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OPTIMAL STOPPING FOR LÉVY PROCESSES WITH ONE-SIDED SOLUTIONS*

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Abstract. An explicit solution of an infinite horizon optimal stopping problem for a Lévy process with a nonmonotonic reward function is given, in terms of the overall supremum of the process, when the solution of the problem is one-sided. The results are obtained via the consideration of the *generalized averaging function* associated with the problem. The method, initially tailored to handle polynomial rewards, has a wide range of applications, as shown in the examples: optimal stopping problems for general polynomial rewards of degree two and three for spectrally negative processes, a quartic polynomial reward and Kou's process, a portfolio of call options, and a trigonometric payoff function. These last two examples are given for general Lévy processes.

Key words. optimal stopping, Lévy processes, one-sided solutions, polynomial rewards

AMS subject classifications. 60G40, 60G51, 62L15

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1. Introduction.

Since the seminal work of Darling, Liggett, and Taylor (1972), which gives the solution to the optimal stopping problem for random walks and reward functions of the form $g(x) = x^+$ and $g(x) = (e^x - 1)^+$, in terms of the distribution of the maximum of the random walk, the possibility of linking these two relevant problems in probability theory became clear: the optimal stopping problem and the computation of the distribution of the overall maximum of a random walk. The natural question that this work posed was the possibility of extending these results to more general classes of processes and to more general reward functions.

The first results for Lévy processes were obtained by Mordecki (2002a, b), where the similar corresponding problems for arbitrary Lévy processes are solved, using a discretization approximation argument, for the same reward functions, with the novelty of the consideration of the decreasing put reward $g(x) = (K - e^x)^+$, with solution in terms of the overall infimum of the process. The first results for general payoffs were obtained by Boyarchenko and Levendorskiĭ (2002). Namely, using the technique of the pseudodifferential operators, these authors obtained solutions to optimal stopping problems considering a large class of reward functions, making clear that the previously obtained results were not based on particular properties of the payoff function but only on the properties of the Lévy processes. Their approach is analytic, based on the decomposition of an operator, which is in certain sense equivalent to the Wiener–Hopf factorization, and imposes certain restrictions on the class of Lévy processes to which the results can be applied. For a general exposition of these results see also Boyarchenko and Levendorskiĭ (2002a). Afterward, Novikov and Shiryaev (2004) solved the optimal stopping problem for arbitrary random walks and reward functions of the form $g(x) = (x^+)^n$, in terms of the Appell polynomials, and Novikov and Shiryaev (2007) proposed the solution to the problem with a power function reward with real and positive exponent, for both random walks and Lévy processes. Alili and Kyprianou (2005) obtained a new proof of the main results in

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Mordecki (2002a) and Kyprianou and Surya (2005) got a generalization of the results in Novikov and Shiryaev (2004) for Lévy processes. These contributions were summarized in the monograph by Kyprianou (2006). Surya (2007) proposes a unified approach, consisting in finding an *averaging function* related to the reward function and the overall maximum of the process. He shows how in each known case, this function can be founded. On the way to the consideration of more general processes, Mordecki and Salminen (2007) obtained a representation of the value function for Hunt processes, that in the case of Lévy processes gave a representation in terms of the maximum of the process. Salminen (2011) applied the representation method (initiated in Salminen (1985)) to the problem of stopping with reward $g(x) = (x^+)^n$, finding the representing measure of the value function, and Mishura and Tomashyk (2011) considered the optimal stopping problem for a general polynomial reward and a random walk. Also in the context of general Markov processes, Christensen, Salminen, and Ta (2013) exploited the excessive property of the maximum to obtain a verification theorem.

It is now clear that the obtained results are mainly based on the probabilistic properties which random walks and Lévy processes share, i.e., the independence and homogeneity of increments, but not on the particular form of the reward functions. Nevertheless, in order to obtain explicit solutions a certain procedure of the ones described above should be applied. Summarizing, the optimal stopping problem for Lévy processes, once the distribution of the maximum is known, can be explicitly solved for call and put rewards (Mordecki (2002a)), linear rewards (Mordecki (2002b)), power functions with integer index (Novikov and Shiryaev (2004), Kyprianou and Surya (2005)), and power function with positive real exponent (Novikov and Shiryaev (2007)).

In the present paper we are interested in obtaining explicit solutions of optimal stopping problems for general Lévy processes and more general payoff functions. In order to handle this problem, we propose to consider problems with one-sided solutions. We pose ourselves in this situation, as it seems that more general stopping regions (as unions of intervals) would require the knowledge of both factors in the Wiener–Hopf factorization (i.e., the distribution of the minimum and the maximum simultaneously) as suggested by the results by Mordecki and Salminen (2007). We are able to exhibit, to the best of our knowledge, the first explicit solutions of problems with nonmonotonous payoffs. In particular, we solve the optimal stopping problem for an arbitrary polynomial reward, generalizing the results by Novikov and Shiryaev (2004) and Kyprianou and Surya (2005), which consider the case of monomials, and also Mishura and Tomashyk (2011), which present the first results for polynomials. (This last paper is written for random walks but can be easily generalized for Lévy processes.)

In order to find the solutions, and to perform the corresponding proofs, we consider the *generalized averaging problem*, showing that it is not necessary to find the averaging function in the whole real line, as proposed by Surya (2007), giving the possibility of finding explicit solutions in a wide set of new situations, which we illustrate with general polynomial rewards, a portfolio of call options, and an example with trigonometric functions.

The content of the paper is as follows. In section 2 we formulate the problem and prove the main results. In section 3 we specialize to polynomial rewards, discuss in detail the optimal stopping problem for Brownian motion and general quadratic and cubic polynomials, and also solve explicitly the optimal stopping problem for a quartic polynomial and a Kou's process. In section 4 we present two more examples:

the valuation of a portfolio of American perpetual call options in a Lévy market and a problem with a trigonometric payoff.

2. Formulation of the problem and main results. Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}_x)$ departing from $X_0 = x$. For $z \in i\mathbb{R}$, the Lévy–Khintchine formula states $\mathbf{E}e^{zX_t} = e^{t\psi(z)}$ with

$$\psi(z) = az + \frac{\sigma^2}{2}z^2 + \int_{\mathbb{R}} (e^{zy} - 1 - zh(y))\,\Pi(dy), \tag{1}$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dy)$ that satisfy $\int_{\mathbb{R}}(1 \wedge y^2)\Pi(dy) < \infty$ conform the characteristic triplet (a, σ, Π) of the process. Here $h(y) = y\mathbf{1}_{\{|y|<1\}}$ is a truncation function. Given the stochastic basis $\mathcal{B}$ the set of stopping times is the set of random variables

$$\mathcal{M} = \{\tau\colon \Omega \to [0,\infty] \text{ such that } \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Observe that we allow the possibility $\tau = \infty$ as for several optimal stopping problems, the optimal stopping time is within this class. A key rôle in the solution of the problem is played by the overall maximum of the process, defined for $r \geq 0$ by

$$M = \sup\{X_t\colon 0 \leq t \leq e(r)\}, \tag{2}$$

where $e(r)$ is an exponential random variable of parameter $r > 0$, independent of X, and we assume $e(0) = \infty$. We further assume throughout the paper that M is a proper random variable. This entails either that $r > 0$ or that $X = \{X_t\}_{t\geq 0}$ drifts to $-\infty$ when $r = 0$ and that $\mathbf{P}_x(M > x) > 0$, excluding the case of the negative of a subordinator, that gives $M = x$ a.s.

Given a nonnegative payoff function $g(x)$, a process $\{X_t\}_{t\geq 0}$ departing from $X_0 = x$ adapted to a filtration $\mathbf{F}$, and a discount factor $r \geq 0$, the optimal stopping problem consists in finding the value function $V(x)$ and the optimal stopping rule τ^* such that

$$V(x) = \sup_{\tau\in\mathcal{M}} \mathbf{E}_x(e^{-r\tau}g(X_\tau)) = \mathbf{E}_x(e^{-r\tau^*}g(X_{\tau^*})). \tag{3}$$

We assume that the payoff received in the set $\{\omega\colon \tau(\omega) = \infty\}$ is

$$\limsup_{t\to\infty} e^{-rt}g(X_t)$$

(see Shiryaev (2008)). In the present paper we are interested in problems with a one-sided solution, i.e., such that the optimal stopping rule is of the form

$$\tau^* = \inf\{t \geq 0\colon X_t \geq x^*\} \tag{4}$$

for some critical threshold x^*. For this reason we assume that

$$\limsup_{x\to-\infty} g(x) = \lim_{x\to-\infty} g(x) = 0.$$

The *generalized averaging problem* for optimal stopping consists in, given a reward function g, a discount rate $r \geq 0$, and a Lévy process with maximum M as in (2), determining a threshold x^* and a function $G^*\colon [x^*,\infty) \to [0,\infty)$ such that

$$\mathbf{E}_x(G^*(M)) = g(x) \text{ for all } x \geq x^*. \tag{5}$$

This equality, combined with the strong Markov property and the invariance of increments of Lévy processes, gives a fluctuation identity that allows us to write the value function of problem (3) in terms of M (see (11) in Lemma 1 below).

THEOREM 1. *Consider a Lévy process $X = \{X_t\}_{t\geq 0}$, a discount rate $r \geq 0$, and a reward function $g\colon \mathbb{R} \to [0,\infty)$ such that $\lim_{x\to-\infty} g(x) = 0$. Assume that there exists a point x^* and a nondecreasing function $G^*\colon [x^*,\infty) \to [0,\infty)$ such that*

$$\mathbf{E}_x(G^*(M)) = g(x) \text{ for all } x \geq x^*.$$

Define the function

$$G(x) = \begin{cases} G^*(x) & \text{if } x \geq x^*, \\ 0 & \text{if } x < x^* \end{cases} \tag{6}$$

and the function $V\colon \mathbb{R} \to \mathbb{R}$ by

$$V(x) = \mathbf{E}_x(G(M)) \quad \text{for all } x \in \mathbb{R}. \tag{7}$$

If the condition

$$V(x) \geq g(x) \quad \text{for all } x < x^* \tag{8}$$

is satisfied, then the optimal stopping problem (3) *has value function $V(x)$ in* (7)*, and* (4) *is an optimal stopping time for the problem.*

Remark 1. Compared to Theorem 4.2 in Surya (2007), Theorem 1 does not require the averaging problem for g and M to be solved on the whole real line, only on a certain set of the form $[x^*,\infty)$. In the examples presented below we appreciate the new possibilities that this fact provides.

As usual in optimal stopping proofs we have to verify two statements:

$$V(x) = \mathbf{E}_x(e^{-r\tau^*} g(X_{\tau^*})), \tag{9}$$

$$V(x) \geq \mathbf{E}_x(e^{-r\tau} g(X_\tau)) \quad \text{for all } \tau \in \mathcal{M}. \tag{10}$$

These two statements are proved based on the following two lemmas, whose proofs follow, with the necessary modifications, the respective proofs of Surya (2007) and Christensen, Salminen, and Ta (2013).

LEMMA 1. *Consider a Lévy process X, a discount rate $r \geq 0$, a reward function g, a threshold x^*, an averaging function G^*, and the extended function G, all these elements as in Theorem* 1*. Then, for any $a \geq x^*$ and $x \in \mathbb{R}$,*

$$\mathbf{E}_x(G(M)\mathbf{1}_{\{M\geq a\}}) = \mathbf{E}_x(e^{-r\tau_a} g(X_{\tau_a})\mathbf{1}_{\{\tau_a<\infty\}}). \tag{11}$$

In particular, when $a = x^*$, for τ^* in (4)*, we have*

$$\mathbf{E}_x(G(M)\mathbf{1}_{\{M\geq x^*\}}) = \mathbf{E}_x(e^{-r\tau^*} g(X_{\tau^*})\mathbf{1}_{\{\tau^*<\infty\}}).$$

Proof. Consider, for $a \geq x^*$, a hitting time of the form

$$\tau_a = \inf\{t \geq 0\colon X_t \geq a\}.$$

As Lévy processes satisfy the homogeneity property of increments in time and space, conditionally to the σ-algebra $\mathcal{F}_{\tau_a}$, and on the set $\{\tau_a < \infty\}$, the process $\tilde{X}_s = X_{\tau_a+s} - X_{\tau_a}$ is independent of $\mathcal{F}_{\tau_a}$ and has the same distribution as X (see Theorem 7,

Chapter 4, in Skorokhod (1986)). Consider first the case $r = 0$. Taking conditional expectation w.r.t. the σ-algebra $\mathcal{F}_{\tau_a}$ and applying the iterated law of conditional expectation, we have

$$\begin{aligned}\mathbf{E}_x(G(M)\mathbf{1}_{\{M\geq a\}}) &= \mathbf{E}_x\Big(G\Big(\sup_{0\leq t<\infty} X_t\Big)\mathbf{1}_{\{\tau_a<\infty\}}\Big)\\ &= \mathbf{E}_x\Big(G\Big(X_{\tau_a}+\sup_{\tau_a\leq t<\infty}(X_t-X_{\tau_a})\Big)\mathbf{1}_{\{\tau_a<\infty\}}\Big)\\ &= \mathbf{E}_x\Big(\mathbf{E}_{X_{\tau_a}}\Big(G\Big(X_{\tau_a}+\sup_{0\leq s<\infty} X_s\Big)\mathbf{1}_{\{\tau_a<\infty\}}\Big)\Big) = \mathbf{E}_x\Big(g\left(X_{\tau_a}\right)\mathbf{1}_{\{\tau_a<\infty\}}\Big).\end{aligned}$$

We proceed now for $r > 0$. In this case, again, taking conditional expectation w.r.t. the σ-algebra $\mathcal{F}_{\tau_a}$ and applying the iterated law of conditional expectation and the memoryless property of exponential random variables, we get

$$\begin{aligned}\mathbf{E}_x(G(M)\mathbf{1}_{\{M\geq a\}}) &= \mathbf{E}_x\Big(G\Big(\sup_{0\leq t<e(r)} X_t\Big)\mathbf{1}_{\{\tau_a<e(r)\}}\Big)\\ &= \mathbf{E}_x\Big(G\Big(X_{\tau_a}+\sup_{\tau_a\leq t<e(r)}(X_t-X_{\tau_a})\Big)\mathbf{1}_{\{\tau_a<e(r)\}}\Big)\\ &= \mathbf{E}_x\Big(\int_{\tau_a}^{\infty} G\Big(X_{\tau_a}+\sup_{0\leq s<t-\tau_a}(X_{\tau_a+s}-X_{\tau_a})\Big)re^{-rt}dt\mathbf{1}_{\{\tau_a<\infty\}}\Big)\\ &= \mathbf{E}_x\Big(e^{-r\tau_a}\int_0^{\infty} G\Big(X_{\tau_a}+\sup_{0\leq s<v}(X_{\tau_a+s}-X_{\tau_a})\Big)re^{-rv}dv\mathbf{1}_{\{\tau_a<\infty\}}\Big)\\ &= \mathbf{E}_x\Big(\mathbf{E}_{X_{\tau_a}}\Big(e^{-r\tau_a}\int_0^{\infty} G\Big(X_{\tau_a}+\sup_{0\leq s<v} X_s\Big)re^{-rv}dv\mathbf{1}_{\{\tau_a<\infty\}}\Big)\Big)\\ &= \mathbf{E}_x\big(e^{-r\tau_a}\mathbf{E}_{X_{\tau_a}}G\big(X_{\tau_a}+M\big)\mathbf{1}_{\{\tau_a<\infty\}}\big) = \mathbf{E}_x(e^{-r\tau_a}g(X_{\tau_a})\mathbf{1}_{\{\tau_a<\infty\}}),\end{aligned}$$

concluding the proof. □

Remark 2. Fluctuation identities as presented in the previous lemma in the case of exponential or related to exponential functions have been obtained by Darling, Liggett, and Taylor (1972) for random walks and by Alili and Kyprianou (2005) for Lévy processes. In the case of power functions with positive integer exponent Novikov and Shiryaev (2004) introduced the Appel polynomials to obtain similar identities for random walks, and Kyprianou and Surya (2005) obtained the corresponding result for Lévy processes. The case of power functions with real positive exponent was considered in Novikov and Shiryaev (2007) for both random walks and Lévy processes. The identity for general functions was obtained by Surya (2007).

LEMMA 2. *Consider a nonnegative nondecreasing function $f(x)$ and a real $r \geq 0$. Then,*

(a) *the function*

$$h(x) = \mathbf{E}_x(f(M)) \quad (x \in \mathbb{R})$$

is r-excessive, and, in consequence,

(b) *the process $\{e^{-rt}h(X_t)\}$ is a supermartingale.*

Proof. The fact that (b) follows from (a) is standard; see, for example, Shiryaev (2008). Statement (a) is a corollary of Lemma 2.2 in Christensen, Salminen, and Ta (2013), as for nondecreasing f we have

$$\sup_{0\leq t\leq e(r)} f(X_t) = f\left(\sup_{0\leq t\leq e(r)} X_t\right) = f(M),$$

concluding the proof. □

Proof of Theorem 1. We first observe that (9) follows from Lemma 1 with $a = x^*$. As, on the one side, for $x \geq x^*$ we have $V(x) = \mathbf{E}_x G(M) = \mathbf{E}_x G^*(M) = g(x)$ and, on the other, condition (8) holds, we conclude that $V(x) \geq g(x)$ for all $x \in \mathbb{R}$. To prove (10) we observe that $V(x)$ is excessive based on Lemma 2 applied to the nondecreasing function $G(x)$ in (6), so, for any stopping time $\tau \in \mathcal{M}$, we have

$$V(x) \geq \mathbf{E}_x(e^{-r\tau} V(X_\tau)) \geq \mathbf{E}_x(e^{-r\tau} g(X_\tau)).$$

This concludes the proof of Theorem 1. □

Remark 3. If the equality in (8) holds also for some $x < x^*$, then, defining the set

$$S = \{x \in \mathbb{R} \colon V(x) = g(x)\},$$

the stopping time

$$\tau^{**} = \inf\{t \geq 0 \colon X_t \in S\}$$

is also an optimal stopping time for problem (3). Observe first that $S \supseteq [x^*, \infty)$, so $\tau^{**} \leq \tau^*$. Then, from the supermartingale property, we have

$$V(x) \geq \mathbf{E}_x(e^{-r\tau^{**}} V(X_{\tau^{**}})) \geq \mathbf{E}_x(e^{-r\tau^{*}} V(X_{\tau^{*}})),$$

obtaining that $\mathbf{E}_x(e^{-r\tau^{**}} V(X_{\tau^{**}})) = \mathbf{E}_x(e^{-r\tau^{*}} g(X_{\tau^{*}}))$.

The following result gives a sufficient condition in order to verify condition (8).

Corollary 1 (Surya (2007)). *Assume that there exists a function* $Q \colon \mathbb{R} \to \mathbb{R}$ *such that*

$$\mathbf{E}_x(Q(M)) = g(x) \text{ for all } x$$

and a real constant x^* *such that whenever*

$$x < x^* < y < z$$

we have

$$Q(x) \leq Q(x^*) = 0 \leq Q(y) \leq Q(z). \tag{12}$$

Then $G^*(x) = Q(x)$ *when* $x \geq x^*$ *verifies the conditions of Theorem* 1, *and* $\lim_{x \to -\infty} g(x) = 0$.

Proof. Let us check first that

$$\lim_{x \to -\infty} g(x) = 0. \tag{13}$$

In fact, if $\ell = \limsup_{x \to -\infty} g(x) > 0$ there exists a decreasing sequence $x_n \to -\infty$ such that $g(x_n) \to \ell$. But

$$\begin{aligned} g(x_n) &= \mathbf{E}(Q(x_n + M)) \\ &= \mathbf{E}(Q(x_n + M)\mathbf{1}_{\{x_n + M \geq x^*\}}) + \mathbf{E}(Q(x_n + M)\mathbf{1}_{\{x_n + M < x^*\}}) \\ &\leq \mathbf{E}(Q(x_n + M)\mathbf{1}_{\{x_n + M \geq x^*\}}) \to 0 \text{ as } n \to \infty, \end{aligned}$$

by dominated convergence, as $Q(x)\mathbf{1}_{\{x + M \geq x^*\}}$ is decreasing in x by hypothesis, giving a contradiction, and concluding (13).

The rest of the proof is immediate as condition (12) implies condition (8). In fact, for G defined in (6), we have

$$V(x) = \mathbf{E}_x(G(M)) \geq \mathbf{E}_x(Q(M)) = g(x),$$

concluding (8) and the proof of the corollary. □

3. Polynomial rewards. Our payoff function is constructed from a polynomial

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x, \tag{14}$$

where we assume that $x = 0$ is a root of $p_n(x)$. The payoff is the positive part of a polynomial for positive values of the variable x:

$$g(x) = \big(p_n(x^+)\big)^+ = \begin{cases} p_n(x)^+ & \text{when } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that problem (3) with reward function $\alpha g(\cdot + x_0)$ has solution $\alpha V(\cdot + x_0)$, so, taking the first coefficient $a_n = 1$ and $x = 0$ as the smallest root of p_n in (14) entails no loss of generality for any polynomial with positive leading coefficient and at least one root.

3.1. The averaging polynomial. We search for a function $P_n(x)$ such that

$$\mathbf{E}_x(P_n(M)) = p_n(x), \ x \in \mathbb{R}. \tag{15}$$

It is not difficult to see that this averaging function can be taken to be a polynomial of order n,

$$P_n(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0. \tag{16}$$

Assume that the first n moments of M are finite and denote them by $\mu_k = \mathbf{E}(M^k)$ ($k = 1, \ldots, n$). Denote $\mu_0 = 1$. With this notation, the left-hand side in (15), after changing the order in the sums, reads

$$\sum_{k=0}^{n} b_k \sum_{\ell=0}^{k} C_k^\ell x^\ell \mu_{k-\ell} = \sum_{\ell=0}^{n} \left(\sum_{k=\ell}^{n} b_k C_k^\ell \mu_{k-\ell} \right) x^\ell,$$

which equating coefficients of equal degree in (15) gives

$$\sum_{k=\ell}^{n} b_k C_k^\ell \mu_{k-\ell} = a_\ell, \ \ell = n, n-1, \ldots, 0. \tag{17}$$

This system of linear equations can be solved recursively backward, i.e.,

$$\begin{aligned} b_n &= 1, \\ b_{n-1} &= a_{n-1} - n\mu_1, \\ b_\ell &= a_\ell - \sum_{k=\ell+1}^{n} b_k C_k^\ell \mu_{k-\ell}, \quad \ell = n-2, \ldots, 0, \end{aligned}$$

where we put $a_0 = 0$.

Remark 4. An equivalent way to obtain the averaging function $P_n(x)$ in (15) is to write it as

$$P_n(x) = \sum_{k=1}^{n} a_k Q_k(x),$$

where the $Q_k(x)$ are the Appell polynomials of the random variable M, introduced in Novikov and Shiryaev (2004) and applied also in Kyprianou and Surya (2005), Salminen (2011), and Mishura and Tomashyk (2011).

We have the following simple result.

PROPOSITION 1. *Consider a polynomial $p_n(x)$ as in* (14).

(a) *The averaging polynomial $P_n(x)$ constructed as in* (16) *has at least one positive root.*

(b) *If x^* denotes the largest root of $P_n(x)$, we have $p_n(x) \geq 0$ for $x \geq x^*$ and $p_n(x) > 0$ for $x > x^*$.*

Proof. (a) If $P_n(x)$ has no positive root for $x \geq 0$, then $P_n(x) > 0$ for all $x > 0$. As $\mathbf{P}_0(M > 0) > 0$, this gives $0 < \mathbf{E}_0 P_n(M) = p_n(0) = 0$, a contradiction.

(b) This is a consequence of

$$p_n(x) = \mathbf{E}_x(P_n(M)) \geq 0 \text{ for } x \geq x^*, \tag{18}$$

as $\mathbf{P}_x(M \geq x^*) > 0$ and $P_n(x) > 0$ for $x \geq x^*$. The condition $\mathbf{P}_x(M > x) > 0$ and inequality $P_n(x) > 0$ for $x > x^*$ give the strict inequality in (18), concluding the proof. □

THEOREM 2. *Let $p_n(x)$ be a polynomial of degree n with leading coefficient $a_n = 1$ and $p_n(0) = 0$. Define as before*

$$g(x) = \left(p_n(x^+)\right)^+.$$

Denote by $P_n(x)$ the averaging polynomial of $p_n(x)$ for the random variable M. Denote by x^* the largest positive root of $P_n(x)$. Define $G(x) = P_n(x)\mathbf{1}_{\{x \geq x^*\}}$, and

$$V(x) = \mathbf{E}_x(G(M)), \quad \tau^* = \inf\{t \geq 0 \colon X_t \geq x^*\}.$$

If $G(x)$ is nondecreasing and $V(x) \geq g(x)$ for $x \leq x^*$, then the pair $V(x)$, τ^* is a solution of the optimal stopping problem (3).

Proof. The result follows directly from the application of Theorem 1. □

3.2. Examples. In order to illustrate our results we first assume that X is the Brownian motion and $r = 1/2$. In this case M has exponential distribution with parameter one (in the general case with parameter $1/\sqrt{2r}$). Its moments satisfy $\mu_n = \Gamma(n+1) = n!$. Observe that for spectrally negative Lévy processes, the random variable M is also exponentially distributed, with parameter $\Phi(r)$, where $\Phi(r)$ is the unique positive root of equation $\psi(\lambda) = r$ and $\psi(\lambda)$ is the Laplace exponent of the corresponding Lévy process (see, for instance, Bertoin (1996)). In this case we can produce similar results.

3.2.1. Example: Quadratic polynomials. Consider $p_2(x) = x^2 + ax$. Solving (17) we obtain

$$P_2(x) = x^2 + (a - 2\mu_1)x + 2(\mu_1)^2 - \mu_2 - a\mu_1.$$

This quadratic polynomial has its largest root

$$x^* = \mu_1 - a/2 + \sqrt{\mu_2 - \mu_1^2 + a^2/4} = \mathbf{E}M - a/2 + \sqrt{\mathbf{var}_0\, M + a^2/4},$$

that is evidently positive, as it can be checked independently of Proposition 1. In the case $a = 0$ we obtain

$$x^* = \mu_1 - a/2 + \sqrt{\mu_2 - \mu_1^2 + a^2/4} = \mathbf{E}M + \sqrt{\mathbf{var}_0\, M},$$

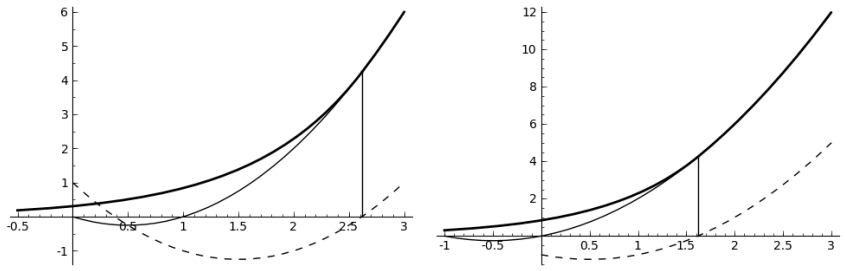


FIG. 1. *Examples with $a = 1$ (left) and $a = -1$ (right). Here $p_2(x)$ is plotted continuously, and $P_2(x)$ (dashed) gives the roots. The thick lines are the respective solutions $V(x)$. Observe that in case $a = 1$ (left) the averaging function P_2 does not remain nonpositive for values smaller than the root $x^* \sim 2.62$.*

which gives the solution found in Novikov and Shiryaev (2004). In our particular case $\mu_1 = 1, \mu_2 = 2$, $P_2(x) = x^2 + (a-2)x - a$, and $x^* = 1 - a/2 + \sqrt{1 + a^2/4}$. For any $a \in \mathbb{R}$ it is evident that $G(x)$ increases after x^* and it is not difficult to calculate $V(x)$:

$$V(x) = (x^2 + ax)\mathbf{1}_{\{x > x^*\}} + ((x^*)^2 + ax^*)e^{x - x^*}\mathbf{1}_{\{x \le x^*\}}.$$

In order to apply Theorem 2, we only need to check the condition $V(x) \ge g(x)$ for $x \le x^*$, but in fact it is only necessary to check this for $-a \le x \le x^*$. Consider the case $a < 0$; the opposite case is considered similarly. So, we need to check the condition

$$(x^2 + ax)e^{-x} \le ((x^*)^2 + ax^*)e^{-x^*} \tag{19}$$

for $-a \le x \le x^*$. The latter inequality holds for $x = -a$ where we have the strict inequality and for $x = x^*$ where we have the equality. Furthermore, function $f(x) = (x^2 + ax)e^{-x}$ has the derivative $f'(x) = -P_2(x)e^{-x}$ which is positive between the roots of $P_2(x)$, the biggest being x^*. Moreover, the smallest root of $P_2(x)$ equals $1 - a/2 - \sqrt{1 + a^2/4} < -a$ for negative a. It means that $f(x)$ increases on $(-a, x^*)$ whence we get (19). So, according to Theorem 2, $(V(x), x^*)$ create a solution of (3). The same is true for $a > 0$. In Figure 1 we plot the solution for $a = -1$ and $a = 1$.

3.2.2. Example: Cubic polynomials. Consider $p_3(x) = x^3 + ax^2 + bx$. Solving (17) we obtain

$$P_3(x) = x^3 + (a - 3\mu_1)x^2 + (b - 3\mu_2 - 2(a - 3\mu_1)\mu_1)x \\ - \mu_3 - (a - 3\mu_1)\mu_2 - (b - 3\mu_2 - 2(a - 3\mu_1)\mu_1)\mu_1.$$

If we further assume that X is the Brownian motion and $r = 1/2$, we have

$$P_3(x) = x^3 + (a-3)x^2 + (b - 2a)x - b$$

and

$$V(x) = (x^3 + ax^2 + bx)\mathbf{1}_{\{x > x^*\}} + ((x^*)^3 + a(x^*)^2 + bx^*)e^{x - x^*}\mathbf{1}_{\{x \le x^*\}}.$$

If $b > 0$, P_3 evidently has at least one positive root since $P_3(0) < 0$ and $P_3(+\infty) = +\infty$. Let $b < 0, a > -2$; then $P_3(1) = -a - 2 < 0$, and at least one positive root exceeds 1. If $b < 0, a \le -2$, then $P(-a) = -a^2 - b(1 + a) < 0$ and at least

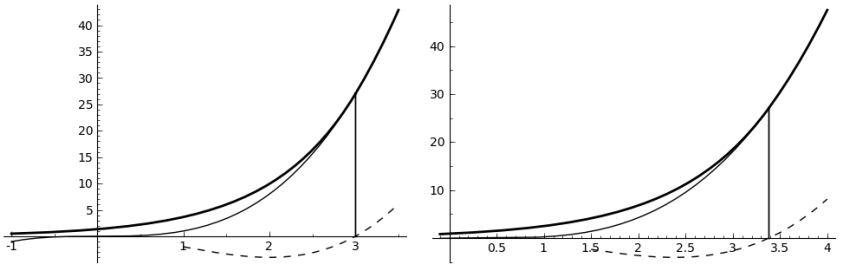


FIG. 2. *Cubic polynomial with $a = b = 0$ (left) and $a = -9/8$, $b = 3/8$ (left). Here $P_3(x)$ is plotted continuously, and $Q_3(x)$ (dashed) gives the root x^* in each case. The thick lines are the respective solutions $V(x)$.*

one positive root exceeds $-a$. So, in any case P_3 has positive roots, in accordance with Proposition 1, but we have checked this independently. Now, in order to apply Theorem 2, consider some particular cases. In the case when $3 < a < \frac{b}{2} \wedge 8$ (the case of positive coefficients, for example, $a = 4, b = 10$), $p_3(x)$ has only one root $x = 0$ because other roots that should equal $x_{1,2} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$ do not exist (discriminant is negative, since $\frac{a^2}{4} - b < \frac{ab}{8} - b < 0$), the derivative $P_3'(x) = 3x^2 + 2(a-3)x + b - 2a$ has two negative roots, therefore it is positive on $[0, \infty)$, and $P_3(x)$ increases on $[x^*, \infty)$, even more, it increases on $[0, \infty)$ being negative on $[0, x^*)$. Moreover, as in the example with quadratic polynomials, we need to check inequality

$$(x^3 + ax^2 + bx)e^{-x} \le ((x^*)^3 + a(x^*)^2 + bx^*)e^{-x^*}$$

on the interval $[0, x^*]$ but on the interval $[0, x^*]$ the derivative of the function $(x^3 + ax^2 + bx)e^{-x}$ being equal $-P_3(x)e^{-x}$ is positive, and therefore both conditions of Theorem 2 hold. In the case $a = b = 0$, i.e., $p_3(x) = x^3$, we have

$$P_3(x) = x^3 - 3x^2$$

with largest root $x^* = 3$ (see Figure 3). Evidently, P_3 increases on $[x^*, \infty)$ because its derivative $3x^2 - 6x$ is positive on the interval $[2, \infty)$ and $-P_3'$ is positive on $(0, 3)$, which supplies both conditions of Theorem 2. An example for Brownian motion with $r = 1/2$ and polynomial with positive b and negative a is shown in Figure 2. We put in this case

$$p_3(x) = x^3 - (9/8)x^2 + 3/8, \qquad P_3(x) = x^3 - \frac{33}{8}\,x^2 + \frac{21}{8}\,x - \frac{3}{8},$$

and the largest root is $x^* = 3.3815$.

3.2.3. Example: Kou's process for a quartic polynomial. A diffusion process $\{X_t\}$ with two-sided exponential jumps, defined by the formula

$$X_t = at + \sigma W_t + \sum_{k=1}^{N_t} Y_k - \sum_{k=1}^{N_t'} Y_k',$$

is known in the financial literature as a Kou's process (see Kou (2002) and Cont and Tankov (2004)). Here $\{N_t\}$ (resp., $\{N_t'\}$) is a Poisson process with parameter μ (resp.,

ν) and $\{Y_k\}$ (resp., $\{Y'_k\}$) is a sequence of independent exponential random variables with parameter α (resp., β). The characteristic exponent (1) of the process is given by

$$\psi(z) = az + \frac{1}{2}\sigma^2 z^2 + \mu\frac{z}{\alpha - z} - \nu\frac{z}{z+\beta},$$

and the density of the maximum M in this case is a mixture of two exponentials

$$f_M(x) = A_1 r_1 e^{-r_1 x} + A_2 r_2 e^{-r_2 x}$$

with coefficients

$$A_1 = \frac{1 - r_1/\beta}{1 - r_1/r_2}, \quad A_2 = \frac{1 - r_2/\beta}{1 - r_2/r_1},$$

where $0 < r_1 < r_2$ are the positive roots of the equation $\psi(z) = r$ (see Mordecki (2003)). In consequence the moments are given by

$$\mu_k = k!\left(\frac{A_1}{r_1^k} + \frac{A_2}{r_2^k}\right).$$

We consider a quartic polynomial

$$p(x) = x^4 + a_3 x^3 + a_2 x^2 + a_1 x.$$

If we denote $P(x) = x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0$, applying (17) we obtain

$$\begin{aligned}
b_3 &= a_3 - 4\mu_1,\\
b_2 &= a_2 - 6\mu_2 - 3b_3\mu_1,\\
b_1 &= a_1 - 4\mu_3 - 3b_3\mu_2 - 2b_2\mu_1,\\
b_0 &= -(\mu_4 + b_3\mu_3 + b_2\mu_2 + b_1\mu_1).
\end{aligned}$$

Assuming that there exists a value x^* that satisfies the conditions of Theorem 2, we can compute for $x < x^*$ the candidate value function:

$$V(x) = \mathbf{E}(P(x+M)\mathbf{1}_{\{x+M\geq x^*\}}) = B_1 e^{r_1(x-x^*)} + B_2 e^{r_2(x-x^*)},$$

where

$$B_1 = A_1 r_1 \int_0^\infty P(z+x^*)e^{-r_1 z}dz, \quad B_2 = A_2 r_2 \int_0^\infty P(z+x^*)e^{-r_2 z}dz.$$

To proceed we choose values for the parameters,

$$a = 2,\ \sigma = 1,\ \mu = 1,\ \nu = 1,\ \alpha = 2,\ \beta = 2,\ r = 6,$$

and choose the polynomial

$$p(x) = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x.$$

We obtain $r_1 = 1.4327$, $r_2 = 2.8740$, giving $A_1 = 0.5656$, $A_2 = 0.4344$. In consequence $x^* = 4.3706$. For this set of parameters the conditions of Theorem 2 are fulfilled; see Figure 3.

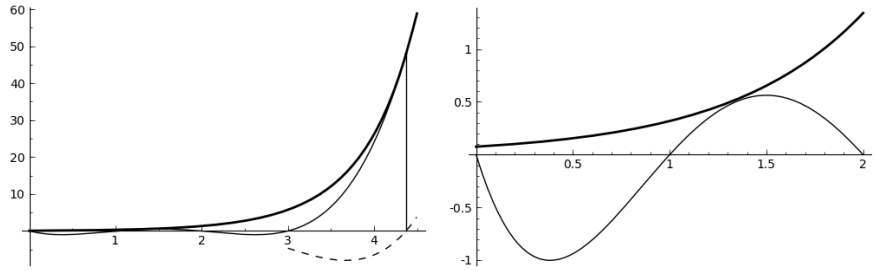


FIG. 3. *On the left $p(x)$ is plotted continuously, and $P(x)$ (dashed) gives the root $x^* = 4.37$. The thick line is the solution $V(x)$. On the right we observe that $V(x) \geq g(x)$.*

4. Other examples.

4.1. American perpetual call-options portfolio. Consider a portfolio with $A_1, \ldots, A_n$ perpetual call options struck at $k_1 < k_2 < \cdots < k_n$, in a financial market model with interest rate $r \geq 0$ and a stock modeled by $S_t = e^{X_t}$, where $X = \{X_t\}_{t \geq 0}$ is a Lévy process. Observe that $S_0 = e^x$. Assume that $\mathbf{E}e^{X_1} < e^r$. This is the situation when pricing American options on a stock that pays dividends under a risk-neutral measure, and also the situation in real option pricing. When this condition is not fulfilled, the problem has a trivial solution (see Mordecki (2002a)).

The pricing of this portfolio requires us to solve the optimal stopping problem with discount r and reward

$$g(x) = \sum_{k=1}^{n} A_i (e^x - k_i)^+$$

for the Lévy process X. Denote $A = \sum_{k=1}^{n} A_i$ and $K = \sum_{k=1}^{n} A_i k_i$. The averaging problem can easily be solved:

$$x^* = \log \mathbf{E}e^M + \log(K/A), \quad G^*(x) = \frac{Ae^x}{\mathbf{E}e^M} - K, \quad x \geq x^*. \tag{20}$$

If condition (8) holds, the solution to the problem is one-sided with optimal threshold x^* and payoff given by (7). Observe that (8) is strictly necessary, as the following simple particular case shows.

To analyze how Theorem 1 works in this context we consider two examples, both portfolios with two options respectively. Choose X a Wiener process with discount $r = 2$; the distribution of the maximum is an exponential with parameter 2 and $\mathbf{E}e^M = 2 < e^2$. In the first example we choose $A_1 = 1$, $A_2 = 2$, $k_1 = 1$, and $k_2 = 5/2$. We obtain $x^* = \log 4$ and

$$G^*(x) = \frac{3}{2}e^x - 6, \quad x \geq \log 4.$$

As condition (8) holds in this example (see the left side in Figure 4), Theorem 1 provides the solution to the problem.

In our second example $A_1 = A_2 = 3$, $k_1 = 1/2$, $k_2 = 5/2$. From (20), we obtain $x^* = \log 3$, with averaging function

$$G^*(x) = 3e^x - 9, \quad x \geq \log 3.$$

But in this case (as it is not difficult to verify) the candidate $V(x)$ defined by (7) does not fulfill (8), as seen in Figure 4 (right).

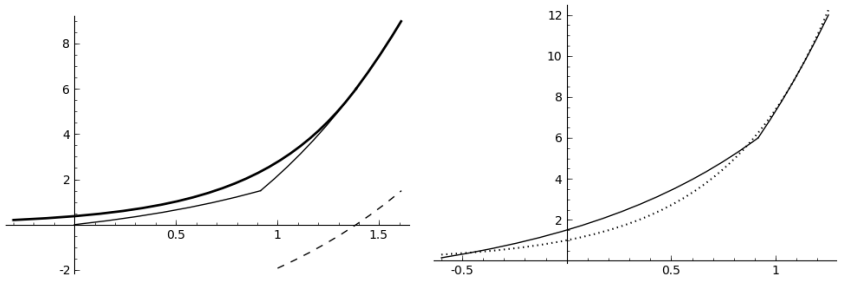


FIG. 4. *On the left the thick line $V(x) \geq g(x)$ and Theorem* 1 *provides the solution. On the right the candidate function (dotted) does not fulfill the relevant condition* (8).

4.2. A trigonometric example. To analyze the versatility of the method, consider β positive, and define the payoff function

$$g(x) = \begin{cases} x + \beta \sin x & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that

$$G(x) = x - \mathbf{E}M + \beta \frac{(\mathbf{E}\cos M)\sin x - (\mathbf{E}\sin M)\cos x}{(\mathbf{E}\sin M)^2 + (\mathbf{E}\cos M)^2}$$

will help to solve the generalized averaging problem. Introduce the quantities

$$\gamma = \sqrt{(\mathbf{E}\sin M)^2 + (\mathbf{E}\cos M)^2} \leq 1, \quad \theta = \arctan\left(\frac{\mathbf{E}\sin M}{\mathbf{E}\cos M}\right) \in (0, \pi/2).$$

We have

$$\cos\theta = \frac{\mathbf{E}\cos M}{\sqrt{(\mathbf{E}\sin M)^2 + (\mathbf{E}\cos M)^2}}, \quad \sin\theta = \frac{\mathbf{E}\sin M}{\sqrt{(\mathbf{E}\sin M)^2 + (\mathbf{E}\cos M)^2}},$$

and

$$G(x) = x - \mathbf{E}M + \frac{\beta}{\gamma}\sin(x - \theta).$$

Observe that

$$G(0) = -\mathbf{E}M - \frac{\beta}{\gamma}\sin\theta < 0,$$

and in case $\beta \leq \gamma$, we have

$$G'(x) = 1 + \frac{\beta}{\gamma}\cos(x - \theta) \geq 0.$$

We then define x^* as the unique positive value such that $G(x^*) = 0$ and define $V(x)$ by (7). If condition (8) holds, we have the solution of the problem.

To illustrate the method we assume $r = 0$ and that the Lévy process is a compound Poisson process with drift:

$$X_t = x - ct + \sum_{k=1}^{N_t} Y_k,$$

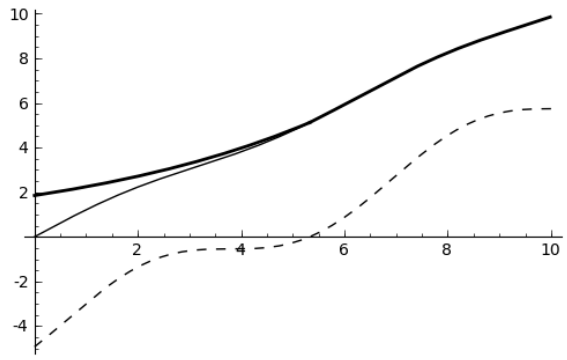


FIG. 5. *The payoff is in black, and the value function is the thick line. G^* is dashed. At $x^* = 5.329$ the value function is not differentiable.*

where $c > 0$, $\{N_t\}$ is a Poisson process with parameter ℓ, and $\{Y_k\}$ is a sequence of independent exponential random variables with parameter α. The condition $\alpha > \ell/c$ ensures $\mathbf{E}M < \infty$, in consequence $\mathbf{P}(M < \infty) = 1$. The density of the maximum (for $x = 0$) is

$$f_M(y) = (1-\mu)\,\delta_0(dy) + \mu\rho e^{-\rho y},$$

where δ_0 is the Dirac function, and

$$\rho = \alpha - \frac{\ell}{c}, \qquad \mu = \frac{\ell}{c\alpha}.$$

Similarly as in Theorem 5.1 in Surya (2007), we see that this model does not satisfy the smooth fit property. We have

$$\mathbf{E}M = \frac{\mu}{\rho}, \quad \mathbf{E}\sin M = \frac{\mu\rho}{\rho^2+1}, \quad \mathbf{E}\cos M = 1 - \mu + \frac{\mu\rho^2}{\rho^2+1},$$

and in consequence

$$\theta = \arctan\left(\frac{\mu\rho}{1+\rho^2-\mu}\right), \quad \gamma = \frac{\sqrt{\mu^2\rho^2 + (1-\mu-\rho^2)^2}}{1+\rho^2}.$$

As in the example in section 3.2.3, the density of the maximum gives us a closed formula for the candidate solution in (7), for $x < x^*$, by

$$V(x) = Be^{\rho(x-x^*)},$$

where

$$B = \mu\rho \int_0^\infty G(x^*+z)e^{-\rho z}dz.$$

In Figure 5 we present an example for the following set of values of the parameters: $\alpha = 1.1, \ell = 1, c = 1.1$, giving $\gamma = 0.2534 := \beta$.

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