopping of functions of the modulus of multidimensional Brownian motion, see Dubins, Shepp and Shiryaev [12], or the classical one-asset-for-other problem of the right to exchange one asset by another introduced by Margrabe [27] whose American counterpart can also be solved by a change of measure, see Gerber and Shiu [18] and Fajardo and Mordecki [16].

Another relevant class of multidimensional optimal stopping problems is the one containing singular diffusions: there is one underlying stochastic process that generates a second stochastic target process (that is usually not Markovian w.r.t. the original filtration). The first example of this type of problems is the Russian option introduced by Shepp and Shiryaev [34], where the target process is the running maximum, and, which gives rise in principle to a two dimensional problem. In some cases, this two dimensional problem can be reduced to a one dimensional one, after a convenient change of measure, as exposed in Shepp and Shiryaev [35]. A similar situation is the one for integral options considered in Kramkov and Mordecki [25]. Other examples in this direction can be found in the recent works by Johnson and Peskir on optimal stopping related to Bessel processes: the quickest detection problem [23], and sequential testing [24], where inherently two-dimensional optimal stopping problems are solved.

In the present work, according to what has been observed above, we are interested in the stopping of a multidimensional process when there is no available reduction. This situation has been by far less explored in the literature. In conclusion and in short, in this paper we concentrate on perpetual discounted optimal stopping problems for multidimensional diffusions, with special emphasis on the d-dimensional Wiener process.

Up to our knowledge, there are a few references on this topic. Hu and Øksendal [20] consider the McDonald–Siegel investment problem originated in a portfolio of several correlated geometric Brownian motions. Unfortunately, it turned out that the explicit solution given there holds only in trivial cases, see Christensen and Irle [6] and Nishide and Rogers [29]. This second reference discusses the more general problem of exchange of baskets, but only bounds for the solution are presented. Villeneuve [37] addresses general multidimensional American options with finite time horizon and characterizes the non-emptiness and some other properties of the stopping set. Christensen and Irle [6] propose a methodology to obtain information about the stopping region for multidimensional problems based on harmonic functions.

The multidimensional McDonald–Siegel investment problem is also studied in Christensen and Salminen [8] where an integral equation is derived which characterizes the stopping region and an ad hoc fixed point method is presented to solve the integral equation. However, the methodology developed in [8] seems also to be restricted to some special subclasses of problems. Crocce [9] obtains some verification theorems for multidimensional Markov processes, using the representation approach. Recently, an elementary approach for multidimensional optimal stopping problems was developed in Christensen and Irle [7] based on the notion of monotone stopping problems. It is shown to be applicable to underlying multidimensional jump processes, but does not seem to work for diffusions.

We refer also to the paper [11] by De Angelis, Federico and Ferrari where a multidimensional optimal stopping problem connected with a multidimensional singular control problem is analyzed. Also here the stopping region is determined via an integral equation.

1.2. Preliminaries and summary

In the present paper, by a *multidimensional diffusion* we understand a homogeneous nonterminating strong Markov process $X = \{X_t\}_{t\geq 0}$ with continuous trajectories, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ and taking values in $\mathbb{R}^d$. For each $x \in \mathbb{R}^d$ we define $\mathbf{P}_x(\cdot) := \mathbf{P}(\cdot \mid X_0 = x)$. We further assume that X satisfies the Feller property (Proposition III.2.4. in Revuz and Yor [32]) and that the filtration $\mathbb{F}$ is the result of the construction detailed in Proposition III.2.10 in [32]. In particular (see Thm. III.2.17 in [32]), the hitting time of any Borel set is a stopping time.

Given a diffusion X, a nonnegative and continuous reward function $g : \mathbb{R}^d \to [0, \infty)$, and a positive discount rate $r > 0$, we consider a perpetual optimal stopping problem that consists of finding the value function $V : \mathbb{R}^d \to [0, \infty)$ and an optimal stopping time τ^* such that

$$V(x) := \sup_{\tau} \mathbf{E}_x \left(e^{-r\tau} g(X_\tau)\right) = \mathbf{E}_x \left(e^{-r\tau^*} g(X_{\tau^*})\right), \tag{1}$$

where the supremum is taken over the class of all stopping times w.r.t. the filtration $\mathbb{F}$. In case $\tau = \infty$, we set $e^{-r\tau} g(X_\tau) := \limsup_{t\to\infty} e^{-rt} g(X_t)$. To guarantee finiteness of the value function, the following *growth condition* is often assumed to hold for all x

$$\mathbf{E}_x \left(\sup_{t\geq 0} e^{-rt} g(X_t)\right) < \infty. \tag{2}$$

This is, however, often difficult to check for specific examples. We discuss more explicit conditions in our particular situation in Section 3.2.

We now briefly describe a free boundary approach to a solution and our contribution in this context. From the general theory of Markovian stopping problems, the optimal stopping time τ^* – if such a time exists – can be characterized as the first entrance time into the stopping set

$$\mathcal{S} := \{x \in \mathbb{R}^d : V(x) = g(x)\}. \tag{3}$$

Denoting by $\mathcal{L}$ the infinitesimal generator of the process X, the value function is furthermore known to fulfill $(r - \mathcal{L})V(x) = 0$ for x in the continuation set $\mathcal{C} = \mathcal{S}^c$ and – if g is nice enough – it holds that $(r - \mathcal{L})V(x) = (r - \mathcal{L})g(x)$ for x in the interior of the stopping set $\mathcal{S}$. Therefore, if we can make sure that V is smooth enough to apply a version of Dynkin's formula also on the

boundary of V (*smooth fit*) and let $t \to \infty$, we obtain under suitable growth conditions,

$$\begin{aligned} V(x) &= \int_0^\infty \mathbf{E}_x \left(e^{-rs}(r-\mathcal{L})g(X_s) 1_{\{X_s \in \mathcal{S}\}} \right) ds \\ &= \int_{\mathcal{S}} (r-\mathcal{L})g(y) G_r(x, dy), \end{aligned} \tag{4}$$

where G_r denotes the r-Green kernel and is for all measurable $H \subseteq \mathbb{R}^d$ given by

$$G_r(x, H) := \int_0^\infty e^{-rs} \mathbf{P}_x(X_s \in H) ds. \tag{5}$$

When this kernel is absolutely continuous with respect to some reference measure, we write $G_r(x, y)$ to denote the respective derivative and call this also the Green kernel. By applying the same reasoning to g, we obtain – on the other hand – the *inversion formula* (see for instance Chapter 1, Prop. 2.1 in Ethier and Kurtz [15])

$$g(x) = \int_{\mathbb{R}^d} (r-\mathcal{L})g(y) G_r(x, dy).$$

In conclusion – under suitable assumptions on g and X – we expect that the value function fulfills

$$0 = g(x) - V(x) = \int_{\mathcal{S}^c} (r-\mathcal{L})g(y) G_r(x, dy), \ x \in \mathcal{S}. \tag{6}$$

In particular, as $\mathcal{S}$ is known to be a closed set under standard assumptions, it holds that

$$\int_{\mathcal{S}^c} (r-\mathcal{L})g(y) G_r(x, dy) = 0, \ x \in \partial\mathcal{S}, \tag{7}$$

which can be interpreted as a (highly nonlinear) integral equation for the unknown boundary of the stopping set. In a next step, the idea is to find a suitable class $\mathbb{U}$ of candidate sets U to characterize the continuation set $U = \mathcal{S}^c$ as the unique solution to equation

$$\int_U (r-\mathcal{L})g(y) G_r(x, dy) = 0, \ x \in \partial U,$$

in the class $\mathbb{U}$. We call (7) the *Green kernel equation* for the stopping set. The previously described approach is nowadays standard in the solution of optimal stopping problems with a free-boundary approach, see [31] for the relevant theory and a whole bunch of examples, mainly with a finite time horizon.

From a mathematical point of view, this is a satisfying solution to the problem. However, as discussed in Section 5, to find a numerical solution to a set of equations as (7) does not seem to be standard. Hence, the aim of this paper is to establish an equivalent characterization which is easier to handle.

To this end, we now specialize to the Wiener process case and proceed inspired by the general Martin boundary theory. More precisely, we do not evaluate (6) at the boundary of S, but take the limits when $\|x\| \to \infty$ along rays choosing a parameter a on the sphere with squared radius $\|a\|^2 = 2r$. Then, for each a, there exists a function h_a such that

$$\frac{G_r(x, y)}{G_r(x, 0)} \to h_a(y).$$

It is proved in Section 2.2 that $h_a(y) = e^{a \cdot y}$ where $\cdot$ denotes the scalar product. Now, if $\mathcal{S}^c$ is bounded, we obtain from Eq. (6) by taking the limit as above

$$\int_{\mathcal{S}^c} e^{a \cdot y} (r-\mathcal{L})g(y) dy = 0 \quad \text{for } a \in \mathbb{R}^d \text{ such that } \|a\|^2 = 2r. \tag{8}$$

We call (8) the *Martin kernel equation* for the stopping set.

In conclusion, the aim of this paper is to develop the theory to use Eq. (8) to solve multidimensional optimal stopping problems. For this, we use a multi-step procedure as follows:

Step 1: Establish the Green kernel equation (7) for the optimal stopping set $\mathcal{S}$ of a multidimensional diffusion (see the discussion above and Section 2).
Step 2: Prove the uniqueness of the solution of the Green kernel equation in a suitable class of candidate sets $\mathbb{U}$ (Section 2.2).
Step 3: When the diffusion is the Wiener process, establish the Martin kernel equation (8) for the optimal stopping set $\mathcal{S}$ (Section 3), and its equivalence to the Green kernel equation.
Step 4: Use step 2 to prove that $\mathcal{S}$ can be characterized as the unique solution to the Martin kernel equation in the class $\mathbb{U}$ (Section 3).

2. The Green kernel equation

2.1. Verification theorem

In this section we formulate a general verification theorem based on the Green kernel for a diffusion X. Denote by h_S the hitting time of a Borel set $S \subseteq \mathbb{R}^d$, defined by

$$h_S := \inf\{t \geq 0 : X_t \in S\}.$$

where, as usual, we consider $h_S = \infty$ if the set is empty. The following results in this subsection are part of Crocce's PhD thesis [9]. For the sake of completeness, we present the corresponding proofs and some auxiliary results in the Appendix.

Theorem 2.1. *Assume that there exists $f : \mathbb{R}^d \to \mathbb{R}$ such that*

$$g(x) = \int_{\mathbb{R}^d} f(y) G_r(x, dy) \quad \text{for all } x \in \mathbb{R}^d. \tag{9}$$

Consider a Borel set S such that $f(x) \geq 0$ for all $x \in S$. Assume that $\widetilde{V} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\widetilde{V}(x) := \int_S f(y) G_r(x, dy), \tag{10}$$

satisfies the following two conditions:

(i) $\widetilde{V}(x) \geq g(x)$ *for all* $x \in S^c$,
(ii) $\widetilde{V}(x) = g(x)$ *for all* $x \in \partial S$ *or equivalently*

$$\int_{S^c} f(y) G_r(x, dy) = 0 \quad \text{for all } x \in \partial S. \tag{11}$$

Then, $\tau^* = h_S$ is an optimal stopping time and $\widetilde{V}$ is the value function of the problem (1).

The following corollary provides a practical way of using Theorem 2.1, based on the notion of extended infinitesimal generator (see [32, Chap. VII, Sect. 1]). Let us, however, mention that the usability of the verification results obtained here is limited in a multidimensional setting as it is not easy to guess a suitable candidate. As outlined above, one standard way to validate the Green kernel equation is to establish the smooth fit condition making a suitable version of Dynkin's lemma applicable. This must, however, be proved on a case-by-case basis, see [8] (or – in the finite time case – [31]) for examples.

Note that instead of considering the discounting factor in our problem formulation, we could alternatively consider the process X killed at an independent, exponential time with parameter r, called r-killed process in the following.

Corollary 2.2. *Assume that $g : \mathbb{R}^d \to \mathbb{R}$ in (9) belongs to the domain of the extended infinitesimal generator associated with the r-killed process, that the growth condition (2) holds, and also that*

$$\mathbf{E}_x\left(\int_0^\infty e^{-rs}|(r-\mathcal{L})g(X_s)|ds\right) < \infty.$$

Suppose that S is a measurable set such that $(r-\mathcal{L})g(x) \geq 0$ for all $x \in S$ and define $\widetilde{V} : \mathbb{R}^d \to \mathbb{R}$ by

$$\widetilde{V}(x) := \int_S (r-\mathcal{L})g(y)G_r(x,dy).$$

Moreover, assume that the following two conditions hold:

(i) $\widetilde{V}(x) \geq g(x)$ *for all* $x \in \mathcal{S}^c$,
(ii) $\widetilde{V}(x) = g(x)$ *for all* $x \in \partial\mathcal{S}$.

Then, S is an optimal stopping region, $\tau^* = h_S$ is an optimal stopping time, and $\widetilde{V}$ is the value function of the problem (1).

2.2. Uniqueness of the solution

We now characterize the unknown stopping set $\mathcal{S}$ as the unique solution to the Green kernel equation (7) in a convenient class of sets. The proof follows the line of argument for corresponding finite time problems, first obtained by Peskir in [30]; more precisely, we use the structure of proof from Peskir [13]. Here, we give a general result in the setup of Theorem 2.1:

Theorem 2.3. *Assume that there exists a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ representing g as in (9) and that the value function $V : \mathbb{R}^d \to \mathbb{R}$ is of the form as in Theorem 2.1, i.e.*

$$V(x) = \int_{\mathcal{S}} f(y)G_r(x,dy),$$

so that the hitting time of $\mathcal{S}$ is optimal. Furthermore, let $\mathbb{U}$ be a class of nonempty closed subsets of $\mathbb{R}^d$ containing $\mathcal{S}$ such that

(a) $f(y) > 0$ *for all* $U \in \mathbb{U}$ *and* $y \in U$.
(b) *For all $U, U' \in \mathbb{U}$ the following implications, where λ denotes the Lebesgue measure, hold true:*

(i) *If* $\mathbf{P}_x(\lambda(\{t \leq h_{U^c} : X_t \notin U'\}) = 0) = 1$ *for all* $x \in U \cap U'$, *then* $U \subseteq U'$.
(ii) *If* $\mathbf{P}_x(\lambda(\{t \leq h_U : X_t \in U'\}) = 0) = 1$ *for all* $x \in U^c$, *then* $U' \subseteq U$.

Then $U = \mathcal{S}$ is the unique set in the class $\mathbb{U}$ which satisfies

$$\int_{U^c} f(y)G_r(x,dy) = 0 \text{ for all } x \in \partial U. \tag{12}$$

Proof. Let $U \in \mathbb{U}$ such that (12) holds and define

$$W(x) := \int_U f(y) G_r(x, dy).$$

Note that by (9) and (12) we have $W(x) = g(x)$ for all $x \in \partial U$ and by Lemma A.2 we also have $W(x) = g(x)$ for all $x \in U$. The rest of the proof is divided into three steps.

Step 1. We first prove that $W(x) \leq V(x)$ for all x:
For $x \in U$, this holds as $W(x) = g(x) \leq V(x)$ as V majorizes g. Now, let $x \in U^c$. Then, as in (30) in Lemma A.2, we obtain

$$W(x) = \mathbf{E}_x \left(e^{-rh_U} W(X_{h_U}) \right) = \mathbf{E}_x \left(e^{-rh_U} g(X_{h_U}) \right) \leq V(x),$$

because $\mathbf{P}_x(X_{h_U} \in \partial U) = 1$.

Step 2. Let $x \in \mathcal{S} \cap U$ and $\gamma := h_{\mathcal{S}^c}$. Using Lemma A.1 and step 1 we have

$$\begin{aligned} g(x) = W(x) &= \mathbf{E}_x \left(e^{-r\gamma} W(X_\gamma) \right) + \mathbf{E}_x \left(\int_0^\gamma e^{-rs} 1_U(X_s) f(X_s) ds \right) \\ &\leq \mathbf{E}_x \left(e^{-r\gamma} V(X_\gamma) \right) + \mathbf{E}_x \left(\int_0^\gamma e^{-rs} 1_U(X_s) f(X_s) ds \right) \end{aligned}$$

and on the other hand

$$\begin{aligned} g(x) = V(x) &= \mathbf{E}_x \left(e^{-r\gamma} V(X_\gamma) \right) + \mathbf{E}_x \left(\int_0^\gamma e^{-rs} 1_{\mathcal{S}}(X_s) f(X_s) ds \right) \\ &= \mathbf{E}_x \left(e^{-r\gamma} V(X_\gamma) \right) + \mathbf{E}_x \left(\int_0^\gamma e^{-rs} f(X_s) ds \right). \end{aligned}$$

Subtracting these formulas yields

$$\begin{aligned} 0 &\geq \mathbf{E}_x \left(\int_0^\gamma e^{-rs} f(X_s) ds \right) - \mathbf{E}_x \left(\int_0^\gamma e^{-rs} 1_U(X_s) f(X_s) ds \right) \\ &= \mathbf{E}_x \left(\int_0^\gamma e^{-rs} 1_{U^c}(X_s) f(X_s) ds \right). \end{aligned}$$

As $f(X_s) > 0$ in the integral above by assumption (a), we obtain that

$$\mathbf{P}_x(\lambda(\{t \leq \gamma : X_t \in U^c\}) = 0) = 1$$

which implies $\mathcal{S} \subseteq U$ by assumption (b(i)).

Step 3. Now assume that $x \in \mathcal{S}^c$. Recall that $\tau^* = \inf\{t : X_t \in \mathcal{S}\}$ is the optimal stopping time. Using Lemma A.1 and step 1 again, we obtain

$$V(x) \geq W(x) = \mathbf{E}_x \left(e^{-r\tau^*} W(X_{\tau^*}) \right) + \mathbf{E}_x \left(\int_0^{\tau^*} e^{-rs} 1_U(X_s) f(X_s) ds \right).$$

As $S \subseteq U$ by step 2 and $W = g$ on U by step 1, it holds that $W(X_{\tau^*}) = g(X_{\tau^*})$ on $\{\tau^* < \infty\}$. We deduce

$$\begin{aligned} V(x) &\geq \mathbf{E}_x \left(e^{-r\tau^*} g(X_{\tau^*}) \right) + \mathbf{E}_x \left(\int_0^{\tau^*} e^{-rs} 1_U(X_s) f(X_s) ds \right) \\ &= V(x) + \mathbf{E}_x \left(\int_0^{\tau^*} e^{-rs} 1_U(X_s) f(X_s) ds \right), \end{aligned}$$

where the optimality of τ^* is also used. Again, as $f(X_s) > 0$ in the integral above by assumption (a), we obtain that

$$\mathbf{P}_x(\lambda(\{t \le \tau^* : X_t \in U\}) = 0) = 1,$$

which yields that $U \subseteq \mathcal{S}$ by assumption (b(ii)). Together with the previous step, we have $U = \mathcal{S}$ as claimed. This concludes the proof. □

3. The Martin kernel equation for the Wiener process

3.1. Martin boundary for the d-dimensional exponentially killed Wiener process

From now on, the underlying diffusion is a standard d-dimensional Wiener process $W = \{W_t : t \ge 0\}$. Recall that the transition density of W is given by

$$p(t; x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{\|x-y\|^2}{2t}\right), \tag{13}$$

and – using formula (29) in Erdelyi et al. [14] p. 146, see also Itô and McKean [22] p. 233 – the Green kernel is given by

$$\begin{aligned} G_r(x, y) &= \int_0^\infty e^{-rt} p(t; x, y) dt \\ &= (2\pi)^{-d/2} 2 \left(\frac{\|x-y\|^2}{2r}\right)^{(2-d)/4} K_{(2-d)/2}\left(\|x-y\|\sqrt{2r}\right), \end{aligned} \tag{14}$$

where $r \ge 0$ (strictly positive in case $d = 1$ or 2) and K_ν is the modified Bessel function of the second kind as defined in Lebedev [26] p. 109 and called the Macdonald function. Recall (see, e.g., [26] p. 136) that for all $\nu \in \mathbb{R}$

$$\lim_{u\to\infty} \sqrt{u}\, \mathrm{e}^u\, K_\nu(u) = \sqrt{\pi/2}. \tag{15}$$

Lemma 3.1. *For y fixed, $d \ge 2$, and all $\|x\|$ large enough it holds*

$$\frac{G_r(x, y)}{G_r(x, 0)} \le 2^{(d+3)/2} \exp\left(\sqrt{2r}\, \|y\|\right).$$

Proof. Let $\nu := (d-2)/2$ and use the fact that $K_{-\nu} = K_\nu$ to obtain

$$\frac{G_r(x, y)}{G_r(x, 0)} = \left(\frac{\|x\|}{\|x-y\|}\right)^\nu \frac{K_\nu(\|x-y\|\sqrt{2r})}{K_\nu(\|x\|\sqrt{2r})}. \tag{16}$$

Applying (15) and the triangle inequality we deduce for $\|x\|$ large enough that

$$\begin{aligned} \frac{G_r(x, y)}{G_r(x, 0)} &\le 4 \left(\frac{\|x\|}{\|x-y\|}\right)^{(2\nu+1)/2} \exp\left(\sqrt{2r}\,(\|x\| - \|x-y\|)\right) \\ &\le 2^{(2\nu+5)/2} \exp\left(\sqrt{2r}\, \|y\|\right), \end{aligned}$$

as claimed. □

In the next proposition the limiting behavior of the ratio of the Green kernels as $\|x\| \to \infty$ along the rays starting from the origin is characterized. The result identifies the Martin boundary of an exponentially killed d-dimensional Wiener process as the unit d-dimensional sphere.

Proposition 3.2. *Let* $(x^{(n)} = (x_1^{(n)}, \dots, x_d^{(n)}))_{n=1}^{\infty}$ *be a sequence of points in* $\mathbb{R}^d$ *such that* $\|x^{(n)}\| \to \infty$. *Then*

$$\lim_{n\to\infty} \frac{G_r(x^{(n)}, y)}{G_r(x^{(n)}, 0)}$$

exists for all $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ *if and only if*

$$\lim_{n\to\infty} \frac{x_i^{(n)}}{\|x^{(n)}\|} = \frac{a_i}{\|a\|}, \quad i = 1, \dots, d$$

for some $a = (a_1, \dots, a_d) \in \mathbb{R}^d \setminus \{0\}$. *When the limit exists it is given by*

$$\frac{G_r(x^{(n)}, y)}{G_r(x^{(n)}, 0)} \to \exp\left(\sqrt{2r}\, \frac{a \cdot y}{\|a\|}\right). \tag{17}$$

Proof. Using (15) in (16) yields

$$\begin{aligned}
&\lim_{n\to\infty} \frac{G_r(x^{(n)}, y)}{G_r(x^{(n)}, 0)} \\
&\qquad = \lim_{n\to\infty} \left(\frac{\|x^{(n)}\|}{\|x^{(n)} - y\|}\right)^{(2\nu+1)/2} \exp\left(\sqrt{2r}\,(\|x^{(n)}\| - \|x^{(n)} - y\|)\right) \\
&\qquad = \lim_{n\to\infty} \exp\left(\sqrt{2r}\,(\|x^{(n)}\| - \|x^{(n)} - y\|)\right),
\end{aligned}$$

since, as is easily seen,

$$\lim_{n\to\infty} \frac{\|x^{(n)}\|}{\|x^{(n)} - y\|} = 1.$$

To conclude the proof, notice that

$$\|x^{(n)}\| - \|x^{(n)} - y\| = \frac{2\, x^{(n)} \cdot y - \|y\|^2}{\|x^{(n)}\| + \|x^{(n)} - y\|}$$

from which the claimed equivalence and formula (17) are readily obtained. □

3.2. Finiteness of the value function

As pointed out above, the finiteness of the value function is not easy to check for an example on hand using condition (2). We circumvent the direct use of (2) for the optimal stopping problem of a Wiener process by giving an estimate in terms of harmonic functions. By the general Martin boundary theory, it is known that the functions

$$x \mapsto \exp(a \cdot x),$$

where $\|a\|^2 = 2r$, found in the previous subsection are (minimal) r-harmonic functions. However, it is also straightforward to check directly that functions of the form

$$f_\mu : x \mapsto \int_{B_{2r}} \exp(a \cdot x)\mu(da) \tag{18}$$

for a finite measure μ on $B_{2r} := \{a : \|a\|^2 = 2r\}$ are r-harmonic, so that in particular, the process $(e^{-rt} f_\mu(W_t))_{t\geq 0}$ is a positive martingale. For a general treatment of Martin boundary theory see II.7 in [3].

Proposition 3.3. *Assume that there exists a finite measure μ on B_{2r} such that the function g/f_μ is bounded. Then the value function in (1) is finite.*

Proof. Let $c > 0$ be such that $g(x)/f_\mu(x) < c$ for all $x \in \mathbb{R}^d$ and consider for each stopping time τ

$$\begin{aligned}\mathbf{E}_x(e^{-r\tau}g(W_\tau)) &= \mathbf{E}_x\left(e^{-r\tau}f_\mu(W_\tau)\frac{g(W_\tau)}{f_\mu(W_\tau)}\right)\\ &\le c\mathbf{E}_x\left(e^{-r\tau}f_\mu(W_\tau)\right)\\ &\le c\, f_\mu(x) < \infty.\end{aligned}$$

Taking supremum over τ proves the claim. □

Corollary 3.4. *In the case $d = 2$, if g is such that*

$$\sqrt{\|x\|}\, e^{-\sqrt{2r}\|x\|}\, g(x) \text{ remains bounded as } \|x\| \to \infty,$$

then the value function in (1) is finite.

Proof. Taking the uniform distribution μ on the sphere in (18) yields

$$\begin{aligned}f_\mu(x) &= \int \exp\Big(\|a\|\,\|x\|\,\cos(\gamma(a,x))\Big)\,\mu(da)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\Big(\sqrt{2r}\,\|x\|\,\cos\theta\Big)d\theta\\ &= \frac{1}{\pi}\int_0^{\pi}\exp\Big(\sqrt{2r}\,\|x\|\,\cos\theta\Big)d\theta\\ &= I_0(\sqrt{2r}\,\|x\|),\end{aligned}$$

where $\gamma(a, x)$ denotes the angle between a and x and I_0 denotes the modified Bessel function of first kind of index 0. The final equality is the definition of I_0, see Abramowitz and Stegun [1]. For I_0 we have the following asymptotic formula

$$I_0(z) \simeq \frac{1}{\sqrt{2\pi z}}e^z \text{ as } z \to \infty,$$

see [1] p. 374 ff., which – when combined with Proposition 3.3 – completes the proof. □

3.3. Equivalence of the Green and Martin kernel equations

The next theorem establishes that the Green kernel equation implies the Martin kernel equation if the continuation set $\mathcal{C}$ is bounded.

Theorem 3.5. *Assume that there exists a bounded measurable set C such that g is C^2 in a neighborhood of C and for all $x \in S := C^c$ the Green kernel equation*

$$\int_C G_r(x, y)(r - \mathcal{L})g(y)dy = 0 \quad \text{for all } x \in S := C^c \tag{19}$$

holds true. Then, the Martin kernel equation

$$\int_C e^{a\cdot y}(r - \mathcal{L})g(y)dy = 0 \quad \text{for all } a \in B_{2r} \tag{20}$$

holds as well.

Proof. Divide by $G_r(x,0)$ in (19) and take limits in x along rays. Combining Lemma 3.1 and Proposition 3.2 with the dominated convergence theorem, the result follows. □

Next, we show that – under natural assumptions – condition (20) implies (19) also. We start with an easy lemma that sheds light on the probabilistic connection of G_r and the Martin boundary.

Lemma 3.6. *Let H be an affine hyperplane of the form $H = \{y \in \mathbb{R}^d : a \cdot y = b\}$ for some $a \in B_{2r}$, $b \in \mathbb{R}$, and introduce $h_H = \inf\{t \geq 0 : W_t \in H\}$. Then, for each $x \in \mathbb{R}^d$,*

$$2r \int_H G_r(x,y)\rho(dy) = \mathbf{E}_x(e^{-rh_H}) = e^{-\|a\cdot x - b\|},$$

where ρ denotes the surface measure on H.

Proof. Note that $h_H = \inf\{t \geq 0 : a \cdot W_t = b\}$ and the process $\widetilde{W} := a \cdot W$ is a one-dimensional Wiener process with infinitesimal variance $2r$. Therefore, the second identity using the well-known expression for the Laplace transform of the first Brownian hitting time, see e.g. Itô and McKean [22] p. 26 or Borodin and Salminen [5] formula 1.2.0.1 p. 204. For the first one, note that for an independent exponential time T with parameter r, we have

$$\begin{aligned}\int_H G_r(x,y)\rho(dy) &= \frac{1}{r}\int_H \frac{\mathbf{P}_x(W_T \in dy)}{dy}\rho(dy) = \frac{1}{r}\frac{\mathbf{P}_x(a\cdot W_T \in db)}{db}\\ &= \frac{1}{r}\frac{\mathbf{P}_{a\cdot x}(\tilde{W}_T \in db)}{db} = \frac{1}{2r}e^{-\|ax-b\|},\end{aligned}$$

where we used the expression of the Green kernel found e.g. in [5] Appendix 1, No 1, p. 124 in the last step. □

Theorem 3.7. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be continuous and Γ be a bounded measurable set such that f is C^2 in a neighborhood of Γ and assume that*

$$\int_\Gamma e^{a\cdot y} f(y)dy = 0, \text{ for all } a \in B_{2r}. \tag{21}$$

Then for all $x \in \mathbb{R}^d \setminus \mathrm{conv}(\Gamma)$

$$\int_\Gamma G_r(x,y) f(y)dy = 0, \tag{22}$$

where $\mathrm{conv}(\Gamma)$ denotes the convex closure of the set Γ. Furthermore, for each $x \in S = \Gamma^c$ such that there exists an analytic curve $\gamma : [0,1] \to S$ with $\gamma(0) = x$ and $\gamma(1)$ is an interior point of $\mathbb{R}^d \setminus \mathrm{conv}(\Gamma)$, (22) holds as well.

Proof. We write

$$f_\Gamma(x) = \int_\Gamma G_r(x,y) f(y)dy,$$

for short. Let $H \subseteq \mathbb{R}^d \setminus \Gamma$ be an arbitrary hyperplane not intersecting Γ, where the parameters a and b are chosen such that $a \cdot y \leq b$ for all $y \in \Gamma$. Then, using the previous lemma,

$$\begin{aligned} 0 =& \frac{1}{2r} \int_\Gamma e^{a\cdot y} f(y)dy = \frac{1}{2r} \int_\Gamma e^b e^{a\cdot y - b} f(y)dy = \int_\Gamma e^b \int_H G_r(y,z)\rho(dz)f(y)dy \\ =& e^b \int_H \int_\Gamma G_r(y,z)f(y)dy\rho(dz) = e^b \int_H \int_\Gamma G_r(z,y)f(y)dy\rho(dz) \\ =& e^b \int_H f_\Gamma(z)\rho(dz). \end{aligned}$$

We obtain that the integral of f_Γ over all hyperplanes not intersecting Γ vanishes. Furthermore, f_Γ is continuous and it is easily checked from (15) and the boundedness of Γ that $\|x\|^k f_\Gamma(x)$ is bounded for each integer $k > 0$. Now, Helgason's support theorem (see, e.g., [19, Corollary 2.8]) yields that $f_\Gamma(x) = 0$ for all $x \in \mathbb{R}^d \setminus \text{conv}(\Gamma)$, proving the first claim.

Now, let $x \in S = \Gamma^c$ be such that there exists an analytic curve $\gamma : [0,1] \to S$ with $\gamma(0) = x$ and $\gamma(1)$ is an interior point of $\mathbb{R}^d \setminus \text{conv}(\Gamma)$. f_Γ is analytic on $\mathbb{R}^d \setminus \Gamma$ as an integral over analytic functions. So, $f := f_\Gamma \circ \gamma$ is analytic as well. Furthermore, $f(t) = 0$ for all t in a neighborhood of 1 by the first claim. By the identity theorem for analytic functions, we have $f \equiv 0$, which proves that $f(0) = f_\Gamma(x) = 0$. □

In many situations of interest, the set Γ in the previous theorem – corresponding to the continuation set – might be convex, so that $conv(\Gamma) = \Gamma$ and the identity (22) immediately holds for all $x \in \Gamma^c$. However, the more general second assumption might be much easier to check as the example below illustrates. We call the set S *analytically connected to* $conv(\Gamma)^c$ in this case.

Adjusting it to the stopping situation, we have the following uniqueness result. Recall that $\mathcal{S}$ denotes the stopping set (cf. (3)).

Corollary 3.8. *In the optimal stopping problem (1) with $X = W$, assume that the reward function g is C^2 and let $\mathbb{U}$ be a class of sets S such that S^c is bounded and S is analytically connected to $conv(S^c)^c$. If $S = \mathcal{S}$ is the unique solution to the Green kernel equation*

$$\int_{S^c} G_r(x,y)(r - \mathcal{L})g(y)dy = 0 \quad \text{for all } x \in S, \tag{23}$$

in the class $\mathbb{U}$, then $S = \mathcal{S}$ is the unique solution to the Martin kernel equation

$$\int_{S^c} e^{a\cdot y}(r - \mathcal{L})g(y)dy = 0 \quad \text{for all } a \in B_{2r} \tag{24}$$

in the class $\mathbb{U}$ as well and vice versa.

Proof. Theorem 3.5 yields that each solution of (23) is also a solution to (24). On the other hand, if C is a solution to (24), then Theorem 3.7 yields that C is also a solution to (23), which proves the uniqueness. □

Remark 3.9. The results we obtained so far are applicable for bounded continuation sets as we are interested in this case for the examples considered below. The generalization of the results to unbounded continuation sets is not straightforward. One first reason is that one cannot expect (20) to hold true for all vectors in the circle $\|a\| = 2r$, but only for those directions a such that there exists a sequence $(x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)}))_{n=1}^\infty$ of points in the stopping set such that

$\|x^{(n)}\| \to \infty$ and $\lim_{n\to\infty} x_i^{(n)}/\|x^{(n)}\| = a_i/\|a\|$, $i = 1, \ldots, d$. To establish a variant for the reverse implication, the main ingredient needed is a variant of Helgason's support theorem for unbounded sets in the spirit of [2] under suitable assumptions on the geometry of the continuation set.

4. Examples

As we have seen, in order to solve an optimal stopping problem we begin by finding the stopping (or the continuation) set. It is clear that the set

$$\mathcal{N} = \{x \in \mathbb{R}^d : (r - \mathcal{L})g(x) \leq 0\},$$

called the *negative set*, is contained in the continuation set, i.e., $\mathcal{N} \subseteq \mathcal{C}$. Hence, in practical examples, once the problem is discretized, a natural proposal is to depart from the negative set and to consider the continuation set $\mathcal{C}$ as an enlargement of $\mathcal{N}$. To illustrate the previous general considerations, we concentrate on reward functions of the form $g(x) = \sum_{i=1}^d \lambda_i x_i^2$, with $\lambda_i > 0$. Due to the symmetry of the law of the Wiener process, any problem with a payoff function which is a positive definite quadratic form can be reduced to have this structure. As scaling does not change the optimal stopping rule, we can furthermore assume without loss of generality that $\lambda_1 = 1$. In the two-dimensional case, we use the parametrization $g(x, y) = x^2 + \alpha^2 y^2$ with a general $\alpha \neq 0$.

4.1. The symmetric case (OSP for Bessel processes)

Let us consider the symmetric case, when $\lambda_i = 1$ ($i = 1, \ldots, d$). The quadratic reward gives then $g(W) = \|W\|^2 = Z^2$ where $Z = \{Z_t = \|W_t\| : t \geq 0\}$ is a Bessel process of dimension d, so the problem has a representation in terms of the one-dimensional optimal stopping problem

$$\sup_\tau \mathbf{E}_x(e^{-r\tau}\tilde{g}(Z_\tau)).$$

The solution to this problem can be found by different methods, for instance following Example 2.1 in Christensen and Irle [6]. We then determine first the increasing fundamental solution $\psi_r(x) = I_\nu(x\sqrt{2r})/x^\nu$ where I_ν is the modified Bessel function of the first kind, and $\nu = d/2-1$ is the order of the process (see Appendix 1.21 p. 133 and Appendix 2.4 p. 638 in [5]) to find that the stopping threshold for the problem is given by

$$x^*(r, d) = \arg\max \frac{x^2}{\psi_r(x)} = \arg\max \frac{x^{2+\nu}}{I_\nu(x\sqrt{2r})} = \frac{1}{2r} \arg\max \frac{x^{2+\nu}}{I_\nu(x)}.$$

In case $d = 2$ we obtain that $x^*(r, 2) = (1/2r) \arg\max \frac{x^2}{I_0(x)}$, and in case $d = 3$, due to the fact that the modified Bessel function can be expressed in terms of hyperbolic functions (see p. 134 in [5]), we obtain also that $\sqrt{2r}x^*(r, 3)$ is the positive root of the equation $\operatorname{artanh}(z) = z/3$ (see p. 80 in [9]).

4.2. The two-dimensional quadratic reward

Now, we consider $g(x, y) = x^2 + \alpha^2 y^2$ for general α. We have

$$\mathcal{N} = \left\{(x, y) : x^2 + \alpha^2 y^2 \leq \beta^2\right\},$$

where

$$\beta = \sqrt{\frac{1+\alpha^2}{r}}.$$

As g is a polynomial, Corollary 3.4 yields the finiteness of the value function. Notice also that arguing for g as is done in the introduction for V it is seen that g can be represented as in (9). In the following lemma, which is proved in the Appendix, we collect some properties of the optimal stopping set $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : V(x, y) = g(x, y)\}$:

Lemma 4.1.

(i) *$\mathcal{S}$ is closed.*
(ii) *$\mathcal{N} \subseteq \mathcal{C} := \mathcal{S}^c$.*
(iii) *$\mathcal{S}$ is symmetric w.r.t. both Cartesian axes, i.e. if $(x, y) \in \mathcal{S}$, then $((-1)^\ell x, (-1)^k y) \in \mathcal{S}$ for all $\ell, k \in \{0, 1\}$.*
(iv) *For any $(x, y) \in \mathcal{S}$ there exists a cone with vertex (x, y) containing the half-line $\{(\lambda x, \lambda y) : \lambda \geq 1\}$ which is a subset of $\mathcal{S}$.*
(v) *$\mathcal{C}$ is star-shaped with origin as a center, i.e., for any $(x, y) \in \mathcal{C}$ the line segment between the origin and (x, y) is also contained in $\mathcal{C}$.*
(vi) *$\mathcal{C}$ is bounded. More precisely, for $\alpha \geq 1$ and (x, y) such that $|x| \geq \alpha^2 x^*(r, 2)$, $|y| \geq x^*(r, 2)$, where $x^*(r, 2)$ denotes the radius of the optimal stopping circle in the symmetric problem discussed in Section 4.1, it holds that $(x, y) \in \mathcal{S}$.*

From the general theory of Markov processes, the r-excessive value function V can be represented as

$$V(x, y) = \int_{\mathcal{S}} G_r((x, y), (x', y'))\eta(dx'dy').$$

for a certain measure η, see [8], Subsection 2.1, for the corresponding arguments for an underlying geometric Brownian motion. Following the lines of Section 1.2 it can more precisely be checked that the value function V is indeed of the form

$$V(x, y) = \int_{\mathcal{S}} G_r((x, y), (x', y'))\,(r - \mathcal{L})\,g(x', y')dx'dy'$$

(cf. [8], Subsection 2.1 and 3.3 for a discussion which can be modified for the present case). Therefore, also the Green kernel equation (7) holds true. To apply the general theory developed above, we need to specify a suitable class $\mathbb{U}$ of sets containing the stopping set. The previous lemma suggests to take the class of all U such that

($\mathbb{U}1$) U is closed.
($\mathbb{U}2$) $\mathcal{N} \subseteq U^c$.
($\mathbb{U}3$) U^c is bounded.
($\mathbb{U}4$) For any $(x, y) \in U$ there exists a cone with vertex (x, y), which is a subset of U.
($\mathbb{U}5$) U^c is star-shaped with origin as a center.

Indeed, it is straightforward to check that the class $\mathbb{U}$ fulfills the assumptions of Theorem 2.3. Therefore, we know that $\mathcal{S}$ is the only set in $\mathbb{U}$ such that for all $(x, y) \in \mathcal{S}$

$$0 = \int_{\mathcal{S}^c} G_r((x, y), (x', y'))\,(r - \mathcal{L})\,g(x', y')dx'dy'.$$

Now, putting pieces together, Corollary 3.8 yields

Proposition 4.2. *The stopping set $\mathcal{S}$ of the stopping problem for quadratic reward is the unique set in class $\mathbb{U}$ such that*

$$\int_{\mathcal{S}^c} e^{a\cdot(x,y)}(r-\mathcal{L})g(x,y)dxdy = 0, \text{ for all } a \in \mathbb{R}^2 \text{ such that } \|a\|^2 = 2r, \tag{25}$$

where $(r-\mathcal{L})g(x,y) = r(x^2+\alpha^2y^2-\beta^2)$.

In order to implement a numerical procedure to find the set $\mathcal{S}$, we change to affine polar coordinates $x = \rho\cos\theta, \alpha y = \rho\sin\theta$ in (25). The negative set becomes

$$\mathcal{N} = \Big\{(\theta,\rho) : 0 \le \theta < 2\pi, 0 \le \rho \le \beta\Big\}.$$

We parametrize the continuation set as

$$\mathcal{C} = \Big\{(\theta,\rho) : 0 \le \theta < 2\pi, 0 \le \rho \le \rho(\theta)\Big\}$$

where $\rho(\theta) \ge \beta$ is an unknown curve. Denoting $a = \sqrt{2r}(\cos\phi, \sin\phi)$, Eq. (25) is

$$\int_0^{2\pi}\int_0^{\rho\theta} e^{\sqrt{2r}(\rho\cos\theta\cos\phi+(\rho/\alpha)\sin\theta\sin\phi)}(r-\mathcal{L})g(\rho,\theta)\frac{1}{\alpha}\rho\, d\rho\, d\theta = 0,$$

which can be rewritten as

$$\int_0^{2\pi} F_2(\rho(\theta),\gamma(\theta,\phi))d\theta = \int_0^{2\pi} F_1(\gamma(\theta,\phi))d\theta,\ 0 \le \phi \le 2\pi. \tag{26}$$

where

$$\begin{aligned}
\gamma(\theta,\phi) &= \sqrt{2r}(\cos\theta\cos\phi + (1/\alpha)\sin\theta\sin\phi),\\
F_1(\gamma) &= \frac{-2\left(\gamma^2\beta^2 - 3\gamma\beta + 3\right)e^{\gamma\beta}}{\gamma^4} + \frac{\gamma^2\beta^2-6}{\gamma^4},\\
F_2(\rho,\gamma) &= \frac{2\left(\beta^2\gamma^2 - 3\beta\gamma + 3\right)e^{\beta\gamma}}{\gamma^4}\\
&\quad + \frac{\left(\left(\beta^2\rho-\rho^3\right)\gamma^3 - \left(\beta^2 - 3\rho^2\right)\gamma^2 - 6\gamma\rho + 6\right)e^{\gamma\rho}}{\gamma^4}.
\end{aligned}$$

Then, given a positive integer N, we discretize (26) by letting $\theta_i = i/(2\pi N)$ $(0 \le i < N)$ and, in order to determine the unknowns $\rho_i = \rho(\theta_i)$, we impose the same number of conditions by putting $\phi_j = j/(2\pi N)$ $(0 \le j < N)$. We then have the discretized problem

$$\sum_{i=0}^{N-1} F_2(\rho_i,\gamma(\theta_i,\phi_j)) = \sum_{i=0}^{N-1} F_1(\gamma(\theta_i,\phi_j)), \quad 0 \le j < N.$$

In order to find the vector $(\rho_i)_i$, we consider the minimization of the objective function

$$\sum_{j=0}^{N-1}\left(\sum_{i=0}^{N-1}\left[F_2(\rho_i,\gamma(\theta_i,\phi_j)) - F_1(\gamma(\theta_i,\phi_j))\right]\right)^2$$

implementing a gradient algorithm. In Fig. 1 we plot the solutions found for $\alpha = 2$ and $r = 1$ and $r = 0.3$, resp., considering $N = 64$. The solution for $\alpha = 1/2$ is, as expected, just a 90 degrees rotation of the one with $\alpha = 2$. The solution we found for $\alpha = 1/3$ and $r = 1$ is presented in Fig. 2.

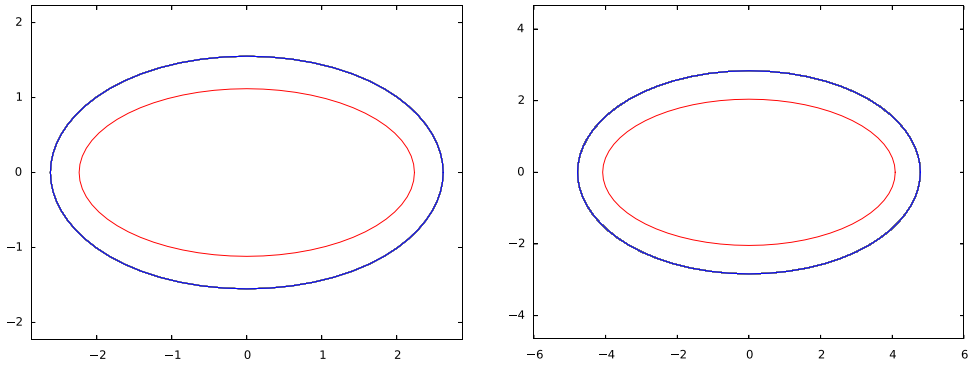


Fig. 1. Example in 4.2 with $\alpha = 2$ and with $r = 1$ (left) and $r = 0.3$ (right): The bounded region delimited in blue is the optimal continuation region. In the region delimited in red $(r - \mathcal{L})g$ is negative. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

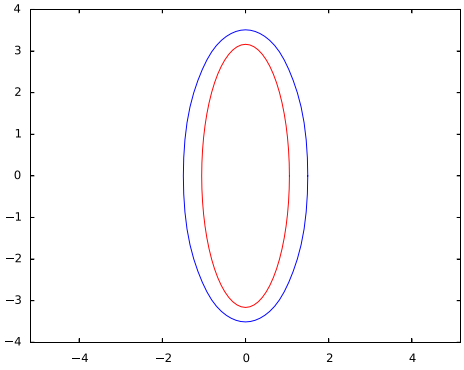


Fig. 2. Example in 4.2 with $\alpha = 1/3$ and $r = 1$: The bounded region delimited in blue is the optimal continuation region. In the region delimited in red $(r - \mathcal{L})g$ is negative. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4.3. The d-dimensional quadratic reward

For quadratic reward functions g for general d-dimensional Wiener process, a similar analysis can be carried out. The adaptation to, say, $d = 3$ is not hard. The quadratic reward considered here has the advantage that numerical methods can be checked for the symmetric reward $x^2 + y^2 + z^2$. For this problem, the analytic solution is known to be a sphere of radius $w/\sqrt{2r}$ where w satisfies $\tanh(w) = w/3$, see Section 4.1. Fig. 3 shows the solution obtained for the payoff $g(x, y, z) = x^2 + 4y^2 + 9z^2$ with $r = 1$, and exhibit the corresponding negative set $\mathcal{N}$ and the continuation region $\mathcal{C}$. The negative set in this case is

$$\mathcal{N} = \{(x, y, z) : (r - \mathcal{L})g \leq 0\} = \{(x, y, z) : x^2 + 4y^4 + 9z^2 \leq 14\}.$$

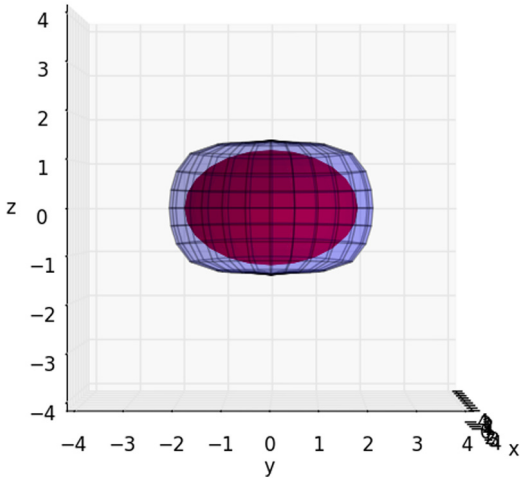


Fig. 3. The negative set $\mathcal{N}$ (in dark) is inside the continuation set $\mathcal{C}$.

5. Discussion

In this paper, we started from the well-established characterization of the optimal stopping set $\mathcal{S}$ via the Green-kernel equation

$$\int_{\mathcal{S}^c} (r - \mathcal{L})g(y)G_r(x, dy) = 0 \quad \text{for all } x \in \mathcal{S}, \tag{27}$$

and proved for underlying Brownian motions – under some assumptions – the equivalence to the Martin kernel equation

$$\int_{\mathcal{S}^c} e^{a\cdot y}(r - \mathcal{L})g(y)dy = 0, \ a \in \mathbb{R}^d \text{ such that } \|a\|^2 = 2r. \tag{28}$$

The new system (28) has many advantages over the Green kernel equation (27). Firstly, the parameters x in (27) depend directly on $\mathcal{S}$; the parameters a in (28) are the same for each choice of $\mathcal{S}$. Secondly, although the Green kernel G_r can be found explicitly for a Wiener process, the exponential functions $e^{a\cdot y}$ have the advantage that they do not have singularities, which makes numerical computations much more stable.

Note that the first two steps (the Green kernel equation-approach) in our approach described are only needed as auxiliary results to establish the characterization via (28). The main contribution of this paper is the latter characterization. The main tool we used to establish equivalence is the famous Helgason support theorem from the theory of Radon transforms, which seems to be new in this context.

The Martin boundary theory has been used already before to solve optimal stopping problems. This approach was initiated by Salminen in [33], where the Martin representation of the value function is the crucial tool to solve optimal stopping problems for one dimensional diffusions. See also [28,9,10,6,8] for further results on the approach based on the Riesz and the Martin representation of the value function. The present paper can be seen as a continuation in the project to study optimal stopping via these tools/concepts from the potential theory (of Markov processes). The theory presented here is, however, different in nature to the previous approaches.

In these, the analysis stopped with equations of the form (4) and (27). The main contribution of this paper is, however, to go one step further and use Martin limits to convert (27) to the more usable form (28).

The system of equations (28) gives rise to an easy to implement numerical scheme for the treatment of multidimensional optimal stopping problems with an infinite time horizon. In contrast to the finite time horizon case, alternative methods are rare as all methods based on backward induction cannot be applied without major modifications. An exception is methods based on Howard's policy improvement such as [21]. These, however, seem to be restricted, as a very rough discretization of the state space is needed, so that even in 2-dimensional problems accurate approximations to the stopping set seems to be hard to obtain. A second alternative is, of course, the use of numerics for free boundary problems for the Laplacian, based, for instance, on Th. 8.1 in Friedman [17]. However, we have not been able to find in the literature a successful implementation of such an approach for multidimensional stopping problems. Nevertheless the method based on (28) has the advantage that it decouples the original problem into two steps: firstly, to find the stopping set, and, secondly, the value function.

Acknowledgment

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Appendix. Proofs of Theorem 2.1, Corollary 2.2, and Lemma 4.1

Lemma A.1. *Let τ be a stopping time, and $f : \mathbb{R}^n \to \mathbb{R}$ a measurable function. If $f(x) \geq 0$ for all $x \in \mathbb{R}^d$ or $\int_{\mathbb{R}^d} |f(y)| G_r(x, dy) < \infty$, define*

$$F(x) = \int_{\mathbb{R}^d} f(y) G_r(x, dy).$$

Then, the Green kernel introduced in (5) satisfies

$$G_r(x, H) = \mathbf{E}_x \left(e^{-r\tau} G_r(X_\tau, H) \right) + \mathbf{E}_x \left(\int_0^\tau e^{-rs} \mathbf{1}_H(X_s) ds \right)$$

for all measurable $H \subseteq \mathbb{R}^d$ and

$$F(x) = \mathbf{E}_x \left(e^{-r\tau} F(X_\tau) \right) + \mathbf{E}_x \left(\int_0^\tau e^{-rs} f(X_s) ds \right).$$

Proof. The first claim immediately follows from the strong Markov property by conditioning on $\mathcal{F}_\tau$. The second claim follows from standard arguments for measurable functions. □

Lemma A.2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable w.r.t. $G_r(x, dy)$ for all $x \in \mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$ a Borel set. Define the functions $W, g : \mathbb{R}^d \to \mathbb{R}$ by*

$$W(x) := \int_S f(y) G_r(x, dy), \tag{29}$$

$$g(x) := \int_{\mathbb{R}^d} f(y) G_r(x, dy).$$

If $g(x) = W(x)$ for all $x \in \partial S$, then, $g(x) = W(x)$ for all $x \in S$. Furthermore, W satisfies

$$W(x) = \mathbf{E}_x \left(e^{-rh_S} W(X_{h_S}) \right) = \mathbf{E}_x \left(e^{-rh_S} g(X_{h_S}) \right) \text{ for all } x \in \mathbb{R}^d. \tag{30}$$

Proof. First observe that from the fact that $g(x) = W(x)$ for all $x \in \partial S$ it follows that

$$g(x) - W(x) = \int_{S^c} f(y) G_r(x, dy) = 0. \tag{31}$$

We can now move on to prove that $g(x) = W(x)$ for $x \in S$. To verify this, we notice that for all x

$$g(x) - W(x) = \int_{S^c} f(y) G_r(x, dy).$$

From Lemma A.1 applied to the function $1_{S^c} f$ we deduce that for $x \in S$

$$\begin{aligned}
&\int_{S^c} f(y) G_r(x, dy) \\
=&\mathbf{E}_x \left(e^{-r h_{S^c}} \int_{S^c} f(y) G_r(X_{h_{S^c}}, dy) \right) + \mathbf{E}_x \left(\int_0^{h_{S^c}} e^{-rs} 1_{S^c}(X_s) f(X_s) ds \right) \\
=&\mathbf{E}_x \left(e^{-r h_{S^c}} \int_{S^c} f(y) G_r(X_{h_{S^c}}, dy) \right).
\end{aligned}$$

Bearing (31) in mind, the integral inside the expected value above on the right-hand side vanishes, as $X_{h_{S^c}} \in \partial S$ by the continuity of the sample paths, thus completing the first part of the proof. The validity of (30) is a direct consequence of Lemma A.1. □

Proof of Theorem 2.1. Applying Lemma A.2 with $W := \widetilde{V}$, we get $\widetilde{V}(x) = g(x)$ for $x \in S$, which in addition to the hypothesis $\widetilde{V}(x) \geq g(x)$ for all $x \in S^c$, yields that $\widetilde{V}$ is a majorant of the reward. By the definition of $\widetilde{V}$, see (10), and taking into account that f is a non-negative function in S, we furthermore deduce evoking the Riesz representation that $\widetilde{V}$ is an r-excessive function. By the general theory we therefore have that $\widetilde{V} \geq V$. Furthermore by Lemma A.2, we have

$$\widetilde{V}(x) = \mathbf{E}_x \left(e^{-r h_S} g(X_{h_S}) \right) \leq \sup_\tau \mathbf{E}_x \left(e^{-r\tau} g(X_\tau) \right),$$

and conclude that the desired equality holds. □

Proof of Corollary 2.2. First observe that the function g satisfies the conditions for Dynkin's characterization (see 1.2.1 in [9]). Since g is in the domain of the extended infinitesimal generator associated with the killed process and the additional hypotheses we made about g, we know that

$$g(x) = \int_{\mathbb{R}^d} (r - \mathcal{L}) g(y) G_r(x, dy) \qquad (x \in \mathbb{R}^d).$$

Therefore, we may apply the previous theorem with $f(y) := (r - \mathcal{L}) g(y)$ to complete the proof. □

Proof of Lemma 4.1. (i) holds as the Wiener process is a strong Feller standard Markov process and therefore the r-excessive functions are lower semicontinuous, see Blumenthal and Getoor [4, p. 77].
(ii) holds as a consequence of the fact that restricted to the set $\mathcal{N}$ the process is a sub-martingale, and, hence, it is optimal not to stop.

For (iii) note that both the distribution of X and the reward function are symmetric with respect to the Cartesian axes. Hence, so is the value function and therefore also the optimal stopping set.

To prove (iv) and (v), first note that by (iii), it is enough to prove the claims for points in the first quadrant. We now use the explicit starting state-dependence of X. For all stopping times τ

and all (x, y) it holds that

$$\begin{aligned} q_\tau(x,y) &:= \mathbf{E}\left(e^{-r\tau} g(x + W_\tau^{(1)}, y + W_\tau^{(2)})\right) - g(x,y) \\ &= (x^2 + \alpha^2 y^2)\left(\mathbf{E}(e^{-r\tau}) - 1\right) + 2x\mathbf{E}\left(e^{-r\tau} W_\tau^{(1)}\right) \\ &\quad + 2\alpha^2 y \mathbf{E}(e^{-r\tau} W_\tau^{(2)}) + \mathbf{E}(e^{-r\tau}({W_\tau^{(1)}}^2 + \alpha^2 {W_\tau^{(2)}}^2)) \\ &= -a_\tau\left((x + d_{1,\tau})^2 + \alpha^2 (y + d_{2,\tau})^2 - c_\tau\right) \end{aligned}$$

for some constants a_τ, $c_\tau \geq 0$, $d_{1,\tau}$, $d_{2,\tau} \in \mathbb{R}$, which depend on τ. Moreover, note that $q_\tau(0,0) \geq 0$ for all τ. Using this notation, (x, y) is in the optimal stopping set iff $q_\tau(x, y) \leq 0$ for all τ. Taking complements, this means that $\mathcal{C}$ is a union of elliptic discs $q_\tau(x, y) > 0$ containing $(0, 0)$, whose axes are parallel to the Cartesian axes. Hence, a point (x, y), $x, y \geq 0$, is in the stopping set $\mathcal{S}$ if (x, y) is outside of all ellipses, i.e. $q_\tau(x, y) \leq 0$ for all τ. But then so are all points in a suitable cone with vertex (x, y), which proves (iv). If, on the other hand, $(x, y) \in \mathcal{C}$, then there exists τ such that (x, y) is inside the ellipse, i.e. $q_\tau(x, y) > 0$. Since, furthermore, the origin is also inside this ellipse, (v) clearly holds.

Finally, consider claim (vi) and assume without loss of generality that $\alpha \geq 1$. By the symmetry property (iii), it is enough to establish the boundedness of $\mathcal{C}$ in the first quadrant. First, recall that the continuation set for the symmetric problem $\alpha = 1$ is a circular disc, see Section 4.1 . Therefore, for $y > 0$ large enough, we know that for all stopping times τ

$$\mathbf{E}_{(0,y)}(e^{-r\tau} g(X_\tau)) \leq \alpha^2 \mathbf{E}_{(0,y)}(e^{-r\tau}({W_\tau^{(1)}}^2 + {W_\tau^{(2)}}^2)) \leq \alpha^2(0^2 + y^2) = g(0, y),$$

so that $(0, y) \in \mathcal{S}$. Next, we establish that also $(\alpha^2 y, 0) \in \mathcal{S}$: For all stopping times τ, it holds

$$\begin{aligned} q_\tau(0, y) &= \alpha^2(\mathbf{E}e^{-r\tau} - 1)y^2 + 2\alpha^2 \mathbf{E}(e^{-r\tau} W_\tau^{(1)})y + d_\tau \\ q_\tau(\alpha^2 y, 0) &= \alpha^4(\mathbf{E}e^{-r\tau} - 1)y^2 + 2\alpha^2 \mathbf{E}(e^{-r\tau} W_\tau^{(2)})y + d_\tau. \end{aligned}$$

Now, fix any stopping time τ and denote by $\tilde{\tau}$ a corresponding stopping time such that $(\tau, W_\tau^{(1)}, W_\tau^{(2)})$ has the same law as $(\tilde{\tau}, W_{\tilde{\tau}}^{(2)}, W_{\tilde{\tau}}^{(1)})$. As $(0, y) \in \mathcal{S}$, it holds that $q_{\tilde{\tau}}(0, y) \leq 0$ and as $\alpha \geq 1$ this also implies that

$$\begin{aligned} q_\tau(\alpha^2 y, 0) &\leq \alpha^2(\mathbf{E}e^{-r\tau} - 1)y^2 + 2\alpha^2 \mathbf{E}(e^{-r\tau} W_\tau^{(2)})y + d_\tau \\ &= \alpha^2(\mathbf{E}e^{-r\tilde{\tau}} - 1)y^2 + 2\alpha^2 \mathbf{E}(e^{-r\tilde{\tau}} W_{\tilde{\tau}}^{(1)})y + d_{\tilde{\tau}} \\ &= q_{\tilde{\tau}}(0, y) \leq 0, \end{aligned}$$

i.e. $(\alpha^2 y, 0) \in \mathcal{S}$ for y large enough. Next, we consider an arbitrary (x, y). Then

$$\begin{aligned} q_\tau(x, y) &= q_\tau(x, 0) + q_\tau(0, y) - \mathbf{E}(e^{-r\tau}({W_\tau^{(1)}}^2 + \alpha^2 {W_\tau^{(2)}}^2)) \\ &\leq q_\tau(x, 0) + q_\tau(0, y), \end{aligned}$$

so for x, y large enough, it holds that $q_\tau(x, y) \leq 0$ for all τ, i.e. $(x, y) \in S$, implying that $\mathcal{C}$ is bounded. □

References

[1] M. Abramowitz, I. Stegun, Mathematical Functions, 9th Printing, Dover publications, Inc., New York, 1970.

[2] T.T. Banh, Support theorems for unbounded convex sets, Complex Variables Theory Appl. 49 (2) (2004) 87–93.

[3] R.F. Bass, Probabilistic Techniques in Analysis, Springer Science Business Media, 1994.

[4] R. Blumenthal, R. Getoor, Markov Processes and Potential Theory, Academic Press, New York, London, 1968.

[5] A. Borodin, P. Salminen, Handbook of Brownian Motion - Facts and Formulae, second ed., Birkhäuser, Basel, Boston, Berlin, 2015, corrected printing.

[6] S. Christensen, A. Irle, A harmonic function technique for the optimal stopping of diffusions, Stochastics 83 (4–6) (2011) 347–363.
[7] S. Christensen, A. Irle, An elementary method for the explicit solution of multidimensional optimal stopping problems, ArXiv e-prints, 1705.01763, 2017.
[8] S. Christensen, P. Salminen, Multidimensional investment problem, Math. Financ. Econ. 12 (1) (2018) 75–95.
[9] F. Crocce, Optimal stopping for strong Markov processes: Explicit solutions and verification theorems for diffusions, multidimensional diffusions, and jump-processes (Ph.D. thesis). Universidad de la República (ArXiv :1405.7539), 2014.
[10] F. Crocce, E. Mordecki, Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions, Stochastics 86 (3) (2014) 491–509.
[11] T. De Angelis, S. Federico, G. Ferrari, Optimal boundary surface for irreversible investment with stochastic costs, Math. Oper. Res. 42 (4) (2017) 897–1312.
[12] L.E. Dubins, L.A. Shepp, A.N. Shiryaev, Optimal stopping rules and maximal inequalities for Bessel processes, Theory Probab. Appl. 38 (2) (1994) 226–261.
[13] J. du Toit, G. Peskir, Selling a stock at the ultimate maximum, Ann. Appl. Probab. 19 (3) (2009) 983–1014.
[14] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, Tables of Integral Transforms, McGraw-Hill, New York, 1954.
[15] S.N. Ethier, T.G. Kurtz, Markov Processes; Characterization and Convergence, in: Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986, p. x+534.
[16] J. Fajardo, E. Mordecki, Pricing derivatives on two-dimensional Lévy processes, Int. J. Theor. Appl. Finance 9 (02) (2006) 185–197.
[17] A. Friedman, Stochastic Differential Equations and Applications. Vol. 2, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976, Probability and Mathematical Statistics, Vol. 28.
[18] H.U. Gerber, E.S. Shiu, Martingale approach to pricing perpetual American options on two stocks, Math. Finance 6 (3) (1996) 303–322.
[19] S. Helgason, The Radon Transform, Birkhauser, Boston, Massachusetts, 1980.
[20] Y. Hu, B. Øksendal, Optimal time to invest when the price processes are geometric Brownian motions, Finance Stoch. 2 (3) (1998) 295–310.
[21] A. Irle, A forward algorithm for solving optimal stopping problems, J. Appl. Probab. 43 (1) (2006) 102–113.
[22] K. Itô, H. McKean, Diffusion Processes and Their Sample Paths, Springer Verlag, Berlin, Heidelberg, 1974.
[23] P. Johnson, G. Peskir, Quickest detection problems for Bessel processes, Ann. Appl. Probab. 27 (2) (2017) 1003–1056.
[24] P. Johnson, G. Peskir, Sequential testing problems for Bessel processes, Trans. Amer. Math. Soc. 370 (3) (2018) 2085–2113.
[25] D.O. Kramkov, E. Mordecki, Integral option, Theory Probab. Appl. 39 (1) (1995) 162–172.
[26] N.N. Lebedev, Special Functions and Their Applications, Dover publications, New York, 1972.
[27] W. Margrabe, The value of an option to exchange one asset for another, J. Finance 33 (1) (1978) 177–186.
[28] E. Mordecki, P. Salminen, Optimal stopping of Hunt and Lévy processes, Stochastics 79 (3–4) (2007) 233–251.
[29] K. Nishide, L.C.G. Rogers, Optimal time to exchange two baskets, J. Appl. Probab. 48 (1) (2011) 21–30.
[30] G. Peskir, On the American option problem, Math. Finance 15 (1) (2005) 169–181.
[31] G. Peskir, A.N. Shiryaev, Optimal Stopping and Free-boundary Problems, in: Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2006.
[32] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, third ed., Springer Verlag, Berlin, Heidelberg, 2001.
[33] P. Salminen, Optimal stopping of one-dimensional diffusions, Math. Nachr. 124 (1985) 85–101.
[34] L. Shepp, A.N. Shiryaev, The Russian option: reduced regret, Ann. Appl. Probab. (1993) 631–640.
[35] L.A. Shepp, A.N. Shiryaev, A new look at pricing of the Russian Option, Theory Probab. Appl. 39 (1) (1995) 103–119.
[36] A.N. Shiryaev, Optimal Stopping Rules, in: Stochastic Modelling and Applied Probability, vol. 8, Springer-Verlag, Berlin, 2008, p. xii+217.
[37] S. Villeneuve, Exercise regions of American options on several assets, Finance Stoch. 3 (3) (1999) 295–322.