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SKEWED LÉVY MODELS AND IMPLIED VOLATILITY SKEW

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We introduce skewed Lévy models, characterized by a symmetric jump measure multiplied by a damping exponential factor. These models exhibit a clear implied volatility pattern, where the damping parameter controls the implied volatility curve's skew, resulting in a measure of the model's skewness. We show that the variation of this parameter produces the typical smirk observed in implied volatility curves. Some theoretical facts supporting these findings are proved.

Keywords: Skewness; Lévy processes; implied volatility smirk.

JEL Classification: C52, G10

1. Introduction

Since the seminal paper by Black & Scholes (1973), many attempts have been made to capture the empirical behavior of the implied volatility surface. The best known stylized fact is the volatility smile, when we look at the volatility as a function of moneyness. Furthermore, the fact that out-of-the money Put options are more expensive than the corresponding out-of-the money Call options has been extensively studied by many authors, among them we recall the work of

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Foresi & Wu (2005), producing the volatility smirk. They establish this fact based on a large set of option prices where the underlyings are major equity indexes from 12 countries. Also, Carr & Wu (2003) analyze the characteristic pattern of implied volatility smirks across maturities using S&P500 index options. Their findings imply that the index has an asymmetric risk neutral distribution.

On the other hand, there is a well-established relationship between the symmetry of the implied volatility and the market symmetry, which is equivalent to the Put–Call symmetry. This has been derived by Fajardo & Mordecki (2006) and Carr & Lee (2009) for Lévy process and local and stochastic volatility models, respectively. Also, Fajardo & Mordecki (2014) showed the relationship between the skewness premium and the market symmetry parameter.

In statistics, skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable. In conclusion, on one side we have volatility, that in statistical terms is a measure of the second centered moment, and we want to incorporate into the analysis a measure of the third centered moment, the skewness.

In finance, we distinguish three different notions of skewness that can be found in the literature: (i) the statistical skewness of the log-prices; (ii) the risk neutral skewness of the log-prices; (iii) the implied volatility skewness or smirkness, that is the depart form symmetry of the implied volatility curve when plotted in terms of the log-moneyness.

In this paper, we present a wide subclass of Lévy processes, with an exponential dampening modeling the skewness. Based on duality techniques we relate the implied volatility skew with a market skewness parameter β. For this class we get a deeper insight into how this particular process generates the skew, that is specially suited for visual analysis.

The paper is organized as follows: in Sec. 2, we introduce the model and through Put–Call duality, we obtain a symmetry property of the implied volatility which extends the results obtained in Fajardo & Mordecki (2006); in Sec. 3, we describe the implied volatility behavior for at the money options, in a neighborhood of the implied volatility symmetric model (i.e. in a neighborhood of $\beta = -1/2$); in Sec. 4, we show different skewed models, and present a complementary example, to show that the obtained implied volatility behavior obtained in Sec. 3 cannot be extended. Section 5 contains the conclusions of the paper, and Appendix A contains the proofs of the results in Sec. 3.

2. Skewed Lévy Processes and Models

Consider a Lévy process $X = \{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The Lévy–Khinchine formula gives the characteristic exponent ψ of X defined by the equation $\mathbb{E}\, e^{zX_t} = e^{t\psi(z)}$ for $z \in i\mathbb{R}$, by the formula

$$\psi(z) = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zh(y))\Pi(dy). \tag{2.1}$$

Here, $h(y) = y\mathbf{1}_{\{|y|<1\}}$ is the (fixed) truncation function, $\gamma \in \mathbb{R}$ is the drift parameter, $\sigma \geq 0$ is the standard deviation of the Gaussian part of the Lévy process, and Π is the Lévy measure, i.e. a positive measure such that $\int(1 \wedge y^2)\Pi(dy) < \infty$.

Given a Lévy process X, we define a *Lévy market model* or simply a *Lévy model*, as a set of two assets (B, S): a nonrisky asset $B = \{B_t\}_{t\geq 0}$, that evolves according to $B_t = e^{rt}$, where $r \geq 0$ is the interest rate in the market, and a risky asset $S = \{S_t\}_{t\geq 0}$ driven by X according to

$$S_t = S_0 e^{X_t + rt}, \quad S_0 > 0. \tag{2.2}$$

Throughout the paper we consider risk neutral Lévy markets, meaning that $\mathbb{E}\, e^{X_1} = 1$ (in particular the previous expectation is finite), which is equivalent to $\psi(1) = 0$.

For general references on Lévy processes see Jacod & Shiryaev (1987), Skorokhod (1991), Bertoin (2006) or Sato (1999). For Lévy processes used in financial models see Boyarchenko & Levendorskiĭ (2002), Schoutens (2003) or Cont & Tankov (2004).

Definition 2.1. We say that X is a *skewed Lévy process* with *skewness parameter* $\beta \in \mathbb{R}$, or equivalently that it is a *β-skewed Lévy process*, when it has a characteristic exponent given by

$$\psi(z;\beta) = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zh(y))\Pi_\beta(dy) \tag{2.3}$$

with a Lévy measure of the form

$$\Pi_\beta(dy) = e^{\beta y}\Pi_0(dy), \tag{2.4}$$

where the Lévy measure Π_0 is symmetric, i.e. $\int_A \Pi_0(dx) = \int_{-A} \Pi_0(dx)$, with $-A = \{x\colon\ -x \in A\}$ the symmetrical image of A.

Definition 2.2. We say that (B, S) is a *skewed Lévy market*, when it is a Lévy market driven by a β-skewed Lévy process X, with characteristic exponent $\psi(z;\beta)$ given in (2.3), with $\gamma = \gamma(\beta)$, defined such that $\psi(1;\beta) = 0$. The probability space is denoted in this case by $(\Omega, \mathscr{F}, \mathbb{Q}_\beta)$ (the letter $\mathbb{Q}$ to emphasize the fact that we consider risk neutral models and the parameter β indicates the skewness of the model).

Remark 2.1. The risk neutral condition in the definition reads

$$\gamma(\beta) = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - h(y))e^{\beta y}\Pi_0(dy). \tag{2.5}$$

We call β the skewness parameter, as it explains the departure of the model from symmetry in two senses: for $\beta = 0$, we obtain a symmetric Lévy process, that shows that the discounted log price is a symmetric process. Second, as shown in Fajardo & Mordecki (2006), for $\beta = -1/2$ the model exhibits a symmetric implied volatility curve, when plotted in terms of the log-moneyness.

2.1. *Implied volatility symmetry in skewed models*

We now consider Call and Put European options with strike K and expiration T, and prices $\mathbb{C} = \mathbb{C}(x, \sigma, \beta)$ and $\mathbb{P}(x, \sigma, \beta)$, respectively, where $x = \log(K/F)$ is the log-moneyness and $F = S_0 e^{rT}$ is the futures price. The dual market model has a risky asset driven by the dual process $\{\widetilde{X}_t = -X_t\}_{t\geq 0}$, and a dual measure $\widetilde{\mathbb{Q}}_\beta$, defined by

$$\frac{d\widetilde{\mathbb{Q}}_\beta}{d\mathbb{Q}_\beta} = e^{X_T}. \tag{2.6}$$

The corresponding dual characteristic exponent of $\widetilde{X}$ under $\widetilde{\mathbb{Q}}_\beta$ is $\widetilde{\psi}(z;\beta) = \psi(1-z;\beta) - \psi(1;\beta) = \psi(1-z;\beta)$. The interesting fact is that if X is a skewed process with parameter β, then $\widetilde{X}$ is a skewed process with parameter $-(\beta+1)$ (see Fajardo & Mordecki (2006)):

$$\widetilde{\psi}(z;\beta) = \psi(z;-(\beta+1)). \tag{2.7}$$

Due to this fact, the Put–Call duality found in Fajardo & Mordecki (2006) reads for skewed models as

$$\mathbb{C}(x,\sigma,\beta) = e^x \mathbb{P}(-x,\sigma,-(\beta+1)). \tag{2.8}$$

We have then the following results for skewed models.

Proposition 2.1. *Consider a skewed Lévy process X with parameters σ, β. The implied volatility surface $\mathbf{v} = \mathbf{v}(x,\beta)$ is symmetric with respect to the point $(0, -1/2)$:*

$$\mathbf{v}(x,\beta) = \mathbf{v}(-x,-(\beta+1)). \tag{2.9}$$

See, Fig. 1, where we plot the implied volatility function for the Variance Gamma model in terms of x and β.

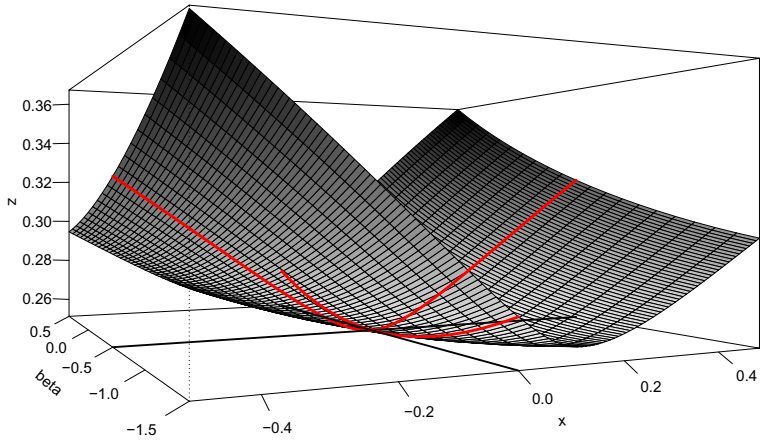


Fig. 1. Variance Gamma Implied volatility as function of x and β. Other parameters: $\alpha = 5$, $\lambda = 1$, $T = 1$, $r = 0.05$.

3. Implied Volatility Behavior

For the most common skewed models in the literature (see Sec. 4), we observe the growth of implied volatility in terms of β when $\beta > -1/2$ and $x > 0$. Consistently with the implied volatility symmetry of Proposition 3.1, we observe the decline when $\beta < -1/2$ and $x < 0$. For example, for the Variance Gamma model, this behavior is shown in Fig. 2. Although this behavior is not a general characteristic of skewed models, as the example in Sec. 4.3 shows, we are able to prove this monotonicity property in a neighborhood of $(x, \beta) = (0, -1/2)$.

This result is accomplished through the computation of the cross-partial derivative of the implied volatility function. For the existence of the derivative with respect to β we need the following condition:

$$\int_{y\geq 1} e^{(1/2+\delta)y}\Pi_0(dy) < \infty, \quad \text{for some } \delta > 0. \tag{3.1}$$

In order to be able to differentiate the implied volatility with respect to x, we need to have a continuous density. The existence of a continuous density $f(x) = f(x;\beta)$ holds true under the condition

$$\int_{iv+\mathbb{R}} |ze^{T\psi(-iz;\beta)}|dz < \infty, \quad \text{for some } v > 1, \tag{3.2}$$

as follows from §27 and §28 in Sato (1999).

Under these two conditions, we obtain two results involving the implied volatility. The first one: the implied volatility as a function of x, in a neighborhood of $x = 0$, is increasing when β is in a right-neighborhood of $-1/2$ and it is decreasing

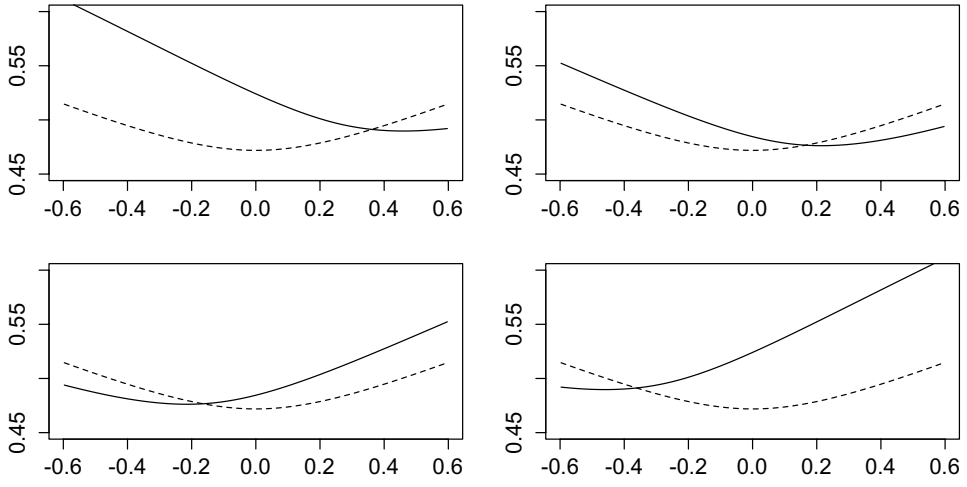


Fig. 2. Variance Gamma implied volatility in terms of x. Dotted line $\beta = -0.5$. Continuous line: top left $\beta = -1.5$, top right $\beta = -1$, bottom left $\beta = 0$, bottom right $\beta = 0.5$. Other parameters: $\alpha = 4$, $\lambda = 2$, $T = 1$, $r = 0.05$.

when β is in a left-neighborhood of $-1/2$. The second one: the implied volatility as a function of β, in a neighborhood of $\beta = -1/2$, is increasing when x is in a right-neighborhood of $x = 0$ and is decreasing when x is in a left-neighborhood of $x = 0$.

Proposition 3.1. *Let be X a skewed model. Assume that conditions* (3.1) *and* (3.2) *hold. Then, for any maturity T*

(1) *There exists $\varepsilon > 0$ such that $\partial_x \mathbf{v}(0, \beta) > 0$ if $\beta \in (-1/2, -1/2 + \varepsilon)$ and $\partial_x \mathbf{v}(0, \beta) < 0$ if $\beta \in (-1/2 - \varepsilon, -1/2)$.*
(2) *There exists $\varepsilon > 0$ such that $\partial_\beta \mathbf{v}(x, -1/2) > 0$ if $x \in (0, \varepsilon)$, and $\partial_\beta \mathbf{v}(x, -1/2) < 0$ if $x \in (-\varepsilon, 0)$.*

This result is obtained from the study of the cross-partial derivative in Appendix A.

4. Examples

The purpose of introducing skewed models is to provide a tool for visual analysis of implied volatility curves and surfaces. The parameter β gains relevance as it represents the quantitative skew, and more important, links the characteristic exponent and its Lévy measure with the shape of the implied volatility curve. We show that many popular models are skewed models, and also present a final example with "bouncing" implied volatility that shows that certain theoretical properties are not available for general skewed models and should be analyzed for each particular model.

4.1. *Infinite activity models*

The generalized hyperbolic model is extensively used in financial modeling, see for instance Eberlein & Keller (1995) or Eberlein (2001), where it is denoted by $GH(\lambda, \alpha, \beta, \delta, \mu)$, characterized by the previous five parameters. They are skewed models with skewness parameter β. The density of X_T (the dependence on T is not explicit) is given by

$$f(x; \beta) = a(\lambda, \alpha, \beta, \delta) e^{\beta(x-\mu)} \frac{K_{\lambda-1/2}(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{(\sqrt{\delta^2 + (x-\mu)^2})^{1/2-\lambda}}, \tag{4.1}$$

where

$$a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}, \tag{4.2}$$

is a norming constant and

$$K_\nu = \frac{1}{2}\int_0^\infty y^{\nu-1} \exp\left(z\left(y + \frac{1}{y}\right)\right) dy, \tag{4.3}$$

is the modified Bessel function of the third kind with index ν (see Eberlein (2001)). This formulation allows to see that for $\delta > 0$ the density is infinitely differentiable.

A relevant limit case not included in this differentiability analysis is the Variance Gamma model. It is the limit of the previous model as $\delta \to 0$. To analyze the differentiability, we observe that the characteristic function of the Variance Gamma model is given by

$$\mathbb{E}\, e^{zX_T} = e^{\mu z}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{2T\lambda}. \tag{4.4}$$

In consequence, for $\lambda T > 1/2$ condition (3.2) gives the existence of a continuous density. For a more detailed analysis of the regularity of the Variance Gamma and other Lévy processes see Cont & Voltchkova (2005). Other related skewed model is the Meixner model (that has a continuous density). In Fig. 2, we show the shape of the implied volatility for the VG model in terms of β.

4.2. *Jump-diffusion models*

Observe that any jump-diffusion model with positive diffusion component ($\sigma > 0$) has a differentiable density of any order (use Theorem 4 V.4 in Feller (1971) and Theorem 2.27 in Folland (1999)). The Merton model has a Lévy measure of the type $\Pi(dy) = e^{\beta y}\Pi_0(dy)$ with Π_0 symmetric, but depending on β:

$$\Pi(dy) = \lambda\frac{1}{\sqrt{2\pi}\sigma_J}\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_J}{\sigma_J}\right)^2\right\}dy \tag{4.5}$$

$$= (\lambda e^{-\frac{1}{2}(\frac{\mu_J}{\sigma_J})^2})e^{\beta y}\frac{1}{\sqrt{2\pi}\sigma_J}\exp\left\{-\frac{y^2}{2\sigma_J^2}\right\}dy, \tag{4.6}$$

where $\beta = \mu_J/\sigma_J^2$ and μ_J depends on β. For this reason, we adapt the Merton model re-parametrizing its Lévy measure.

We define the *skewed Merton model* as a Lévy process with triplet $(\gamma, \sigma, \Pi(dy))$, where

$$\Pi(dy) = \lambda e^{\beta y}\frac{1}{\sqrt{2\pi}\sigma_J}\exp\left\{-\frac{y^2}{2\sigma_J^2}\right\}dy. \tag{4.7}$$

In Fig. 3, we show the behavior of the implied volatility in a neighborhood of $\beta = -1/2$.

The Kou model has a jump density given by $g(y) = p\theta_1 e^{\theta_1 y}\mathbf{1}_{\{y\le 0\}} + (1-p)\theta_2 e^{-\theta_2 y}\mathbf{1}_{\{y>0\}}$, where $p \in (0,1)$ and $\theta_1, \theta_2 > 0$. The following particular case constitutes a skewed model.

We define the *skewed Kou model* as a Lévy process with triplet $(\gamma, \sigma, \Pi(dy))$, where

$$\Pi(dy) = \lambda e^{\beta y - \alpha|y|}dy. \tag{4.8}$$

The relations between the parameters are $\lambda = p\theta_1 = (1-p)\theta_2$, $\theta_1 = \beta + \alpha$ and $\theta_2 = \beta - \alpha$.

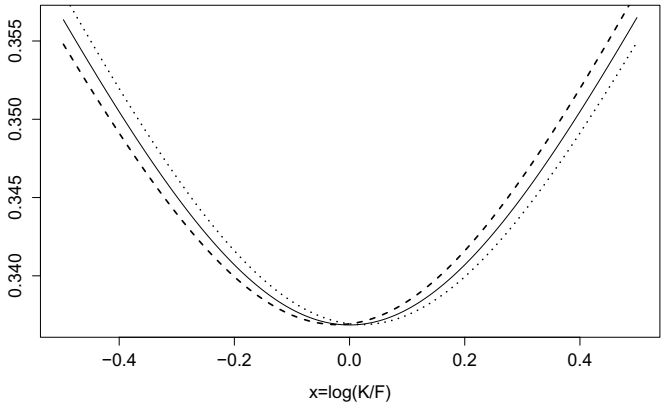


Fig. 3. Skewed Merton Implied volatility with: $T = 1$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 2$, $\sigma_J = 0.2$. Continuous line $\beta = -0.5$, dotted line: $\beta = -0.6$, dashed line: $\beta = -0.4$.

4.3. *Bouncing volatility model*

In this section, we show an example of a Lévy process where the corresponding model verifies the regularity conditions of Proposition 3.1, but does not verify $\partial \mathbf{v}(0,\beta)/\partial\beta > 0$ for all $\beta > -1/2$ neither $\partial \mathbf{v}(x,-1/2)/\partial\beta > 0$ for all $x > 0$. This model is Example 2 in Cont & Voltchkova (2005).

Definition 4.1 (Bouncing process). We define the *bouncing process* as a Lévy process where the jump measure is given by

$$\Pi_\beta(dy) = \lambda e^{\beta y}(\delta_a(y) + \delta_{-a}(y))dy. \tag{4.9}$$

If the bouncing process has a positive diffusion part, then it satisfies the assumptions of Proposition 3.1. However, we show in Fig. 5 that $\partial_x \mathbf{v}(0,\beta)$ is not positive,

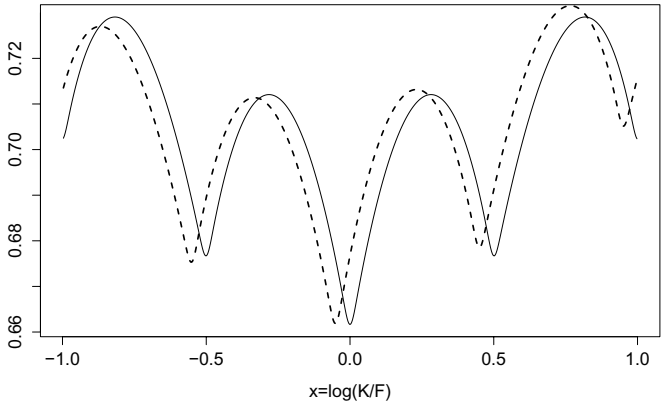


Fig. 4. Implied Volatility under diffusion with a two-sided Poisson jump process with $\sigma = 0.01$, $a = 0.5$ and $\lambda = 2$. Continuous line: $\beta = -0.5$, dashed line: $\beta = -0.4$.

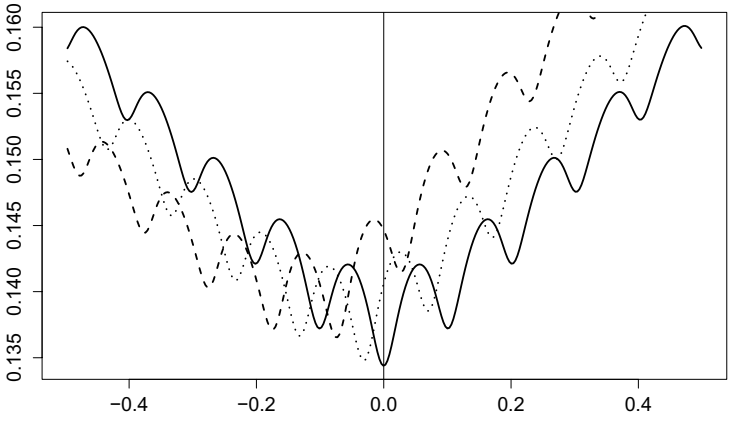


Fig. 5. Implied Volatility under diffusion with a two-sided Poisson jump process with $\sigma = 0.01$, $a = 0.1$ $\lambda = 2$, $T = 1$ and $r = 0.05$. Continuous line: $\beta = -0.5$, dotted line: $\beta = 1$ and dashed line: $\beta = 3$.

neither negative[a] for all $\beta > -1/2$. And we show in Fig. 4 that $\mathbf{v}$ has local minima at $x = a$ and $x = -a$, where the results of Proposition 3.1 cannot be extended to the entire real line.

It is clear that this setting is not interesting as a realistic model, however it is an important example when we try to derive results on implied volatility in the context of Lévy processes.

5. Conclusions

The skewed Lévy models introduced in this paper are characterized by a symmetric jump measure multiplied by a damping exponential factor that depends on a parameter β. This parameter quantifies the skewness of the model, and its variation captures the typical smirk feature observed in implied volatility curves. We show that skewed models with continuous density exhibit a precise monotonicity behavior of the implied volatility surface, as a function of the log-moneyness and the skewness parameter. These results are independent of the option's maturity, and apply to many main models in literature. Other popular models can easily be adapted to be skewed. Finally, we present a simple example that shows that the obtained monotonicity behavior cannot be further extended.

Our proposal establishes a link between the visual analysis of the implied volatility curves and its mathematical modeling, through the skewness parameter β. It also raises at least two questions regarding monotonicity properties of the implied volatility surface: (i) about the possibility of proving that the surface has a minimum

[a]Observe that here we consider any maturity, in contrast with the results obtained in Gerhold *et al.* (2014) for small-maturity options.

at the point $x = 0$ and $\beta = -1/2$, (ii) about obtaining the monotonicity property of Proposition 3.1 for a larger set of values for particular models.

Appendix A

A.1. *Proof of Proposition* 3.1

In order to formulate the following results, we use the Black–Scholes formula in terms of x as

$$\mathrm{BS}(K_x, \mathbf{v}(x,\beta)) = S_0 N(d_1) - K_x e^{-rT} N(d_2),$$

where

$$d_1 = \frac{-x + \mathbf{v}^2(x,\beta)\dfrac{T}{2}}{\mathbf{v}(x,\beta)\sqrt{T}} \quad \text{and} \quad d_2 = \frac{-x - \mathbf{v}^2(x,\beta)\dfrac{T}{2}}{\mathbf{v}(x,\beta)\sqrt{T}}, \tag{A.1}$$

and the strike in terms of the log-moneyness x is

$$K_x = Fe^x = S_0 e^{x+rT}.$$

Now, differentiating with respect to x, we obtain

$$\frac{\partial \mathrm{BS}(K_x, \mathbf{v}(x,\beta))}{\partial x} = \frac{\partial \mathrm{BS}(K_x, \mathbf{v})}{\partial K_x}\frac{\partial K_x}{\partial x} + \frac{\partial \mathrm{BS}(K_x, \mathbf{v})}{\partial \mathbf{v}}\frac{\partial \mathbf{v}(x,\beta)}{\partial x} \tag{A.2}$$

$$= -S_0 e^x N(d_2) + \sqrt{T}\phi(d_1)\frac{\partial \mathbf{v}(x,\beta)}{\partial x}, \tag{A.3}$$

where N and ϕ denote the standard normal cdf and density, respectively. On the other hand, the value of a Call option under the Lévy process is denoted by $\mathbb{C}$, and given by

$$\mathbb{C}(x,\beta) = e^{-rT}\,\mathbb{E}_\beta(S_T - K_x)^+ = S_0 \int_x^{+\infty} (e^y - e^x) d\mathbb{Q}_\beta. \tag{A.4}$$

In order to obtain the differentiability of the implied volatility with respect to x, we need the differentiability of the function $\mathbb{C}(x,\beta)$ with respect to x. In view of (A.4), the existence of a continuous density of the measure $\mathbb{Q}_\beta$ is a sufficient condition for the differentiability of $\mathbb{C}$, and this fact holds under (3.2). We then have

$$\frac{\partial \mathbb{C}(x,\beta)}{\partial x} = -S_0 e^x \mathbb{Q}_\beta(X_T > x), \tag{A.5}$$

and from the equality $\mathrm{BS}(K_x, \mathbf{v}(x,\beta)) = \mathbb{C}(x,\beta)$, we obtain

$$\frac{\partial \mathbf{v}(x,\beta)}{\partial x} = \frac{S_0 e^x (N(d_2) - \mathbb{Q}_\beta(X_T > x))}{\phi(d_1)\sqrt{T}} = S_0 \frac{N(d_2) - \mathbb{Q}_\beta(X_T > x)}{\phi(d_2)\sqrt{T}}. \tag{A.6}$$

For the proof of Proposition 3.1, we introduce some preliminary results.

Lemma A.1. *Let $\{X_t, \mathbb{Q}_\beta\}$ be a skewed model. Then,*

$$\psi(z;\beta) = \psi(z+\beta;0) - \psi(\beta;0) - z\mu_\beta, \tag{A.7}$$

where $\mu_\beta = \beta\sigma^2 + \int_{\mathbb{R}}(e^y - 1)(e^{\beta y} - 1)\Pi_0(dy)$. *If there exists a continuous density we have*

$$f_t(x;\beta) = e^{\beta(x+t\mu_\beta)-t\psi(\beta;0)} f_t(x + t\mu_\beta; 0). \tag{A.8}$$

Proof. We define the measure $\hat{\mathbb{Q}}_\beta$ such that

$$\frac{d\hat{\mathbb{Q}}_\beta}{d\mathbb{Q}_0} = e^{\beta X_t - t\psi(\beta;0)}. \tag{A.9}$$

Then, from the Esscher transform, $\{X_t, \hat{\mathbb{Q}}_\beta\}$ is a Lévy process with triplet $(\hat{\gamma}, \sigma, e^{\beta y}\Pi_0(dy))$ with $\hat{\gamma} = \beta\sigma^2 + \gamma_0 + \int_{\mathbb{R}}(e^{\beta y} - 1)h(y)\Pi_0(dy)$. We observe that $\{X_t, \hat{\mathbb{Q}}_\beta\}$ and $\{X_t, \mathbb{Q}_\beta\}$ are both Lévy processes that differ only in the drift term, therefore

$$\{X_t, \mathbb{Q}_\beta\} \overset{\mathscr{L}}{=} \{X_t - t\mu_\beta, \hat{\mathbb{Q}}_\beta\}, \tag{A.10}$$

where $\mu_\beta = \beta\sigma^2 + \gamma_0 - \gamma_\beta + \int_{\mathbb{R}}(e^{\beta y} - 1)h(y)\Pi_0(dy)$. From (A.9) and (A.10), we obtain

$$\mathbb{E}_\beta(e^{zX_t}) = \hat{\mathbb{E}}_\beta(e^{z[X_t - t\mu_\beta]}) = \mathbb{E}_0(e^{(z+\beta)X_t - t[\psi(\beta;0)+z\mu_\beta]}), \tag{A.11}$$

and from (A.11) the result follows. On the other hand, if a continuous density exists we have

$$\begin{aligned}
f_t(x;\beta) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx} e^{t\psi(iz;\beta)} dz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx} e^{t[\psi(iz+\beta;\beta) - \psi(\beta;0) - iz\mu_\beta]} dz \\
&= e^{-t\psi(\beta;0)} \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iz(x+t\mu_\beta)} e^{t\psi(iz+\beta;0)} dz \\
&= e^{-t\psi(\beta;0)} \frac{1}{2\pi}\int_{\mathbb{R}} e^{-(iw-\beta)(x+t\mu_\beta)} e^{t\psi(iw;0)} dw \\
&= e^{\beta(x+t\mu_\beta)-t\psi(\beta;0)} \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iw(x+t\mu_\beta)} e^{t\psi(iw;0)} dw \\
&= e^{\beta(x+t\mu_\beta)-t\psi(\beta;0)} f_t(x + t\mu_\beta; 0).
\end{aligned}$$

□

We observe that for $\beta = 0$, the process $X_t - t\gamma_0$ is symmetric because of the symmetry of the jump measure. If there exists a density, we have

$$f_t(x + t\gamma_0; 0) = f_t(-x + t\gamma_0; 0). \tag{A.12}$$

Lemma A.2. *If* $\{X_t, \mathbb{Q}_\beta\}$ *is a skewed model with continuous density, then*

$$f_t\left(x; -\frac{1}{2}\right) \leq f_t\left(0; -\frac{1}{2}\right) e^{-1/2x} \quad \forall\, x \in \mathbb{R}. \tag{A.13}$$

Proof. Observe that

$$\begin{aligned}\mu_{-1/2} &= -\sigma^2/2 + \gamma_0 - \gamma_{-1/2} + \int_{\mathbb{R}} h(y)(e^{-y/2} - 1)\Pi_0(dy) \\ &= \gamma_0 + \int_{\mathbb{R}} ((e^y - 1)e^{-y/2} - h(y))\Pi_0(dy) = \gamma_0, \end{aligned} \tag{A.14}$$

because $(e^y - 1)e^{-y/2} - h(y)$ is an odd function. On the other hand,

$$\begin{aligned} f_t(x + t\gamma_0; 0) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iz(x+t\gamma_0)} e^{t\psi(iz;0)} dz \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx} e^{t[-iz\gamma_0 + \psi(iz;0)]} dz \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \cos(zx) e^{t[-iz\gamma_0 + \psi(iz;0)]} dz \qquad \text{(A.15)} \\ &\leq \frac{1}{2\pi}\int_{\mathbb{R}} e^{t[-iz\gamma_0 + \psi(iz;0)]} dz \\ &= f_t(t\gamma_0; 0), \qquad \text{(A.16)} \end{aligned}$$

where in (A.15) we use that the imaginary part of $-iz\gamma_0 + \psi(iz;0) = -z^2\sigma^2/2 + \int_{\mathbb{R}}(e^{izy} - 1 - izh(y))\Pi_0(dy)$ vanishes.

From Lemmas A.1 and (A.14), after some computations, we have

$$\begin{aligned} f_t\left(x; -\frac{1}{2}\right) &= e^{-1/2(x+t\gamma_0) - t\psi(-1/2;0)} f_t(x + t\gamma_0; 0) \\ &= f_t\left(0; -\frac{1}{2}\right) e^{-x/2} \frac{f_t(x + t\gamma_0; 0)}{f_t(t\gamma_0; 0)}. \end{aligned} \tag{A.17}$$

Observe that from (A.16), $f_t(t\gamma_0; 0) \neq 0$. From (A.16) and (A.17), we obtain the result. □

In order to differentiate with respect to β observe that in view of (3.1) and taking into account the symmetry property of the measure Π_0, the characteristic exponent is defined for $\beta \in (-(1/2 + \delta), 1/2 + \delta)$, what is enough for our purpose, as the set

$$\mathbb{C}_0 = \left\{ z = p + iq \in \mathbb{C} \colon \int_{\{|y|>1\}} e^{py} \Pi_0(dy) < \infty \right\}. \tag{A.18}$$

is a vertical strip in the complex plane, that contains the line $i\mathbb{R}$, and consists of all complex numbers $z = p + iq$ such that $\mathbb{E}\, e^{pX_t} < \infty$ for some $t > 0$. In other words, the characteristic exponent $\psi(z; \beta)$ can be defined as an analytic function on the strip $\mathbb{C}_0$. The corresponding proofs can be found in §25 in Sato (1999).

The next result admit an interpretation in terms of a digital call option. Under a skewed model, the price of a digital call *at the money* is decreasing with β at $\beta = -1/2$.

Lemma A.3. *If* $\{X_t, \mathbb{Q}_\beta\}$ *is a skewed model, then*

$$\left.\frac{\partial \mathbb{Q}_\beta(X_t > 0)}{\partial \beta}\right|_{-1/2} < 0. \tag{A.19}$$

Proof. For simplicity, we denote $f_t(y) := f_t(y; -1/2)$, the density of the distribution of X_t at $\beta = -1/2$. From the Lewis formula for a Digital Call option (see Lewis (2001)), for $v > 0$, we have

$$\mathbb{Q}_\beta(X_t > x) = -\frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{izx}e^{t\psi(-iz;\beta)}}{iz} dz. \tag{A.20}$$

Then, differentiating with respect to β, we obtain

$$\left.\frac{\partial \mathbb{Q}_\beta(X_t > 0)}{\partial \beta}\right|_{-1/2}$$

$$= -\frac{t}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{t\psi(-iz;\beta)}}{iz}\left.\frac{\partial \psi(-iz;\beta)}{\partial \beta}\right|_{-1/2} dz \tag{A.21}$$

$$= -\frac{t}{2\pi}\int_{iv+\mathbb{R}}\int_{\mathbb{R}} \left[\frac{e^{t\psi(-iz;\beta)}}{iz}(e^{-izy} - 1 + iz(e^y - 1))ye^{-y/2}\right]\Pi_0(dy)dz \tag{A.22}$$

$$= t\int_{\mathbb{R}} ye^{-y/2}\left[\mathbb{Q}_{-1/2}(X_t > -y) - \mathbb{Q}_{-1/2}(X_t > 0) - (e^y - 1)f_t(0)\right]\Pi_0(dy) \tag{A.23}$$

$$= t\int_0^{+\infty} ye^{-y/2}[\mathbb{Q}_{-1/2}(X_t > -y) - \mathbb{Q}_{-1/2}(X_t > 0) - (e^y - 1)f_t(0)]\Pi_0(dy)$$

$$- t\int_0^{+\infty} ye^{y/2}[\mathbb{Q}_{-1/2}(X_t > y) - \mathbb{Q}_{-1/2}(X_t > 0) - (e^{-y} - 1)f_t(0)]\Pi_0(dy) \tag{A.24}$$

$$= t\int_0^{+\infty} ye^{-y/2}[\mathbb{Q}_{-1/2}(-y < X_t < 0) - (e^y - 1)f_t(0)]\Pi_0(dy)$$

$$- t\int_0^{+\infty} ye^{y/2}[-\mathbb{Q}_{-1/2}(0 < X_t < y) - (e^{-y} - 1)f_t(0)]\Pi_0(dy)$$

$$= t\int_0^{+\infty} ye^{-y/2}[\mathbb{Q}_{-1/2}(-y < X_t < 0) + e^y\mathbb{Q}_{-1/2}(0 < X_t < y)$$

$$- 2(e^y - 1)f_t(0)]\Pi_0(dy)].$$

To obtain (A.21), (A.22) and (A.23), we use that the integrand in (A.22) is in $L^1(\Pi_0(dy)\times dz)$ which is a consequence of (3.2) and in (A.24), we use the symmetry property of Π_0.

Let $g(y) = \mathbb{Q}_{-1/2}(-y < X_t < 0) + e^y \mathbb{Q}_{-1/2}(0 < X_t < y) - 2(e^y - 1)f_t(0)$. From the dual model we have $f_t(y) = e^{-y} f_t(-y)$, then

$$\begin{aligned} g'(y) &= f_t(-y) + e^y \mathbb{Q}_{-1/2}(0 < X_t < y) + e^y f_t(y) - 2e^y f_t(0) \\ &= e^y [\mathbb{Q}_{-1/2}(0 < X_t < y) + 2[f_t(y) - f_t(0)]]. \end{aligned} \tag{A.25}$$

From Lemma A.2 and (A.25), we have

$$\begin{aligned} e^{-y} g'(y) &= \int_0^y f_t(s)ds + 2[f_t(y) - f_t(0)] \\ &\leq f_t(0) \int_0^y e^{-1/2\, s} ds + 2[f_t(y) - f_t(0)] \\ &= -2f_t(0)e^{-1/2\, y} + 2f_t(y) \leq 0. \end{aligned} \tag{A.26}$$

Observe that $f_t(0)e^{-1/2\, y}$ cannot be equal to $f_t(y)$ for every $y > 0$ because in this case, from $f_t(y) = e^{-y} f_t(-y)$, we obtain $f_t(-y) = f_t(0)e^{1/2\, y}$ for $y > 0$ that is a contradiction. Then, there exists an open interval $I \subset \mathbb{R}^+$ such that $g'(y) < 0$ for $y \in I$.

Since g is decreasing, $g(y) \leq g(0) = 0$ for $y > 0$ and $g(y) < 0$ for $y \in I$, that implies

$$\left.\frac{\partial \mathbb{Q}_\beta(X_t > 0)}{\partial \beta}\right|_{-1/2} = t \int_0^{+\infty} y e^{-y/2} g(y) \Pi_0(dy) < 0. \tag{A.27}$$

□

Proof. (Proof of Proposition 3.1)

From (A.6), we obtain

$$\frac{\partial^2 \mathbf{v}(x,\beta)}{\partial \beta \partial x} = \frac{\partial}{\partial \beta} \frac{\partial \mathbf{v}(x,\beta)}{\partial x} \tag{A.28}$$

$$= S_0 \frac{\left[\phi(d_2)\dfrac{\partial d_2}{\partial \beta} - \dfrac{\partial \mathbb{Q}_\beta(X_T > x)}{\partial \beta}\right] - [N(d_2) - \mathbb{Q}_\beta(X_T > x)](-d_2)\dfrac{\partial d_2}{\partial \beta}}{\phi(d_2)\sqrt{T}}. \tag{A.29}$$

We observe that

$$\frac{\partial d_2}{\partial \beta} = -\frac{T^{3/2}\mathbf{v}^2(x,\beta)\dfrac{\partial \mathbf{v}(x,\beta)}{\partial \beta} - (x + \mathbf{v}^2(x,\beta)\dfrac{T}{2})\dfrac{\partial \mathbf{v}(x,\beta)}{\partial \beta}}{\mathbf{v}^2(x,\beta)T}. \tag{A.30}$$

From Proposition 2.1, we have $\partial_\beta \mathbf{v}(0,-1/2)=0$ and then $\partial_\beta d_2(0,-1/2)=0$, where we conclude that, from Lemma A.3

$$\frac{\partial^2 \mathbf{v}\left(0,-\frac{1}{2}\right)}{\partial\beta\partial x} = -\frac{S_0}{\phi\left(\mathbf{v}\left(0,-\frac{1}{2}\right)\frac{\sqrt{T}}{2}\right)\sqrt{T}}\frac{\partial \mathbb{Q}_\beta(X_T>0)}{\partial\beta}\Bigg|_{\beta=-1/2} > 0. \quad \text{(A.31)}$$

From (A.31), we have

$$\frac{\partial}{\partial\beta}\frac{\partial \mathbf{v}(0,\beta)}{\partial x}\Bigg|_{\beta=-1/2} > 0, \quad \text{(A.32)}$$

then, there exists $\varepsilon>0$ such that $\partial_x \mathbf{v}(0,\beta)$ is increasing as a function of $\beta \in (-\frac{1}{2}-\varepsilon; -\frac{1}{2}+\varepsilon)$. The first part of Proposition 3.1 is obtained from $\partial_x \mathbf{v}(0,-1/2)=0$.

Under condition (3.2), using expression (A.4) and Lemma A.1, we obtain $\frac{\partial^2 \mathbf{v}(x,\beta)}{\partial\beta\partial x} = \frac{\partial^2 \mathbf{v}(x,\beta)}{\partial x\partial\beta}$. Then from (A.31), we have

$$\frac{\partial}{\partial x}\frac{\partial \mathbf{v}}{\partial\beta}\left(0,-\frac{1}{2}\right) = \frac{\partial^2 \mathbf{v}\left(0,-\frac{1}{2}\right)}{\partial x\partial\beta} > 0. \quad \text{(A.33)}$$

From (A.33), the second part follows analogously to the first. □

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