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Stochastic Processes and their Applications 129 (2019) 1349–1370

stochastic processes and their applications

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Asymptotic normality of high level-large time crossings of a Gaussian process

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Received 15 June 2017; received in revised form 2 May 2018; accepted 3 May 2018
Available online 16 May 2018

Abstract

We prove the asymptotic normality of the standardized number of crossings of a centered stationary mixing Gaussian process when both the level and the time horizon go to infinity in such a way that the expected number of crossings also goes to infinity.
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MSC: 60F05; 60G15

Keywords: High-level crossings; Rice formula; Mixing process; Dependent CLT

1. Introduction

The number of crossings of a stochastic process through a level over a time interval gives important information about the geometry of the trajectories and, for large values of the level, about the behavior of the tail of the maximum of the process. Since the exact distribution of this

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https://doi.org/10.1016/j.spa.2018.05.003
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functional is not known but in very particular cases, the study of its asymptotic behavior, under different asymptotic schemes, has become a classical subject of research.

In the case when the level u and the time horizon T both go to infinity in such a way that the expectation of the number of crossings through u on $[0, T]$ remains fixed, the classical results by Volkonskiĭ and Rozanov state that the conveniently normalized number of up-crossings asymptotically behaves like a standardized Poisson process [16,17] where the intensity is the constant expectation of the number of up-crossings. Since the standardized Poisson distribution approximates the normal distribution when this intensity tends to infinity, it is natural to ask for the asymptotic normality when its intensity tends to infinity.

A classical way to prove asymptotic normality is based on the computation of different moments of the underlying random variables. In the case of the number of crossings of a smooth stochastic process, this task can be carried out with the help of Rice formulas [13]. However, the explicit computation and the analysis of the asymptotic behavior of moments of higher order than the second is in general a difficult task, in particular taking into account that the level is not fixed. A general picture of the field can be found in the books by Cramér and Leadbetter [4] and Azaïs and Wschebor [1]. An alternative approach to prove asymptotic normality consists in the study of the chaotic expansion of the crossings in the Wiener space (see for instance [14]), using the corresponding limit theorem results as the one exposed in the books by Peccati and Taqqu [12] or Nourdin and Peccati [11]. Wiener chaos techniques have the advantage of avoiding higher moments than the second and, sometimes, of giving rates of convergence. The first results in this direction were obtained by Malevič [9], Cuzick [5], Slud [15] and Kratz and León [8] where – within other results – the normal asymptotic behavior of the standardized number of crossing of a smooth stationary Gaussian process is obtained for a fixed level u as the time horizon goes to infinity. In order to obtain this result, the chaotic expansion of the number of crossings and the approximation of the process by m-dependent processes are used. In his recent PhD thesis [10], Mourareau analyzes the chaos expansion of the crossings in the case where the level, the time horizon and the mean number of crossings go to infinity, obtaining the asymptotic normality of the normalized number of crossings for an m-dependent Gaussian process. However, the usual scheme of translating this approximation to more general Gaussian processes is not carried out. Mourareau's work points out the sophisticated nature of this situation. While most of the asymptotic distributions found via the Wiener chaos techniques rely on the fact that some – possibly every – component of the chaotic expansion of the functional have variance of the same order than the global variance, in the present case the variance of each chaotic component of the number of crossings tends to zero.

In the present paper, we study the normal asymptotic behavior of the normalized number of crossings of a class of stationary Gaussian processes when both the level and the observation time go to infinity, in such a way that the expectation of the number of crossings also goes to infinity (see Theorem 2). The basic idea to obtain our result is to use the Bernstein block method [3] for dependent random variables in the Central Limit Theorem, with the formulation presented in [2]. In particular, this approach requires the analysis of the asymptotic behavior of the second and third moments of the number of crossings over an increasing time interval, when the level of the crossings also goes to infinity in a regulated way. This task is accomplished with the help of the corresponding Rice formulas. The computations of the third moment for zero level (i.e. roots) in the context of stationary random polynomials can be found in [7], see also section 5.2 in [1]. To our knowledge, the use of the Rice formula for the third factorial moment at an arbitrary level has not been used previously. In the way of our proof, we obtain the equivalence of the asymptotic behavior of the expectation and the variance of the crossings under a very general asymptotic scheme (see Theorem 1).

The rest of the paper is organized as follows: Section 2 presents the problem, the main results and some motivating partial results. Section 3 introduces some preliminary results. Section 4 contains the proof of Theorem 1, Section 5 the proof of Theorem 2 and Section 6 the proof of Theorem 3.

Notation: We use the usual notations $f(t) \sim g(t)$ to indicate that $\lim \frac{f(t)}{g(t)} = 1$; $f(t) = o(g(t))$ for $\lim \frac{f(t)}{g(t)} = 0$.

2. Main results

Assume that $\mathcal{X} = \{X(t) : t \in \mathbb{R}\}$ is a mean zero variance one stationary Gaussian process with smooth paths. Denote the covariance function of $\mathcal{X}$ by

$$r(\tau) = \mathbf{E}(X(t)X(t+\tau)).$$

Without loss of generality we assume that $r(0) = 1$, and that $\mathbf{var}\dot{X}(0) = -\ddot{r}(0) = 1$. Define the number of crossings through level u by the process $\mathcal{X}$ over the time interval $I \subset \mathbb{R}$ by

$$N(I, u) = \#\{t \in I : X(t) = u\},$$

and denote $N(T, u) = N([0, T], u)$. For simplicity of notation we always assume $u > 0$. Set

$$C(u) = \mathbf{E}\, N(1, u) = \frac{1}{\pi} \exp(-u^2/2),$$
$$v(T, u) = \mathbf{E}\, N(T, u) = TC(u) = \frac{T}{\pi} \exp(-u^2/2).$$

We present now our main results. The first theorem states that when the level u tends to infinity, uniformly in T bounded away from zero, the mean and the variance of $N(T, u)$ are of the same order. Wiener chaos techniques are used in Proposition 7.4.2 of [10] to prove a similar result under a more restrictive time-level asymptotic scheme. Here we use an approach based on Rice formula.

Theorem 1. *Assume that $\mathcal{X}$ satisfies Geman's condition:*

(G) *If $r(\tau) = 1 - \tau^2/2 + \theta(\tau)$, then, for some $\delta > 0$, the integral*

$$\int_0^\delta \frac{\theta'(\tau)}{\tau^2} d\tau < \infty.$$

Furthermore, assume that $r(\tau) \to 0$, $\dot{r}(\tau) \to 0$ as $\tau \to \infty$, and that the integral

$$\int_0^\infty \big(|r(\tau)| + |\dot{r}(\tau)| + |\ddot{r}(\tau)|\big) d\tau < \infty. \tag{1}$$

Hence, for any fixed $t_0 > 0$, as u tends to infinity, we have

$$\frac{\mathbf{var}N(T, u)}{\mathbf{E}\, N(T, u)} \to 2, \tag{2}$$

uniformly in $T \in [t_0, \infty)$.

The second result states the asymptotic normality of the standardized number of crossings $N(T, u)$. Extra conditions on the process $\mathcal{X}$ are imposed and now u depends on T, we write u_T to emphasize this dependence. We need the following definition.

Definition 1. For $t > 0$ and $-\infty \le a \le b \le \infty$, let $\mathcal{F}_a^b$ be the σ-algebra generated by the random variables $\{X_s : a \le s \le b\}$. The α-mixing coefficient is defined as

$$\alpha(t) = \sup\{|\mathbf{P}(U)\mathbf{P}(V) - \mathbf{P}(U \cap V)| : U \in \mathcal{F}_{-\infty}^0,\ V \in \mathcal{F}_t^\infty\},$$

and the ρ-mixing coefficient is

$$\rho(t) = \sup\{|\mathbf{cov}(X, Y)| : X \in \mathbb{L}^2(\Omega, \mathcal{F}_{-\infty}^0, \mathbf{P}),\ Y \in \mathbb{L}^2(\Omega, \mathcal{F}_t^\infty, \mathbf{P})\}.$$

We say that the process is α-mixing (resp. ρ-mixing) if $\alpha(t) \to_{t\to\infty} 0$ (resp. $\rho(t) \to_{t\to\infty} 0$).

Theorem 2. *Assume the following conditions on the process $\mathcal{X}$: For any $t_1 < t_2 < t_3$, the distribution of the vector $(X(t_1), X(t_2), X(t_3))$ is non-degenerated. The process $\mathcal{X}$ is α-mixing with a polynomial rate $\mu > 4$, i.e.*

$$\alpha(T) \sim T^{-\mu}, \quad (T \to \infty). \tag{3}$$

The covariance function verifies:

$$r(\tau) = 1 - \frac{1}{2}\tau^2 + d\tau^4 + e\tau^6 + o(\tau^6), \quad (\tau \to 0). \tag{4}$$

Besides, assume that there exists $0 < \gamma < 1$ such that $\mu\gamma > 4$ and

$$\nu(T, u_T) \sim T^\gamma, \quad (T \to \infty).$$

Then, the standardized number of crossings converges in distribution towards the standard normal distribution as $T \to \infty$, that is:

$$\frac{N(T, u_T) - \nu(T, u_T)}{\sqrt{2\nu(T, u_T)}} \Rightarrow \mathcal{N}(0, 1), \quad (T \to \infty). \tag{5}$$

Remark 1. A sufficient condition for the non-degeneracy of the finite-distributions of the mean zero stationary Gaussian process $\mathcal{X}$ is that the support of the spectral measure μ of r has some accumulation point. Indeed, if Σ denotes the variance matrix of $(X(t_1), \ldots, X(t_k))$; it suffices to see that the associated quadratic form $F(z) = z^t \Sigma z$, $z \in \mathbb{R}^k$ is positive definite. But $F(z)$ can be expressed in terms of the spectral measure μ by

$$F(z) = \int_{\mathbb{R}} \left| \sum_{i=1}^k e^{ixz_i} z_i \right|^2 d\mu(x),$$

using that $z \mapsto \sum_{i=1}^k e^{ixz_i} z_i$ is an analytic function and that the support of μ has some accumulation point it follows that $F(z) > 0$ for $z \neq 0$, see page 82 of [1].

Let us finish this section discussing a direct approach based on the celebrated Volkonskiĭ – Rozanov Theorem. This approach yields some partial results and the motivation for Theorem 2, see [16,17] or Th. 10.1 in [1]. Roughly speaking, Volkonskiĭ – Rozanov Theorem states that when the number of up-crossings

$$U(T, u) = \#\{t \in [0, T] : X(t) = u,\ X'(t) > 0\},$$

satisfy $\mathbf{E}\, U(T, u) \to \ell$, once normalized, they converge towards a Poisson distribution with parameter ℓ. As said above, the motivation for Theorem 2 is provided by the fact that the normalized Poisson distribution with parameter ℓ converges to the normal distribution as $\ell \to \infty$. Theorem 3, using this approach, states the asymptotic normality of $N(T, u)$ but only for some

sequences (T_n, u_n) in a non-constructive way. Theorem 2 gives a more satisfactory result under more restrictive conditions on the process $\mathcal{X}$. In order to formulate Theorem 3, whose hypothesis are the same as those of Volkonskiĭ – Rozanov Theorem (Th. 10.1 in [1]), we need to introduce the following condition:

(B) Berman's condition: $r(\tau)\log(\tau) \to 0$ as $\tau \to \infty$.

Theorem 3 (*Existence of a Good Sequence*)**.** *Assume that $\mathcal{X}$ satisfies conditions $-\ddot{r}(0) < \infty$, (B) and (G). Then, there exists a sequence $(T_n, u_n) \to (\infty, \infty)$ such that $\nu_n := \nu(T_n, u_n) \to \infty$ and*

$$\frac{N(T_n, u_n) - \nu_n}{\sqrt{2\nu_n}} \Rightarrow \mathcal{N}(0, 1), \text{ as } n \to \infty.$$

Remark 2. As the process has continuous trajectories, we have the relation $|N(T, u) - 2U(T, u)| \leq 1$, and all asymptotic results for large number of crossings can be expressed in terms of crossings or up-crossings. Observe that this is not the case in Volkonskiĭ–Rozanov scheme.

3. Preliminaries

3.1. The covariance function and its derivatives

We present some basic results that are used throughout the paper.

Lemma 1. *Assume that the process $\mathcal{X}$ has a twice differentiable covariance.*
(a) Under the conditions $r(0) = -\ddot{r}(0) = 1$, we have $|r(t)| \leq 1$, $|\dot{r}(t)| \leq 1$, $|\ddot{r}(t)| \leq 1$.
(b) Under (1) the functions r, $\dot{r}$ and $\ddot{r}$ are in $L^p(\mathbb{R})$ for all $p \geq 1$.
(c) The α-mixing condition (3) implies that the functions r, $\dot{r}$ and $\ddot{r}$ are in $L^p(\mathbb{R})$ for all $p \geq 1$. In particular, condition (1) holds.

Proof. *Proof of (a).* First observe that, according to Proposition 1.13 in [1], the process $\{\dot{X}(t)\}$ exists, as the derivative in quadratic mean of $\mathcal{X}$. We have

$$r(t) = \mathbf{E}(X(0)X(t)), \quad \dot{r}(t) = \mathbf{E}(X(0)\dot{X}(t)), \quad \ddot{r}(t) = -\mathbf{E}(\dot{X}(0)\dot{X}(t)).$$

As $\mathbf{E}(X(t)^2) = \mathbf{E}(\dot{X}(t)^2) = 1$, the statements of (a) follow from the application of the Cauchy–Schwarz inequality.
Proof of (b). In view of (a) it is direct.
Proof of (c). By using that $X(0)$ and $\dot{X}(0)$ belong to $\mathcal{F}^0_{-\infty}$, and that $X(t)$ and $\dot{X}(t)$ belong to $\mathcal{F}^\infty_t$, we get

$$|r(t)| \leq \rho(t), \quad |\dot{r}(t)| \leq \rho(t), \quad |\ddot{r}(t)| \leq \rho(t).$$

As by Kolmogorov–Rozanov inequality (see pag. 57 in [6]), $\rho(t) \leq \mathbf{C}\alpha(t)$ and that by our hypothesis $\int_0^\infty \alpha(t)dt < \infty$, we get that the three functions $r, \dot{r}, \ddot{r}$ are integrable. As by (a) they are bounded, the statement of (c) follows. □

Remark 3. The parameters in (4) are not completely free, in particular

$$r^{(iv)}(0) - \ddot{r}(0)^2 = 24d - 1 \geq 0.$$

In fact, $\ddot{r}(t) = \mathbf{E}(X(0)\ddot{X}(t))$, so taking $t = 0$, and applying Cauchy–Schwarz inequality, we obtain $|\ddot{r}(0)| = |\mathbf{E}(X(0)\ddot{X}(0))| \leq \sqrt{r(0)r^{(iv)}(0)}$. Since $r(0) = \ddot{r}(0) = 1$ we get the desired inequality.

3.2. Rice formula

In the sequel we need to deal with the second and the third moments of the number of crossings $N = N(T, u)$. The main tool is the celebrated Rice formula which we now present in its general form, see [1] for the details.

Let $n \geq 1$, $N^{[n]}(T, u) = \prod_{k=0}^{n-1}(N(T, u) - k)$, then we have

$$\mathbf{E}\big(N^{[n]}(T, u)\big) = \int_{[0,T]^n} \mathbf{E}\left[\left|\prod_{k=1}^{n} \dot{X}(s_k)\right| \mid X = \mathbf{u}\right] p_X(\mathbf{u})ds, \tag{6}$$

with $s = [s_1, \ldots, s_n]$, $X = [X(s_1), \ldots, X(s_n)]^t$ and $\mathbf{u} = [u, \ldots, u]^t$. To deal with the conditional expectation in this formula we use the following result, that has a direct proof.

Lemma 2 (*Gaussian Regression*). *Let $R > 0$. Consider the times $s_1 \leq \cdots \leq s_n \leq R$. Denote $\dot{X} = [\dot{X}(s_1), \ldots, \dot{X}(s_n)]^t$,*

$$\Sigma = \mathbf{var}X = \mathbf{E}(XX^t), \quad \Sigma_{10} = \mathbf{cov}(\dot{X}, X) = \mathbf{E}(\dot{X}X^t), \quad \Sigma_{11} = \mathbf{E}(\dot{X}\dot{X}^t) = \mathbf{var}\dot{X},$$

and

$$\mathbf{\Psi} = \begin{bmatrix} \Psi_1 & \ldots & \Psi_n \end{bmatrix}^t := \Sigma_{10}\Sigma^{-1}.$$

Then: (a) The Gaussian vector $Y = [Y_1, \ldots, Y_n]^t$ defined by

$$Y = \dot{X} - \mathbf{\Psi}X \tag{7}$$

is independent from X, centered, and has a variance matrix given by

$$\Sigma_Y = \mathbf{var}Y = \Sigma_{11} + \mathbf{\Psi}\Sigma_{10} = \Sigma_{11} - \mathbf{\Psi}\Sigma_{10}^t,$$

since $\Sigma_{10}^t = -\Sigma_{10}$. Since $\Psi\Sigma_{10}^t = \Sigma_{10}\Sigma^{-1}\Sigma_{10}^t$ and Σ^{-1} is positive definite, it follows from (7) that $\mathbf{var}Y_i \leq \mathbf{var}\dot{X}(s_i) = 1$ for $i = 1, 2, \ldots, n$.
(b) The random vector

$$(Y_1 + \Psi_1\mathbf{u}, \ldots, Y_n + \Psi_n\mathbf{u}),$$

has the same distribution as $\dot{X}$ conditional on the set $\{X = \mathbf{u}\}$.

4. Proof of Theorem 1

In this section we prove that the asymptotic expectation and variance of $N(T, u)$ are of the same order. We begin by specializing Lemma 2 to the case $n = 2$, see pg. 76 in [1]. For $\tau = s_2 - s_1$ we have

$$\Sigma = \begin{bmatrix} 1 & r(\tau) \\ r(\tau) & 1 \end{bmatrix}.$$

Thus –omiting the τ in the notation –, we have

$$\Sigma_{10} = \dot{r}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \Sigma_{11} = \begin{bmatrix} 1 & -\ddot{r} \\ -\ddot{r} & 1 \end{bmatrix}.$$

Hence,

$$\mathbf{\Psi} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \frac{\dot{r}}{1 - r^2}\begin{bmatrix} r & -1 \\ 1 & -r \end{bmatrix}, \quad \mathbf{var}\mathbf{\Psi}X = \frac{\dot{r}^2}{1 - r^2}\Sigma.$$

$$\mathbf{var}Y = \begin{bmatrix} 1 & -\ddot{r} \\ -\ddot{r} & 1 \end{bmatrix} - \frac{\dot{r}^2}{1-r^2}\begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}.$$

Therefore, we find the following expressions

$$\mathbf{var}Y_1 = \mathbf{var}Y_2 = 1 - \frac{\dot{r}^2}{1-r^2}, \quad \mathbf{cov}(Y_1, Y_2) = -\ddot{r} - \frac{r\dot{r}^2}{1-r^2},$$
$$\rho := \frac{\mathbf{cov}(Y_1, Y_2)}{\mathbf{var}Y_1} = -\frac{(1-r^2)\ddot{r} + r\dot{r}^2}{1-r^2-\dot{r}^2}. \tag{8}$$

We also have

$$\Psi\mathbf{u} = \frac{\dot{r}}{1+r}\begin{bmatrix} -u \\ u \end{bmatrix} \sim \frac{-\tau}{2}\begin{bmatrix} -u \\ u \end{bmatrix}, \ (\tau \to 0).$$

Then we denote

$$\psi u = \Psi_1\mathbf{u} = -\Psi_2\mathbf{u} = -\frac{\dot{r}}{1+r}u. \tag{9}$$

We now turn to the proof of Theorem 1.

Proof of Theorem 1. With $N = N(T, u)$, we have

$$\frac{\mathbf{var}N}{\mathbf{E}N} = \frac{\mathbf{E}(N(N-1)) - \mathbf{E}^2N}{\mathbf{E}N} + 1. \tag{10}$$

The condition $r(\tau) \to 0$ $(\tau \to \infty)$ implies that $|r(\tau)| \neq 1$ for $\tau \neq 0$. Hence, we can apply Rice formula. By Rice formula and the stationarity of $\mathcal{X}$ we have

$$\mathbf{E}(N(N-1)) = 2\int_0^T (T-\tau)\mathbf{E}(|\dot{X}(0)\dot{X}(\tau)| \mid X(0) = X(\tau) = u)p_{X(0),X(\tau)}(u,u)d\tau.$$

Here,

$$p_{X(0),X(\tau)}(u,u) = \frac{1}{2\pi}\frac{1}{\sqrt{1-r^2(\tau)}}e^{-\frac{u^2}{1+r(\tau)}}.$$

Also by Rice formula $\mathbf{E}N = TC(u) = T\mathbf{E}(|\dot{X}(0)|)p_{X(0)}(u)$, since $T^2 = 2\int_0^T (T-\tau)d\tau$, we have

$$\mathbf{E}^2N = 2\int_0^T (T-\tau)\mathbf{E}^2(|\dot{X}(0)|)p^2_{X(0)}(u)d\tau.$$

Hence, using the Gaussian regression in Lemma 2, the first term on the r.h.s. of (10) can be written as $\int_0^T \mathcal{J}(T,\tau,u)d\tau$, being

$$\mathcal{J}(T,\tau,u) = \frac{T-\tau}{T}\left(\mathbf{E}(|Y_1 + \psi u|\,|Y_2 - \psi u|)\frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} - \mathbf{E}^2(|\dot{X}(0)|)e^{-\frac{u^2}{2}}\right).$$

We have to prove that

$$\lim_{u\to\infty}\int_0^T \mathcal{J}(T,\tau,u)d\tau = 1,$$

uniformly on $T \geq t_0$ for any $t_0 > 0$. We divide the proof in three steps, corresponding to small, medium and large values of τ in the integral.

First step. For an arbitrary small $\delta > 0$ to be chosen, smaller than t_0, uniformly in T, we have

$$\lim_{u\to\infty} \int_0^\delta \mathcal{J}(T,\tau,u)d\tau = 1.$$

It is no hard to verify the following limits for any $\delta > 0$:

$$\lim_{u\to\infty} \int_0^\delta \frac{T-\tau}{T} \mathbf{E}^2(|\dot{X}(0)|) e^{-\frac{u^2}{2}} d\tau = 0,$$

$$\lim_{u\to\infty} \int_0^\delta \frac{T-\tau}{T} \mathbf{E}|Y_1 Y_2| \frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} d\tau = 0,$$

$$\lim_{u\to\infty} \int_0^\delta \frac{T-\tau}{T} \mathbf{E}|(Y_1 - Y_2)\psi u| \frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} d\tau = 0.$$

Note that Geman's condition guarantees the finiteness of the second integral since $\frac{\theta'(\tau)}{\tau^2} \sim \frac{\mathbf{var} Y_i}{\sqrt{1-r^2(\tau)}}$ as $\tau \to 0$, see page 99 of [1]. From these results, it follows that

$$\lim_{u\to\infty} \int_0^\delta \mathcal{J}(T,\tau,u)d\tau = \lim_{u\to\infty} \int_0^\delta \frac{T-\tau}{T} (\psi u)^2 \frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} d\tau.$$

Now, using (9), we have to prove that

$$\lim_{u\to\infty} \int_0^\delta \frac{T-\tau}{T} \left(\frac{\dot{r}(\tau)u}{1+r(\tau)}\right)^2 \frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} d\tau = 1.$$

We now observe that, as

$$\lim_{\tau\to 0} \frac{T-\tau}{T} \frac{-\dot{r}(\tau)}{\sqrt{1-r^2(\tau)}} = 1,$$

and δ is arbitrarily small, we have to prove that

$$\lim_{u\to\infty} \int_0^\delta \frac{-\dot{r}(\tau)u^2}{(1+r(\tau))^2} e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}} d\tau = 1.$$

We change variables according to

$$v = \frac{u^2}{2}\left(\frac{1-r(\tau)}{1+r(\tau)}\right),$$

that is monotonous for δ conveniently small. So

$$\begin{aligned}\int_0^\delta \frac{-\dot{r}(\tau)u^2}{(1+r(\tau))^2} e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}} d\tau &= \int_0^{\frac{u^2}{2}\left(\frac{1-r(\delta)}{1+r(\delta)}\right)} e^{-v} dv \\ &= 1 - \exp\left(-\frac{u^2}{2}\left(\frac{1-r(\delta)}{1+r(\delta)}\right)\right) \to 1 \ (u\to\infty),\end{aligned}$$

concluding the first step.

Second step. For all $T_0 > 0$

$$\lim_{u\to\infty} \int_{\delta}^{T_0} \mathcal{J}(T,\tau,u)d\tau = 0. \tag{11}$$

We now work for $\tau \geq \delta > 0$. As both $r(\tau)$ and $\dot r(\tau)$ converge to zero as $\tau \to \infty$, are continuous functions, and $|r(\tau)| \neq 1$ for $\tau > 0$, we obtain the existence a constant, say $r_0 < 1$, $|r(\tau)| \leq r_0$. As $|\dot r(\tau)| \leq 1$, we have

$$\begin{aligned}\mathbf{E}(|\dot X(0)\dot X(\tau)| \mid X(0) = X(\tau) = u) &= \mathbf{E}(|Y_1 + \psi u|\,|Y_2 - \psi u|)\\ &\leq 1 + \frac{2}{1-r_0}u + \left(\frac{1}{1-r_0}\right)^2 u^2 \leq A + Bu^2.\end{aligned}$$

It follows that, for all fixed $\tau > \delta$ and uniformly in T, we have

$$\lim_{u\to\infty} \mathcal{J}(T,\tau,u) = 0.$$

Furthermore,

$$\frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} \leq \frac{e^{-\frac{u^2}{2}\frac{1-r_0}{1+r_0}}}{\sqrt{1-r_0^2}},$$

so the integrand in (11) is uniformly bounded in u, concluding the second step by dominated convergence.

Third step. There exists $T_0 > 0$ such that

$$\lim_{u\to\infty} \int_{T_0}^{\infty} \mathcal{J}(T,\tau,u)d\tau = 0.$$

Coming back to the difference,

$$\begin{aligned}\int_{T_0}^{\infty} \mathcal{J}(T,\tau,u)d\tau &= \int_{T_0}^{\infty} \frac{T-\tau}{T} \frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} \Big(\mathbf{E}(|Y_1 + \psi u|\,|Y_2 - \psi u|) - \mathbf{E}^2(|\dot X(0)|)\Big)d\tau\\ &\quad + \frac{2}{\pi}\int_{T_0}^{\infty} \frac{T-\tau}{T}\Big(\frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} - e^{-\frac{u^2}{2}}\Big)d\tau =: I_1 + I_2,\end{aligned}$$

where I_1, I_2 denote respectively the first and second addends. Now, we apply the triangle inequality. For the second integral we get

$$I_2 \leq \frac{2}{\pi}\int_{T_0}^{\infty} \left|\frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} - e^{-\frac{u^2}{2}}\right| d\tau.$$

We have

$$\frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}}}{\sqrt{1-r^2(\tau)}} - e^{-\frac{u^2}{2}} = \frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}} - e^{-\frac{u^2}{2}}}{\sqrt{1-r^2(\tau)}} + \left[\frac{1}{\sqrt{1-r^2(\tau)}} - 1\right] e^{-\frac{u^2}{2}}. \tag{12}$$

The second term is equivalent, as $\tau \to \infty$, to $\frac{1}{2}r^2(\tau)e^{-u^2/2}$, because $r(\tau) \to 0$ as $\tau \to \infty$. In conclusion its integral vanishes because $\int_0^{\infty} r^2(\tau)d\tau < \infty$ (see (b) in Lemma 1). For the first term on the r.h.s in (12), we denote $\beta = \frac{1-r(\tau)}{1+r(\tau)}$. As $\beta \to 1$, we take T_0 large enough such that $|\beta - 1| \leq 1/2$, and, simultaneously, as $r(\tau) \to 0$, such that $\sqrt{1-r^2(\tau)} \geq 1/2$, and

also $|r(\tau)| \le 1/2$, these three inequalities for all $\tau \ge T_0$. We apply Lagrange's Formula to the difference inside the integrand,

$$e^{-\frac{u^2}{2}\beta} - e^{-\frac{u^2}{2}} = e^{-\frac{u^2}{2}\theta}\left(-\frac{u^2}{2}\right)\left(\frac{1-r(\tau)}{1+r(\tau)} - 1\right) = e^{-\frac{u^2}{2}\theta}\frac{u^2 r(\tau)}{1+r(\tau)}.$$

The value θ is such that $|\theta - 1| \le |\beta - 1| \le 1/2$, so $\theta \ge 1/2$. Then

$$e^{-\frac{u^2}{2}\theta} \le e^{-\frac{u^2}{4}}.$$

In conclusion, we have the following bound for the integrand:

$$\frac{1}{\sqrt{1-r^2(\tau)}}\left|e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}} - e^{-\frac{u^2}{2}}\right| \le 2e^{-\frac{u^2}{4}}u^2|r(\tau)|,$$

and, as $r(\tau)$ is integrable, we conclude that

$$\int_{T_0}^{\infty}\left|\frac{e^{-\frac{u^2}{2}\frac{1-r(\tau)}{1+r(\tau)}} - e^{-\frac{u^2}{2}}}{\sqrt{1-r^2(\tau)}}\right|d\tau \le 2e^{-\frac{u^2}{4}}u^2\int_{T_0}^{\infty}|r(\tau)|d\tau \to 0 \quad (u \to \infty).$$

Let us look at the first integral I_1. With similar arguments we have the following bound:

$$I_1 \le 2e^{-\frac{u^2}{4}}\int_{T_0}^{\infty}\left|\mathbf{E}(|Y_1 + \psi u|\,|Y_2 - \psi u|) - \mathbf{E}^2(|\dot{X}(0)|)\right|d\tau.$$

Denote $\sigma^2 = \mathbf{var}Y_1 = \mathbf{var}Y_2$, and $Z_i = Y_i/\sigma$ $i = 1, 2$. We have

$$\begin{aligned}
\left|\mathbf{E}(|Y_1 + \psi u||Y_2 - \psi u|) - \mathbf{E}^2(|\dot{X}(0)|)\right| &\le |\mathbf{E}(|Y_1 + \psi u|\,|Y_2 - \psi u|) - \mathbf{E}(|Y_1 + \psi u|\,|Y_2|)| \\
&+ |\mathbf{E}(|Y_1 + \psi u|\,|Y_2|) - \mathbf{E}(|Y_1|\,|Y_2|)| + \left|\mathbf{E}(|Y_1|\,|Y_2|) - \sigma^2\mathbf{E}(|Z_1|)\mathbf{E}(|Z_2|)\right| \\
&+ \left|1 - \sigma^2\right|\mathbf{E}^2(|\dot{X}(0)|) \\
\le \mathbf{E}(|Y_1 + \psi u|\,|\psi u|) &+ \mathbf{E}(|Y_2|\,|\psi u|) + \sigma^2\,|\mathbf{E}(|Z_1 Z_2|) - \mathbf{E}(|Z_1|)\mathbf{E}(|Z_2|)| \\
&+ \left|1 - \sigma^2\right|\mathbf{E}^2(|\dot{X}(0)|).
\end{aligned}$$

Let us analyze term by term. As for $i = 1, 2$, we have

$$|\psi u| = \frac{|\dot{r}|}{1+r}u \le 2|\dot{r}|u.$$

Then, for the first two terms

$$\begin{aligned}
\mathbf{E}(|Y_1 + \psi u|\,|\psi u|) &\le \mathbf{E}(|Y_1|)|\psi u| + |\psi u \psi u| \le 6|\dot{r}|u^2, \\
\mathbf{E}(|Y_2|\,|\psi u|) &\le 2|\dot{r}|u.
\end{aligned}$$

For the third term, we have $\sigma^2 \le 1$ and denoting $\rho = \mathbf{cor}(Y_1, Y_2)$ and Z a standard Gaussian random variable, independent from Z_1

$$\begin{aligned}
|\mathbf{E}(|Z_1 Z_2|) - \mathbf{E}(|Z_1|)\mathbf{E}(|Z|)| &= \left|\mathbf{E}\left(|Z_1|\,|\rho Z_1 + \sqrt{1-\rho^2}Z|\right) - \mathbf{E}(|Z_1|)\mathbf{E}(|Z|)\right| \\
&= \left|\mathbf{E}\left(|Z_1|\left|\rho Z_1 + \sqrt{1-\rho^2}Z\right| - |Z_1|\,|Z|\right)\right| \le \left|\mathbf{E}\left(|Z_1|\left|\rho Z_1 + \left(\sqrt{1-\rho^2} - 1\right)Z\right|\right)\right| \\
&\le |\rho|\mathbf{E}(Z_1^2) + \left(\sqrt{1-\rho^2} - 1\right)\mathbf{E}^2(|Z|).
\end{aligned}$$

Besides, using Eq. (8)

$$|\rho| \le 2\left(|\ddot{r}| + \dot{r}^2\right),$$
$$\left|\sqrt{1-\rho^2} - 1\right| = \frac{\rho^2}{\sqrt{1-\rho^2}+1} \le |\rho|.$$

Gathering all the terms, and observing that the integrals of $\dot{r}$ and $\ddot{r}$ are absolutely convergent, enlarging T_0 if necessary, we obtain the bound

$$I_1 \le K u^2 e^{-\frac{u^2}{4}} \to 0, \quad (u \to \infty),$$

where K is a convenient constant. This concludes the proof of Theorem 1. □

5. Proof of Theorem 2

We begin by the construction of the corresponding blocks in Bernstein's scheme. Consider b, c such that $0 < c < b < \gamma/4$ and $\mu b > 1$. Define

$$n = n_T = T^{1-b}, \quad p_T = T^b, \quad q_T = T^c.$$

Then, as $T \to \infty$, $n_T, p_T, q_T \to \infty$,

$$n_T(p_T + q_T) = T + o(T),$$

and

$$\frac{n_T p_T}{T} \to 1, \qquad \frac{n_T q_T}{T} \to 0. \tag{13}$$

For $k = 1, \dots, n_T$ denote

$$I_k = [(k-1)(p_T+q_T), kp_T + (k-1)q_T], \quad J_k = [kp_T + (k-1)q_T, k(p_T+q_T)].$$

$$X_k^{n_T} = \frac{N(I_k,u) - p_T C(u)}{\sqrt{2v(T,u_T)}}, \quad Y_k^{n_T} = \frac{N(J_k,u) - q_T C(u)}{\sqrt{2v(T,u_T)}}.$$

Our strategy to establish (5), is to prove

$$\sum_{k=1}^{n_T} X_k^{n_T} \Rightarrow \mathcal{N}(0,1), \tag{14}$$

$$\sum_{k=1}^{n_T} Y_k^{n_T} \xrightarrow{P} 0, \tag{15}$$

where $\xrightarrow{P}$ stands for convergence in probability.

In both cases we need a bound for the (factorial) third moment of $N(I,u)$. The next result gives such a bound. As the proof requires the use of the Rice formula for the third (factorial) moment, it is deferred to Section 5.3.

Lemma 3. *For $R > 0$, there exist constants A, B, independent of R and u, such that*

$$\mathbf{E}\left[N(R,u)^{[3]}\right] \le R^3(A + Bu^3)e^{-u^2/2}.$$

5.1. Proof of (15) — Small blocks

The convergence in (15) is a consequence of the following lemma.

Lemma 4. *(a) Let $\ell(T) = o(T)$ as $T \to \infty$. We have*

$$\frac{N(\ell(T), u) - \ell(T)C(u)}{\sqrt{\nu(T, u_T)}} \xrightarrow{P} 0, \quad (T \to \infty).$$

(b) Under the mixing condition (3), we have

$$\mathbf{E}\left(\sum_{k=1}^{n} Y_k^n\right)^2 \to 0, \quad (T \to \infty).$$

Proof. In order to verify (a), we write

$$\mathbf{E}\left(\frac{N(\ell(T), u) - \ell(T)C(u)}{\sqrt{\nu(T, u_T)}}\right)^2 = \frac{\mathbf{var} N(\ell(T), u)}{\nu(T, u_T)} \sim \frac{2\ell(T)}{T} \to 0.$$

To see (b), we use a moment inequality under mixing (see 1.2.2 in [6]), that in our context reads

$$|\mathbf{E}(Y_k^n, Y_{k+\ell}^n)| \le 8\alpha(\ell p_T)^{1/3}\left(\mathbf{E}(|Y_1^n|^3)\right)^{2/3}.$$

So, applying Lemma 3 and the mixing rate, we obtain

$$\begin{aligned}
|\mathbf{E}(Y_k^n, Y_{k+\ell}^n)| &\le 8\ell^{-\mu/3}T^{-b\mu/3}\left[\frac{(A + Bu^3)q_T^3 e^{-u^2/2}}{\nu(T, u_T)^{3/2}}\right]^{2/3} \\
&\le 8\ell^{-\mu/3}T^{-b\mu/3}\left[\frac{(A + Bu^2)q_T^2 e^{-u^2/3}}{\nu(T, u_T)}\right] \\
&\le 8\ell^{-\mu/3}\left[(A + Bu^2)T^{2c-b\mu/3-\gamma/3-2/3}\right] \le 8\ell^{-\mu/3}T^{2b-b\mu/3-\gamma/3-2/3+\epsilon}.
\end{aligned}$$

Note that as $\mu > 4$ the series $\sum_{\ell=1}^{\infty} \ell^{-\mu/3} < \infty$. So

$$\begin{aligned}
\mathbf{E}\left(\sum_{k=1}^{n} Y_k^n\right)^2 &= n\mathbf{E}(Y_1^n)^2 + 2(n-1)\mathbf{E}(Y_1^n Y_2^n) + \cdots + 4\mathbf{E}(Y_1^n Y_{n-2}^n) + 2\mathbf{E}(Y_1^n Y_{n-1}^n) \\
&\le \frac{1}{\nu(T, u_T)} Knq_T e^{-u^2/2} + 16nT^{2b-b\mu/3-\gamma/3-2/3+\epsilon} \sum_{\ell=1}^{\infty} \ell^{-\mu/3} \\
&\le \frac{1}{\nu(T, u_T)} Knq_T e^{-u^2/2} + KT^{1/3+b-b\mu/3-\gamma/3+\epsilon} \to 0,
\end{aligned}$$

because $\epsilon > 0$ can be chosen arbitrarily small and $\mu b > 1$. This concludes the proof of the lemma. □

5.2. *Proof of (14) — Big blocks*

In order to prove (14), we use the Central Limit Theorem proved by Lindeberg's method as presented in Theorem 1 in [2]. The next three lemmas verify the hypothesis of Theorem 1 in [2].

Lemma 5. *The mixing rate of the process $\mathcal{X}$ implies that as $T \to \infty$:*

$$\sum_{k=1}^{n_T} |\mathbf{cov}(e^{it(X_1^n + \cdots + X_{k-1}^n)}, e^{itX_k^n})| \to 0.$$

Proof. We use Kolmogorov–Rozanov inequality as in the proof of (c) in Lemma 1. Since $1-b-\mu c \le 1-b-\mu b < -b < 0$ by (13), we have

$$\sum_{k=1}^{n_T} |\mathbf{cov}(e^{it(X_1^n+\cdots+X_{k-1}^n)}, e^{itX_k^n})| \le \mathbf{C} n_T \alpha(q_T) \sim T^{1-b-\mu c} \to 0. \quad \square$$

Lemma 6. *As $T \to \infty$:*

$$\sum_{k=1}^{n} \mathbf{var} X_k^n = n \mathbf{var} X_1^n \sim \frac{n p_T C(u)}{\nu(T,u_T)} \to 1.$$

In the next lemma we check hypothesis H_δ of Theorem 1 in [2] with $\delta = 1$.

Lemma 7 (H_δ *for* $\delta = 1$)**.** *As $n \to \infty$, we have*

$$A_n = \sum_{k=1}^{n} \mathbf{E}|X_k^n|^3 \le \frac{n}{\nu(T,u_T)^{3/2}} \mathbf{E}|N(p_T,u) - p_T C(u)|^3 \to 0.$$

Proof. Denote $N = N(I_1, u)$. We use the following rough bound

$$\mathbf{E}|N - \mathbf{E}N|^3 \le 8\left[\mathbf{E}(N^3) + (\mathbf{E}N)^3\right] \le 16\mathbf{E}(N^3).$$

where we rely on $(A+B)^3 \le 8(A^3+B^3)$, for $A, B > 0$; and Jensen's inequality for the convex function $f(x) = |x|^3$. Then

$$\mathbf{E}|X_1^n|^3 \le \frac{16}{\nu(T,u_T)^{3/2}} \mathbf{E}(N^3).$$

The proof is then split into several steps.

1. We expand the third factorial moment of the crossings

$$\mathbf{E}N^{[3]} := \mathbf{E}(N(N-1)(N-2)) = \mathbf{E}N^3 - 3\mathbf{E}N^2 + 2\mathbf{E}N.$$

2. We verify

$$\frac{n\mathbf{E}N}{\nu(T,u_T)^{3/2}} = \frac{n p_T C(u)}{\nu(T,u_T)^{3/2}} \sim \frac{1}{\sqrt{\nu(T,u_T)}} \to 0.$$

3. We verify

$$\begin{aligned}\frac{n\mathbf{E}N^2}{\nu(T,u_T)^{3/2}} &= \frac{n(\mathbf{var}N + (\mathbf{E}N)^2)}{\nu(T,u_T)^{3/2}} \sim \frac{n(\mathbf{E}N + (\mathbf{E}N)^2)}{\nu(T,u_T)^{3/2}} \sim \frac{n(\mathbf{E}N)^2}{\nu(T,u_T)^{3/2}} \\ &\sim T^{1-b} p_T^2 C(u)^2 T^{-3\gamma/2} \sim T^{1-b} T^{2b} T^{2(\gamma-1)} T^{-3\gamma/2} \sim T^{-1+b+\gamma/2} \to 0,\end{aligned}$$

as $b < \gamma/4$.

4. It remains to prove

$$\frac{n\mathbf{E}N^{[3]}}{\nu(T,u_T)^{3/2}} \to 0.$$

This step is a consequence of Lemma 3. In fact,

$$\frac{n\mathbf{E}(N^{[3]})}{\nu(T,u_T)^{3/2}} \le \frac{n p_T^3 (A + Bu^3) e^{-u^2/2}}{T^{3\gamma/2}} \le K T^{1+2b+\gamma-1-3\gamma/2+\epsilon} \le T^{2b-\gamma/2+\epsilon} \to 0,$$

because $4b < \gamma$ and ϵ can be taken arbitrarily small. $\square$

5.3. Proof of Lemma 3

Below, based on Rice formula, we present the proof for the bound on the third moment given in Lemma 3. In this section K, K' stands for meaningless constants whose value may change from line to line.

Proof of Lemma 3. Let $\mathbf{u} = (u, u, u)$, $\mathcal{C} = \{X(s_1) = X(s_2) = X(s_3) = u\}$ and

$$A(s_1, s_2, s_3) = \mathbf{E}(|\dot{X}(s_1)\dot{X}(s_2)\dot{X}(s_3)| \mid \mathcal{C}) p_{X(s_1),X(s_2),X(s_3)}(\mathbf{u}).$$

Rice formula (6) for $n = 3$ states

$$\mathbf{E}(N(R, u)^{[3]}) = \iiint_{[0,R]^3} A(s_1, s_2, s_3) ds_1 ds_2 ds_3.$$

As the process is stationary, we have $A(s_1, s_2, s_3) = A(0, s_2 - s_1, s_3 - s_1)$, so we change variables according to

$$s = s_1, \quad h = s_2 - s_1, \quad k = s_3 - s_2.$$

We have

$$\begin{aligned}
\mathbf{E}(N^{[3]}(R, u)) &= 6 \iiint_{\{0 \le s_1 \le s_2 \le s_3 \le R\}} A(0, s_2 - s_1, s_3 - s_1) ds_1 ds_2 ds_3 \\
&= 6 \iint_{\{0 \le h+k \le R\}} \left(\int_0^{R-h-k} A(0, h, h + k) ds \right) dh dk \\
&= 6 \int_0^R dk \int_0^{R-k} (R - h - k) A(0, h, h + k) dh \\
&\le 6 \int_0^\delta dk \int_0^\delta (R - h - k) A(0, h, h + k) dh \\
&\quad + 12 \int_\delta^R dk \int_0^\delta (R - h - k) A(0, h, h + k) dh \\
&\quad + 6 \int_\delta^R dk \int_\delta^{R-k} (R - h - k) A(0, h, h + k) dh =: I_1 + I_2 + I_3.
\end{aligned}$$

In conclusion, we have to bound the three integrals that appear above. The proof is divided into several steps.

First step: General facts. Since $(X(0), X(h), X(h + k))$ is non-degenerated by hypothesis, its covariance matrix

$$\Sigma = \begin{bmatrix} 1 & r(h) & r(h+k) \\ r(h) & 1 & r(k) \\ r(h+k) & r(k) & 1 \end{bmatrix}$$

is not singular.

The density is

$$p_{X(0),X(h),X(h+k)}(\mathbf{u}) = \frac{1}{(2\pi)^{3/2}\sqrt{\Delta}} \exp\left(-\frac{1}{2} \mathbf{u} \Sigma^{-1} \mathbf{u}^t \right).$$

Claim. *We have*

$$\mathbf{u} \Sigma^{-1} \mathbf{u}^t > u^2. \tag{16}$$

Proof of the Claim. As Σ is symmetric and positive definite, we denote its three eigenvalues by

$$0 < \kappa_1 \le \kappa_2 \le \kappa_3,$$

and introduce the diagonal matrix $D = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3)$. Observe that

$$\kappa_1 + \kappa_2 + \kappa_3 = \mathrm{trace}(\Sigma) = 3,$$

so $\kappa_3 < 3$. If P denotes the corresponding orthogonal matrix, such that $\Sigma P = PD$, then $\Sigma^{-1} = PD^{-1}P^t$. So, denoting $\mathbf{v} = (v_1, v_2, v_3) = \mathbf{u}P$,

$$\mathbf{u}\Sigma^{-1}\mathbf{u}^t = \mathbf{v}D^{-1}\mathbf{v}^t = \frac{v_1^2}{\kappa_1} + \frac{v_2^2}{\kappa_2} + \frac{v_3^2}{\kappa_3} \ge \frac{1}{\kappa_3}\mathbf{v}\mathbf{v}^t > \frac{1}{3}\mathbf{u}PP^t\mathbf{u}^t = \frac{1}{3}3u^2 = u^2,$$

completes the proof. □

Hence, the exponential factor in the density is bounded by e^{-u^2}. The denominator of this density must be bounded jointly with the remaining factors in the integrals.

We now specialize Lemma 2 to the case $n = 3$. We use the notation

$$\mathbf{\Psi} = \Sigma_{10}\Sigma^{-1} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix},$$

$$r_{ij} = \mathbf{E}X_iX_j, \quad \dot r_{ij} = \mathbf{E}(\dot X_i X_j), \quad \ddot r_{ij} = \mathbf{E}(\dot X_i \dot X_j).$$

$$\Sigma_{10} = \begin{bmatrix} \dot r_{11} & \dot r_{12} & \dot r_{13} \\ \dot r_{21} & \dot r_{22} & \dot r_{23} \\ \dot r_{31} & \dot r_{32} & \dot r_{33} \end{bmatrix} = \begin{bmatrix} 0 & \dot r_{12} & \dot r_{13} \\ -\dot r_{12} & 0 & \dot r_{23} \\ -\dot r_{13} & -\dot r_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot r(h) & -\dot r(h+k) \\ \dot r(h) & 0 & -\dot r(k) \\ \dot r(h+k) & \dot r(k) & 0 \end{bmatrix},$$

$$\Sigma_{11} = \begin{bmatrix} \ddot r_{11} & \ddot r_{12} & \ddot r_{13} \\ \ddot r_{21} & \ddot r_{22} & \ddot r_{23} \\ \ddot r_{31} & \ddot r_{32} & \ddot r_{33} \end{bmatrix} = \begin{bmatrix} -1 & -\ddot r(h) & -\ddot r(h+k) \\ -\ddot r(h) & -1 & -\ddot r(k) \\ -\ddot r(h+k) & -\ddot r(k) & -1 \end{bmatrix}.$$

Recall that from Lemma 2 we have

$$\mathbf{E}(|\dot X(s_1)\dot X(s_2)\dot X(s_3)| \mid \mathcal{C}) = \mathbf{E}(|Y_1 - \Psi_1\mathbf{u}\|Y_2 - \Psi_2\mathbf{u}\|Y_3 - \Psi_3\mathbf{u}|).$$

Second step: First integral. We make the Taylor expansion of each component of the integrand as $(h, k) \to (0, 0)$ in order to prove that the integral is convergent. As a consequence we obtain a bound of the desired form for this term. The following result is taken from Proposition 5.9, equation (5.22) in [1]. We have

$$\Delta := \det(\Sigma) = 1 + 2r(h)r(k)r(h+k) - r(h)^2 - r(k)^2 - r(h+k)^2,$$

that, for $(h, k) \to (0, 0)$ satisfies

$$\Delta \sim \left(\frac{24d-1}{4}\right) h^2k^2(h+k)^2. \tag{17}$$

Claim (*On the Regression Coefficients in* $\mathbf{\Psi}$). *Denote* $\alpha_i = \Psi_i(1,1,1)^t = \alpha_{1i} + \alpha_{2i} + \alpha_{3i}$, $(i = 1, 2, 3)$. *As* $(h, k) \to 0$ *we have*

$$\alpha_1 \sim -\left(\frac{8d^2 + 10e}{24d - 1}\right) h(h+k)(2h+k),$$

$$\alpha_2 \sim \left(\frac{8d^2+10e}{24d-1}\right) hk(h-k),$$
$$\alpha_3 \sim -\left(\frac{8d^2+10e}{24d-1}\right) k(h+k)(2k+h).$$

Proof of the Claim. We have

$$\Sigma^{-1} = \frac{1}{\Delta}\begin{bmatrix} 1-r^2(k) & r(h+k)r(k)-r(h) & r(h)r(k)-r(h+k) \\ r(h+k)r(k)-r(h) & 1-r^2(h+k) & r(h)r(h+k)-r(k) \\ r(h)r(k)-r(h+k) & r(h)r(h+k)-r(k) & 1-r^2(h) \end{bmatrix},$$

so

$$\begin{aligned}
\Delta\alpha_1 &= -[1-r(h+k)][1+r(h+k)-r(h)-r(k)]\dot r(h) \\
&\quad -(1-r(h))(1+r(h)-r(k)-r(h+k))\dot r(h+k), \\
\Delta\alpha_2 &= (1-r(k))(1+r(k)-r(h)-r(h+k))\dot r(h) \\
&\quad -(1-r(h))(1+r(h)-r(k)-r(h+k))\dot r(k), \\
\Delta\alpha_3 &= (1-r(k))(1+r(k)-r(h)-r(h+k))\dot r(h+k) \\
&\quad +(1-r(h+k))(1+r(h+k)-r(h)-r(k))\dot r(k).
\end{aligned} \tag{18}$$

We perform the Taylor expansions for $\alpha_i = \alpha_i(h,k)$ $(i=1,2,3)$:

$$\Delta\alpha_1(h,k) \sim -\left(\frac{4d^2+5e}{2}\right) h^3k^2(h+k)^3(2h+k),$$
$$\Delta\alpha_2(h,k) \sim 2d^2h^3k^3(h+k)^2(h-k),$$
$$\Delta\alpha_3(h,k) \sim -\left(\frac{4d^2+5e}{2}\right) h^3k^3(h+k)^2(k-h),$$

that in view of (17), give the results.

Claim. *For $i=1$ and $i=3$ we have*

$$\mathbf{var}Y_i \sim \frac{4(16d^2-d-10e)}{24d-1}(h^2+hk+k^2),$$

and for $i=2$,

$$\mathbf{var}Y_2 \sim \frac{4d(16d-1)}{24d-1}(h^2+hk+k^2).$$

Proof of the Claim. We depart from

$$1 = \mathbf{var}\dot X(s_1) = \mathbf{var}Y_1 - (\alpha_{12}\dot r_{12} + \alpha_{13}\dot r_{13}),$$

and expand the expression

$$\begin{aligned}
\Delta(\alpha_{12}\dot r_{12}+\alpha_{13}\dot r_{13})(h,k) &= -\left(16d^2-d-10e\right) h^2k^2(h+k)^2(h^2+hk+k^2) \\
&\quad +\frac{1}{4}\left(24d-1\right) h^2k^2(h+k)^2.
\end{aligned}$$

We conclude

$$\mathbf{var}Y_1 = 1-(\alpha_{12}\dot r_{12}+\alpha_{13}\dot r_{13})(h,k) \sim \frac{4(16d^2-d-10e)}{24d-1}(h^2+hk+k^2).$$

Similar computations hold for $i = 3$. For $i = 2$, we compute

$$1 = \mathbf{var}\dot{X}_i = \mathbf{var}Y_2 - (\alpha_{21}\dot{r}_{21} + \alpha_{23}\dot{r}_{23}).$$

Besides,

$$\begin{aligned}\Delta(\alpha_{21}\dot{r}_{21} + \alpha_{23}\dot{r}_{23})(h,k) &\sim \frac{1}{4}(24d-1)h^2k^2(h+k)^2\\ &\quad -(16d^2-d)h^2k^2(h+k)^2(h^2+hk+k^2),\end{aligned}$$

and in consequence

$$\mathbf{var}Y_2 = 1 - (\alpha_{21}\dot{r}_{21} + \alpha_{23}\dot{r}_{23})(h,k) \sim \frac{4d(16d-1)}{24d-1}(h^2+hk+k^2).$$

This proves the claim. □

Claim. *For $u > 1$ there exist constants A_1, B_1 such that*

$$\mathbf{E}(|\dot{X}(s_1)\dot{X}(s_2)\dot{X}(s_3)| \mid \mathcal{C}) \le (A_1 + B_1u^3)E(h,k),$$

with $E(h,k) = o(h^2k^2(h+k)^2)$ as $(h,k) \to (0,0)$.

Proof of the Claim. We have

$$\begin{aligned}\mathbf{E}(|\dot{X}(s_1)\dot{X}(s_2)\dot{X}(s_3)| \mid \mathcal{C}) &= \mathbf{E}(|(Y_1+\alpha_1u)(Y_2+\alpha_2u)(Y_3+\alpha_3u)|)\\ &\le J_1 + J_2u + J_3u^2 + |\alpha_1\alpha_2\alpha_3|u^3,\end{aligned} \tag{19}$$

being

$$J_1 = \mathbf{E}(|Y_1Y_2Y_3|) = \iiint_{\mathbb{R}^3} |y_1y_2y_3| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2}y\Sigma_Y^{-1}y^t}dy,$$

$$J_2 = \sum_{i=1}^3 |\alpha_i| \mathbf{E}\left(\prod_{k\neq i}^3 |Y_k|\right) = \sum_{i=1}^3 |\alpha_i| \iiint_{\mathbb{R}^3} \left(\prod_{k\neq i}^3 |y_k|\right) \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2}y\Sigma_Y^{-1}y^t}dy,$$

$$J_3 = \sum_{i=1}^3 \left(\prod_{k\neq i}^3 |\alpha_k|\right) \mathbf{E}(|Y_i|) = \sum_{i=1}^3 \left(\prod_{k\neq i}^3 |\alpha_k|\right) \iiint_{\mathbb{R}^3} |y_i| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2}y\Sigma_Y^{-1}y^t}dy.$$

To deal with them we perform the change of variables

$$y_j = \gamma_j z_j, \qquad \gamma_j = \prod_{i\neq j}^3 (s_i - s_j)^2; \qquad j = 1,2,3.$$

Furthermore, we use the equivalents above and the fact that (see pg. 147 in [1])

$$\det(\Sigma_Y) \sim K \prod_{1\le i<j\le 3} (s_j - s_i)^6.$$

The first integral J_1 is the same as the one that appears for the case $u = 0$ in Prop. 5.10 in [1]:

$$J_1 = \prod_{1\le i<j\le 3} (s_j - s_i)^8 \iiint_{\mathbb{R}^3} |z_1z_2z_3| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2}zGz^t}dz \sim Kh^5k^5(h+k)^5,$$

where $G = D\Sigma_Y^{-1}D$ and $D = diag(\gamma_1, \gamma_2, \gamma_3)$. The second term J_2 is

$$\begin{aligned}
J_2 &= |\alpha_1| \iiint_{\mathbb{R}^3} |z_2 z_3| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2\det(\Sigma_Y)} zGz'} dz \\
&\quad + |\alpha_2| \iiint_{\mathbb{R}^3} |z_1 z_3| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2\det(\Sigma_Y)} zGz'} dz \\
&\quad + |\alpha_3| \iiint_{\mathbb{R}^3} |z_1 z_2| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2\det(\Sigma_Y)} zGz'} dz \\
&\sim K_1(s_3 - s_2)^5(s_2 - s_1)^4(s_3 - s_1)^4(2s_3 - s_2 - s_1) \\
&\quad + K_2(s_3 - s_1)^5(s_3 - s_2)^4(s_1 - s_2)^4(2s_2 - s_3 - s_1) \\
&\quad + K_3(s_2 - s_1)^5(s_1 - s_3)^4(s_2 - s_3)^4(2s_3 - s_2 - s_1) \\
&= h^4k^4(h+k)^4[K_1k(2h+k) + K_2(h+k)(h-k) + K_3h(2h+k)].
\end{aligned}$$

The third term J_3 is

$$\begin{aligned}
J_3 &= |\alpha_2\alpha_3| \iiint_{\mathbb{R}^3} |z_1| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2\det(\Sigma_Y)} zGz'} dz \\
&\quad + |\alpha_1\alpha_3| \iiint_{\mathbb{R}^3} |z_2| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2\det(\Sigma_Y)} zGz'} dz \\
&\quad + |\alpha_1\alpha_2| \iiint_{\mathbb{R}^3} |z_3| \frac{1}{(2\pi)^{3/2}\sqrt{\det(\Sigma_Y)}} e^{-\frac{1}{2\det(\Sigma_Y)} zGz'} dz \\
&\sim h^3k^3(h+k)^3[K_1h(h+k)(h-k)(2k+h) + K_2hk(2k+h)(2h+k) \\
&\quad + K_3k(h+k)(2h+k)].
\end{aligned}$$

Finally,

$$|\alpha_1\alpha_2\alpha_3| \sim Kh^2k^2(h+k)^2(2k+h)(2h+k)(h-k).$$

Gathering all these terms and defining $A_1 = J_1$, $B_1 = J_2 + J_3 + |\alpha_1\alpha_2\alpha_3|$ we conclude the proof of the Claim. □

In conclusion, we have

$$A(0, h, h+k) \le (A_1 + B_1u^3)\frac{E(h,k)}{\sqrt{\Delta}} e^{-\frac{1}{2}u^2}, \quad (h,k) \to (0,0),$$

and $\frac{E(h,k)}{\sqrt{\Delta}} = o(hk(h+k))$ as $(h,k) \to (0,0)$. Consequently

$$I_1 \le \int_0^\delta dk \int_0^\delta (R-h-k)(A_1 + B_1u^3)\frac{E(h,k)}{\sqrt{\Delta}} e^{-\frac{1}{2}u^2} dhdk \le K\, R(A_1 + B_1u^3)e^{-\frac{1}{2}u^2},$$

Third step: Second integral. We have $h < \delta < k \le R$. Then, we expect Y_1 and Y_2 to be small, as s_1 is close to s_2.

We proceed as in (19). Besides, we have $\mathbf{var}Y_i \le 1$ and when h is bounded away from 0 is easy to bound α_i, $i = 1, 2, 3$ by a constant, see (18). Thus, also J_1, J_2 and J_3 are bounded by constants.

The result follows provided the convergence of the integral at $h = 0$.

If we consider $\Delta = \Delta(h, k)$, we have

$$\partial_h\Delta(0,k) = \partial_k\Delta(0,k) = \partial_{hk}\Delta(0,k) = \partial_{kk}\Delta(0,k) = 0,$$

and

$$\Delta(h,k) \sim \left[1 - r(k)^2 - \dot{r}(k)^2\right] h^2, \quad (h,k) \to (0,k).$$

Now, the Taylor expansion gives

$$\Delta\alpha_1 \sim [(r(k)-1)\ddot{r}(k) - \dot{r}(k)^2 - r(k) + 1]\frac{h^3}{2},$$

that in view of the expansion of the determinant gives

$$\alpha_1 \sim C_1(k)h,$$

where $C_1(k)$ is a continuous and bounded function of $k \geq \delta$. By similar computations we have

$$\alpha_2 \sim C_2(k)h,$$

being $C_2(k)$ continuous and bounded on $k \geq \delta$. For the variance of Y_i, we have $\mathbf{var}Y_1 = 1 - (\alpha_{12}\dot{r}_{12} + \alpha_{13}\dot{r}_{13})$. We have

$$\Delta(\alpha_{12}\dot{r}_{12} + \alpha_{13}\dot{r}_{13}) = \left(1 - r(k)^2 - \dot{r}(k)^2\right) h^2 - (r(k)\dot{r}(k) + \dot{r}(k)\ddot{r}(k))\, h^3,$$

so

$$\alpha_{12}\dot{r}_{12} + \alpha_{13}\dot{r}_{13} = 1 - \frac{r(k)\dot{r}(k) + \dot{r}(k)\ddot{r}(k)}{1 - r(k)^2 - \dot{r}(k)^2}h.$$

So

$$\mathbf{var}Y_1 \sim C_1'(k)h.$$

By similar arguments, we have

$$\mathbf{var}Y_2 \sim C_2'(k)h.$$

Both C_1' and C_2' are continuous and bounded on $k \geq \delta$. We have the bound

$$\mathbf{E}(|\dot{X}(s_1)\dot{X}(s_2)\dot{X}(s_3)| \mid \mathcal{C}) \leq \prod_{k=1}^{3} \left(\mathbf{E}(|\dot{X}(s_i)|^3 \mid \mathcal{C})\right)^{1/3}.$$

The first two expectations depend on h, so, for $i = 1, 2$

$$\mathbf{E}(|\dot{X}(s_i)|^3 \mid \mathcal{C}) = \mathbf{E}|Y_i - \alpha_i u|^3 \leq 8\mathbf{var}Y_i^{3/2} + 8|\alpha_i|^3u^3 \sim Kh^{3/2} + K'u^3h^3.$$

Furthermore,

$$\mathbf{E}(|\dot{X}(s_3)|^3 \mid \mathcal{C}) = \mathbf{E}|Y_3 - \alpha_3 u|^3 \leq K + K'u^3.$$

As $\sqrt{\Delta} \sim \sqrt{1 - r(k)^2 - \dot{r}(k)^2}h$ when $(h,k) \to (0,k)$, we obtain that for these values

$$A(0, h, h+k) \leq (K + K'u^3)e^{-u^2/2}.$$

In conclusion

$$I_2 \leq (A_2 + B_2u^3)R^2e^{-u^2/2}.$$

Fourth step: Third integral. We consider the off-diagonal term.

Claim. *When $\delta < \min(h,k)$, we have*

$$\mathbf{E}(|\dot{X}(s_1)\dot{X}(s_2)\dot{X}(s_3)| \mid \mathcal{C}) \leq A_3 + B_3u^3, \tag{20}$$

where A_3, B_3 are constants, depending on δ.

Proof of the Claim.

$$\mathbf{E}(|\dot{X}(s_1)\dot{X}(s_2)\dot{X}(s_3)| \mid \mathcal{C}) \le \prod_{k=1}^{3} \left(\mathbf{E}(|\dot{X}(s_i)|^3 \mid \mathcal{C})\right)^{1/3}.$$

We have

$$\mathbf{E}(|\dot{X}(s_i)|^3 \mid \mathcal{C}) = \mathbf{E}|Y_i - \alpha_i u|^3 \le 8\mathbf{E}|Y_i|^3 + 8|\alpha_i|^3 u^3.$$

By (a) in Lemma 2, we know that $\mathbf{var} Y_i \le 1$. We need some rough bounds. Based on (18) and (a) in Lemma 1, we obtain that $|\Delta\alpha_i| \le 16$. As in the domain of integration, there exists $\Delta_0 > 0$ such that $\Delta \ge \Delta_0$, we obtain that

$$|\alpha_i| \le \frac{16}{\Delta_0}, \quad i = 1, 2, 3.$$

This concludes the proof of the claim.

Taking into account (20) and the bound of the density of (16):

$$I_3 \le K \int_\delta^R dk \int_\delta^{R-k} (R - h - k)(A + Bu^3)e^{-\frac{1}{2}u^2} dh \le K R^3 (A + Bu^3)e^{-\frac{1}{2}u^2}.$$

This concludes the proof of Lemma 3. □

6. Proof of Theorem 3

Proof. We begin the proof considering the up-crossings. Then, relation (2) reads as

$$\frac{\mathbf{var} U(T, u)}{\mathbf{E}\, U(T, u)} \to 1,$$

uniformly in $T \in [t_0, \infty)$. The normalized number of up-crossings is

$$Z(T, u) = \frac{U(T, u) - \mathbf{E}U(T, u)}{\sqrt{\mathbf{E}U(T, u)}},$$

and denote its probability distribution by

$$F(T, u) = F_{Z(T,u)}(\cdot).$$

Denote by $\pi(F, G)$ the Prohorov distance between the probability measures induced by the distributions F and G in $\mathbb{R}$. Denote now by $N(\ell)$ a Poisson random variable with parameter ℓ, and

$$Y(\ell) = \frac{N(\ell) - \ell}{\sqrt{\ell}},$$

and the corresponding distribution

$$G(\ell) = G_{Y(\ell)}(\cdot).$$

By the Central Limit Theorem for Poisson random variables, we know that

$$\pi(G(\ell), \Phi) \to 0, \quad (\ell \to \infty),$$

where Φ stands for the standard normal distribution function. This means that there exists ℓ_n such that

$$\pi(G(\ell), \Phi) < 1/(2n), \quad \text{for } \ell \ge \ell_n.$$

We have obtained a sequence $\ell_n \to \infty$. We now observe that we are under the hypothesis of the Volkonskiĭ – Rozanov Theorem that states that, as processes indexed by $\ell > 0$, we have

$$U(\ell C(u)^{-1}, u) \Rightarrow P(\ell), \quad (u \to \infty),$$

where $P = \{P(\ell) : \ell \geq 0\}$ denotes a standard Poisson process. We then have, for each n,

$$U(\ell_n C(u)^{-1}, u) \Rightarrow P(\ell_n), \quad (u \to \infty),$$

that is equivalent to

$$\frac{U(\ell_n C(u)^{-1}, u) - \ell_n}{\sqrt{\ell_n}} \Rightarrow \frac{P(\ell_n) - \ell_n}{\sqrt{\ell_n}} = Y(\ell_n), \quad (u \to \infty).$$

In consequence, there exists $u_n \to \infty$ such that

$$\pi(F(\ell_n C(u)^{-1}, u), G(\ell_n)) < \frac{1}{2n}, \text{ for } u \geq u_n.$$

Denote $T_n = \ell_n C(u_n)^{-1}$. As ℓ_n and u_n tend to infinity it follows that $T_n \to \infty$. We obtain that

$$\pi(F(T_n, u_n), \Phi) \leq \pi(F(T_n, u_n), G(\ell_n)) + \pi(G(\ell_n), \Phi) < \frac{1}{n},$$

concluding the proof for up-crossings. Now, based on Remark 2, we have

$$\frac{N(T_n, u_n) - \nu(T_n, u_n)}{\sqrt{2\nu(T_n, u_n)}} = \frac{U(T_n, u_n) - \mathbf{E}U(T_n, u_n)}{\sqrt{\mathbf{E}U(T_n, u_n)}} + \frac{r(T_n, u_n)}{\sqrt{\mathbf{E}U(T_n, u_n)}},$$

where $|r(T_n, u_n)| \leq 1/2$. As $\mathbf{E}U(T_n, u_n) \to \infty$, the result follows. □

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