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Keywords (separated by "-")	Uruguayan sovereign bonds - Stochastic processes - Yield curve - Term structure surface - Interest rate models - Forward-rate models - Arbitrage possibilities	

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Chapter 17
Modelling the Uruguayan Debt Through Gaussians Models

Ernesto Mordecki and Andrés Sosa

Abstract We model bond price curves corresponding to the sovereign Uruguayan debt nominated in USD, as an alternative to the official bond price publication released by the Central Bank of Uruguay (CBU). Four different Gaussian models are fitted, based on historical data issued by the CBU, corresponding to some of the more frequently traded bonds. The main difficulty we approach is the absence of liquidity in the bond market. Nevertheless the adjustment is relatively good, giving the possibility of non-arbitrage pricing of the whole family of nontraded instruments and also the possibility of pricing derivative securities.

Keywords Uruguayan sovereign bonds • Stochastic processes • Yield curve • Term structure surface • Interest rate models • Forward-rate models • Arbitrage possibilities

17.1 Introduction

Bond price curves (or equivalently yield curves) constitute a major tool in debt analysis and perspective of the sovereign debt of a country. Term structure models have therefore been used in different ways by different classes of market participants. In the monetary policy context, the term structure is an indicator of the market's expectations regarding interest rate and inflation rates. From a financial point of view, the existence of a bond price curve helps the development of the domestic capital market, both for the primary and secondary market.

There are essentially two approaches to model the term structure. The general equilibrium approach starts from a description of the economy and derives the term structure of interest rate endogenously, as used, for example, by Cox et al. (1985). In contrast, the arbitrage approach starts from assumptions about the stochastic evolution of one or more interest rates and derives prices of contingent claims by imposing the no-arbitrage condition; this is, for example, the pioneering approach proposed by Vasicek (1977).

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In this first approximation to the problem of the Uruguayan USD nominated debt, we use the second approach, and further we restrict our analysis to four Gaussian models due to their analytical and numerical tractability. Factor models assume that the term structure of interest rate is driven by a set of state variables named as "factors". As one factor generally explains a large proportion of the yield curve movement, it may tempting to reduce the analysis to one factor models. Nevertheless, the consideration of one factor involves, in some cases, perfect correlation between log prices of bonds. Although empirical data shows high correlation, it is never the case for this perfect correlation. Therefore, we use multiple factor model and in particular two factor models.

The first two models we choose are short-rate models, the classical Vasicek model (Vasicek 1977), and the more flexible $G2++$ model (Brigo and Mercurio 2006). The parameter estimation is carried out based on maximum likelihood, following (Chen and Scott 1993). The second couple of models belong to the forward-rate model family, based on the methodology proposed by Heath et al. (1992). Here we again choose first the simplest possible one, the Ho–Lee model with constant volatility (Ho and Lee 1986), and second the more flexible Hull–White model with tempered volatility (Hull and White 1993). In these two last cases, we calibrate the models with the help of minimization of squared differences between theoretical and quoted prices of bonds.

Our contribution is to provide an arbitrage-free set of bond prices for the Uruguayan USD debt that can be used for portfolio valuation and derivative pricing.

The rest of the paper is organized as follows. Sections 17.2 and 17.3 describe the short-rate model and the forward-rate models, respectively. Section 17.4 provides information about the Uruguayan debt and the methodology of computation of bond prices used by the CBU. Section 17.5 describes the estimation methodology through maximum likelihood and calibration and presents the results obtained. Section 17.6 concludes.

17.2 Short-Rate Models

When modelling yield curves through stochastic processes, the classical approach consists in modelling the short rate through a certain amount of sources of uncertainty, under the denomination of factors. When these factors are used to construct the short interest rate, we obtain a large variety of models, described in a vast part of the literature in fixed income mathematical finance (Bjork 2009; Brigo and Mercurio 2006; Filipovic 2009; Musiela and Rutkowski 2005).

In this part of the work, we use two Gaussian models, the one proposed by Vasicek in 1977 (Vasicek 1977) that is seminal in the literature. The second model we apply is the G2++ (Brigo and Mercurio 2006), a modification of the previous one, having two factors and including the initial price curve, avoiding in this way arbitrage at the initial time. This model has in fact a more general version, under the denomination of Gn++ model (Di Francesco 2012), including an

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arbitrary amount of factors. However, the choice of the number of factors involves a compromise between numerically efficient implementation and capability of the model to represent realistic correlation patterns and to fit satisfactorily enough market data.

17.2.1 The Vasicek Model

In his classical paper (Vasicek 1977), Vasicek proposes a model for the short rate through a stochastic differential equation driven by a Wiener process,

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \qquad r(0) = r_0; \tag{17.1}$$

where a, b and σ are positive constants and $\{W(t)\}$ is a standard Wiener process defined in a stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P})$. The solution to (17.1) is known as the Ornstein–Uhlenbeck process. It defines an elastic random walk around a trend, with a mean-reverting characteristic. Given the set of information at time s, the short rate $r(t)$ is normally distributed with

$$\mathbf{E}\,(r_t|\mathscr{F}_s) = r_s e^{-a(t-s)} + b\left(1 - e^{-a(t-s)}\right)$$

$$\mathbf{var}(r_t|\mathscr{F}_s) = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right).$$

The bond price can be obtained by computing the discounted expected terminal value of the bond with respect to a risk-neutral probability measure Q. This quantity can be explicitly computed concluding that the Vasicek model is an affine model whose solution is

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}; \tag{17.2}$$

where

$$A(t,T) = \exp\left(\left(b - \frac{\sigma^2}{2a^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4a}B(t,T)^2\right);$$

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right).$$

17.2.2 The G2++ Model

As mentioned above, the price correlation given by the Vasicek model to log prices of different bonds is one. For this reason, as observed prices do not show such high correlation; as a result the idea of considering models with more factors that imply

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more parameters that would allow to better fit to the data emerged. One of these proposals is the Gn++ model, proposed in Di Francesco (2012) that we use with two factors. Another relevant characteristic of this proposal is that it takes into account the whole initial price curve.

The dynamics of the short-rate process in this model is given by

$$r(t) = x(t) + y(t) + \varphi(t), \qquad r(0) = r_0;$$

where the process $x(t)$ and $y(t)$ is driven by a correlated two-dimensional Wiener process $\{W_1(t), W_2(t)\}$, by the equations

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \qquad dy(t) = -by(t)dt + \eta dW_2(t). \tag{17.3}$$

We have $dW_1(t)dW_2(t) = \rho dt$ where $-1 \le \rho \le 1$ and the constants $r_0,\ a,\ b, \sigma, \eta$ are positive. The function $\varphi(t)$ is deterministic and well defined in the time interval $[0, T]$, which is added in order to fit exactly the initial zero-coupon curve. Given the set of information at time s, the short rate $r(t)$ is normally distributed with

$$\begin{aligned}\mathbf{E}\,(r(t)|\mathscr{F}_s) &= x(s)e^{-a(t-s)} + y(s)e^{-b(t-s)} + \varphi(t);\\ \mathbf{var}(r(t)|\mathscr{F}_s) &= \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})\\ &\quad + \frac{\eta^2}{2b}(1 - e^{-2b(t-s)}) + \frac{2\rho\sigma\eta}{a+b}(1 - \exp^{-(a+b)(t-s)}).\end{aligned}$$

The price at time t of a zero-coupon bond with maturity at time T is

$$\begin{aligned}P(t,T) = \exp\Bigg(&-\int_t^T \varphi(u)du - \frac{1 - e^{-a(T-t)}}{a}x(t)\\ &-\frac{1 - e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T)\Bigg);\end{aligned} \tag{17.4}$$

where

$$\begin{aligned}V(t,T) =& \frac{\sigma^2}{a^2}\left(T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\right)\\ &+ \frac{\eta^2}{b^2}\left(T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}\right)\\ &+ \frac{2\rho\sigma\eta}{ab}\left(T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b}\right).\end{aligned} \tag{17.5}$$

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Even though the previous formula (17.4) gives bond prices in the model, it is necessary to estimate the function φ. In order to do this, it is necessary to assume that the initial price curve $T \mapsto P^M(0,T)$ is known. The model fits the currently observed term structure if $P^{Model}(0,T) = P^M(0,T)$; therefore, the price at time t of a zero-coupon bond maturity at time T is

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp(A(t,T)); \tag{17.6}$$

where

$$A(t,T) = \frac{1}{2}\big(V(t,T) - V(0,T) + V(0,t)\big) - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t).$$

17.3 Forward-Rate Models

In the market we do not have a real instantaneous interest rate. In certain cases, the one (or three)-month interest rate series is used as a proxy to estimate this interest rate. It is not convenient to use the overnight rate, as it has very high volatility due to economical factors, as, for instance, the daily liquidity in the market that can give distortions on the structure of the yield curve.

By this reason, and also to avoid arbitrage opportunities in a systematic way, Heath, Jarrow and Morton introduce a new methodology (Heath et al. 1992) (referred to as HJM models) that models the *forward instantaneous rate* at time t by a stochastic differential equation driven by a Wiener process

$$f(s,t) = f(0,t) + \int_0^t \alpha(u,t)du + \int_0^t \sigma(u,t)dW_u;$$

where W_t is a Wiener process.

We depart from the free arbitrage model that assumes that the coefficients $\alpha(u,t)$, $0 \le u \le t$ and $\sigma(u,t)$, $0 \le u \le t$ are adapted processes defined in an underlying stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_u\}, \mathbf{Q})$, and $\mathbf{Q}$ is a risk-neutral probability measure. The key aspect of HJM techniques lies in the recognition that the drift of the no-arbitrage evolution of certain variables can be expressed as functions of their volatilities and the correlations among themselves, giving

$$\alpha(s,t) = \sigma(s,t)\int_s^t \sigma(s,u)du. \tag{17.7}$$

Therefore, in order to specify an HJM model, the initial forward rate curve $f(0,t)$ and the volatility structure $\sigma(s,t)$ should be given, because no drift estimation is needed. It should be observed that as long as the function σ is deterministic, by condition (17.7), the drift is also deterministic; in consequence, the forward rates are Gaussian.

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17.3.1 The Ho–Lee Model and Hull–White Model

In the applications that follow, we first choose the simplest possible alternative for the volatility. We assume $\sigma(s,u) = \sigma > 0$ is a positive constant, and this gives

$$f(s,t) = f(0,t) + \sigma(st - s^2/2) + \sigma W_t;$$

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the so-called Ho–Lee model (Ho and Lee 1986). The price of a zero-coupon bond in this model is given by

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\Big((T-t)f^M(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r(t)\Big). \tag{17.8}$$

In the second model, we assume that the volatility is also deterministic but in this case is time dependent, given by $\sigma(s,u) = \sigma e^{-a(u-s)}$, where a and σ are positive constants. This gives

$$r(t) = f(0,t) + \frac{\sigma^2}{2a^2}(e^{-at} - 1)^2 + \sigma \int_0^t e^{-a(t-u)} dW_u.$$

The price of a zero-coupon bond in this case is

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left(B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2 - B(t,T)r(t)\right). \tag{17.9}$$

where the function $B(t,T)$ is

$$B(t,T) = \frac{1}{a}\big(1 - e^{-a(T-t)}\big).$$

This corresponds to the Hull–White model (Hull and White 1993) with time-dependent parameters.

17.4 About the Uruguayan Debt

The Uruguayan government issues debt through different financial instruments, on different currencies and expirations. In present times, the most relevant circulating instruments according to terms and currencies are:

- the *Treasury Notes* in local currency and linked to CPI (consumer price index),
- the *Local Bonds* (issued in the country) linked to CPI,
- the *Local Bonds* (issued in the country) in USD,
- the *Global Bonds* (issued mainly in United States of America) in USD.

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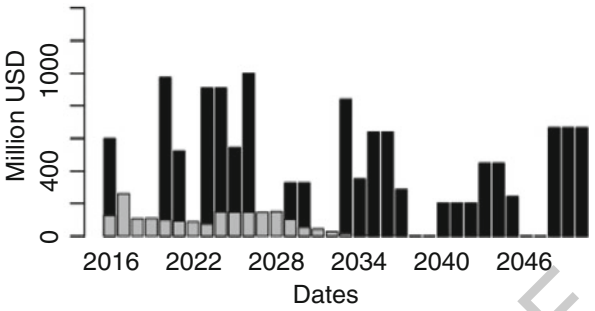


Fig. 17.1 Central government debt profile

To complete the analysis of the debt, it should be mentioned that the government also holds obligations with multilateral financial institutions, estimated in less than the 10 % of the global debt amount. In Fig. 17.1 we observe the debt's profile of the Uruguayan debt across maturities. The last expiration dates correspond to the 2050 bond, recently issued, paying its face value on thirds, in the years 2048, 2049 and 2050.

Regarding the currency composition, more than one half of the Uruguayan debt is nominated in local currency (part of it linked to CPI), 45 % is issued in USD, and the rest corresponds to bonds issued in Euros and Yens. Almost all of the debt (94 %) pays fixed coupons, and the rest is mainly adjusted to Libor rate.

17.4.1 Curve Price Estimation

In most of the countries, the corresponding regulating agencies and some financial corporations release yield curves corresponding to the sovereign debt. The most popular methods to produce these curves include parametric methods [as the ones proposed by Nelson and Siegel (1987) and Svensson (1994)]. Non-parametric methods are also used.

Regarding Uruguay, the CBU does not release yield curves but daily issues a report with prices for all the financial instruments issued by the government. The two authorized stock exchange institutions release their respective yield curves. The *Electronic Stock Exchange of Uruguay (Bolsa Electrónica de Valores del Uruguay (BEVSA))* uses B-splines to produce the yield curves corresponding to local currency, linked to CPI, and to USD. The *Montevideo Stock Exchange (Bolsa de Valores de Montevideo (BVM))* emits a single curve (linked to CPI) based on the methodology proposed by Svensson (1994).

The purpose of the present paper is to propose an alternative methodology for bond valuation, for the debt issued in USD, to the one used by the CBU and also by the BVM and BEVSA. In the next subsection, we briefly describe the procedure used by the monetary authority (CBU) to compute the prices of the different financial instruments.

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17.4.2 CBU Pricing Methodology

The CBU issues daily reports containing prices of all the financial instruments issued by the Uruguayan government. This set of prices is known as the "price vector". The pension funds and insurance companies that are relevant participants in the domestic bond market are obliged by law to use these prices in order to report their respective portfolio values.

This vector of prices is computed through a methodology that includes four different criteria, according to whether the bond has been traded or not and according also to its expiration, and applies with a hierarchical scheme. At the end of each business day, the CBU releases information about bond prices, according to the following procedure:

1. For bonds that have been negotiated in the day (according to certain minimal amounts), the prices are computed as a weighted mean of the respective negotiation prices in the stock exchanges.
2. For bonds with less than a year to expire, an interpolation procedure between the previous price and the face value is applied to compute the prices.
3. An *index* $I(t)$ where t represents the current date is computed in order to compute the prices of the other bonds. This amount represents the value of an "ideal" mean bond.
4. For all other bonds, the new price is obtained from the previous one multiplying by the ratio $I(t)/I(t-1)$.

The most relevant characteristic to be taken into account when analysing this procedure is that the stock exchange of long expiration debt instruments issued by the government (bonds) is really not liquid. This implies that most of the prices published by the CBU result from the application of 4, i.e. are computed instead of negotiated, and this can happen for some instruments consecutively, AQ7 during a relatively long period of time. This can lead to (theoretical) arbitrage opportunities, for instance, giving larger yields for bonds with smaller maturities, which is equivalent to larger prices of zero-coupon bond with smaller maturity than other zero-coupon bonds. Of course, these arbitrage opportunities do not exist in real negotiations, as they do not follow the released bond prices, but are present, for instance, when evaluating portfolios. Although the intention of the methodology is to follow the movements of the market, one should always take into account this fact.

17.5 Empirical Results

17.5.1 Model Fitting

In relatively developed markets, prices of financial derivatives on bonds are available in order to estimate the parameters of the model; see Brigo and Mercurio (2006),

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Filipovic (2009) and the references therein. Our main difficulties are the absence of derivatives market and the lack of liquid market prices of bonds due to the absence of frequent transactions. This is the reason why we develop two different methodologies of model adjustment, first, the classical maximum likelihood estimation, and, second, we propose a calibration procedure for models with fewer parameters that seems to give relatively good results.

Our first approach is closer to Chen and Scott (1993), where model adjustment is carried out departing from bond prices. In our case we do not have the need to introduce measurement errors in order to fit the model, as we develop a first approximation to the data. In the second approach, we perform a conventional calibration model using daily prices of bonds.

17.5.1.1 Maximum Likelihood Estimation

Observe that in formulas (17.2) and (17.6), the log prices are expressed as a linear function of the state variables. In order to determine these state variables in both models, it is necessary to use one bond price time series in Vasicek model and two time series in G2++ model. But to carry out this procedure, the values of the parameters (to be estimated) are necessary. This is why we use the maximum likelihood (ML) method with the help of a change of variables that gives the corresponding Jacobian term in the ML expression. In both models, based on the Markov property of the processes, the joint density is written as a product of conditional densities, each of which has normal distribution, with three parameters in the first case (a, b, σ) and five parameters in the second $(a, b, \sigma, \eta, \rho)$.

Given the panel data set $P_t^M = \left(P^M(t, T^{(i)})\right)$, $i = 1, \ldots, I$ and $t = 1, \ldots, T$, where $P^M(t, T^{(i)})$ is the price at time t of the zero-coupon bond with maturity $T^{(i)}$, denote by $X_t = \left(x_t^{(i)}\right)$, $t = 1, \ldots, T$ the state vector.

The joint density of $P_2^M, \ldots, P_T^M$ satisfies

$$f(P_2^M, \ldots, P_T^M | P_1^M, \Theta) = \prod_{k=2}^{T} f(P_k^M | P_{k-1}^M, \Theta)$$

by the Markov property. Changing variables in each conditional density, we obtain

$$\begin{aligned} f(P_k^M | P_{k-1}^M, \Theta) &= f(X_k | X_{k-1}, \Theta) \left| \frac{\partial X_k}{\partial P_k} \right| \\ &= f(X_k | X_{k-1}, \Theta) \left| \frac{\partial P_k}{\partial X_k} \right|^{-1} = f(X_k | X_{k-1}, \Theta) \frac{1}{|J_k|}, \end{aligned}$$

where J_k is the Jacobian of the change of variables.

We find adequate in our situation to use as many bond price time series as number of factors. But this is not strictly necessary. In some cases, when more time series than state variables are used, it is possible to introduce additional random variables in order to estimate the model (Chen and Scott 1993).

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17.5.1.2 Calibration

In order to use our models in practice, to price contingent claims or with other aims, we have to calibrate its parameters departing from market data. To do this we adopt a cross-sectional approach. Assume that we observe a cross section of market prices of contingent claims, that is to say, the prices of a set of N zero-coupon bonds corresponding to the same day. For time t, let us denote the vector prices $P_t^M = \left(P^M(t, T^{(i)})\right)$, $i = 1, \ldots, I$, where $P^M(t, T^{(i)})$ is the price at time t the zero-coupon bond with maturity at time $T^{(i)}$ and assume that we are able to compute the price vector corresponding to the model, denoted by $P_t^{\text{Model}} = \left(P^{\text{Model}}(t, T^{(i)})\right)$, $i = 1, \ldots, I$. Model prices are functions of the parameter vector Θ of the respective models; see (17.8) for the Ho–Lee model and (17.9) for the Hull–White model.

The idea is to find the values for Θ minimizing the difference between market prices and model prices. To do this we have to solve the least-square problem,

$$\min_{\Theta} \frac{1}{I} \sum_{i=1}^{i=I} \left(P^M(t, T^{(i)}) - P^{\text{Model}}(t, T^{(i)})\right)^2.$$

17.5.2 Results in Short-Rate Models

17.5.2.1 Data

The information used in the construction of the yield curves is the one provided by the CBU, for USD nominated bond prices, traded both in the domestic market and in foreign exchanges. It should be noticed that besides the liquidity problem of the bond market, there are no derivatives on these instruments, so all the available information is provided by the bond prices.

The data to be used in the estimation procedure corresponds to the time period from January 4, 2010 to October 30, 2013. In the Vasicek model, we use the bond *BE330115P*, expiring in January 2033, and, in the G2++ model, we use the same one and add the bond *BE250928F* with expiry in September 2025. We choose these bonds as they are the most frequently traded. Each time series is processed according to the coupon payment scheme (amount and frequency) to obtain the corresponding yields that give the zero-coupon bond prices used in the estimation.

17.5.2.2 Estimation in Vasicek Model

In order to evaluate the influence of the nonliquid computed prices (as explained above), we proceed in this case in estimating the parameters corresponding to the Vasicek model in two situations. We first estimate parameters using all the available information, i.e. we use weekly bond prices taken on Wednesdays. The obtained

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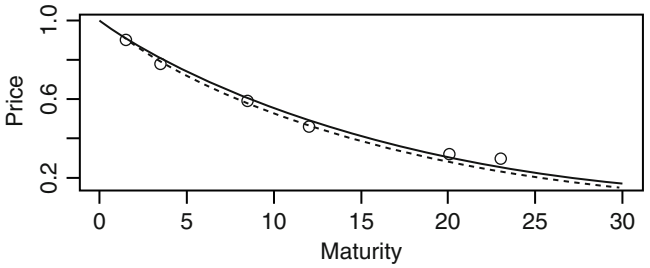


Fig. 17.2 The *solid line* shows the first estimation with Vasicek model and the *dotted* one the second (that uses only negotiated prices). The *dots* correspond to market prices of zero-coupon bonds

parameters in (17.1) are:

$$a = 1.7051; \qquad b = 0.0937; \qquad \sigma = 0.3721.$$

In the second case, we use only negotiated prices (504 observations). This implies that the time intervals between prices are not regular, leading to a slightly more complicated estimation scheme. The obtained parameters in (17.1) are:

$$a = 1.7145; \qquad b = 0.0896; \qquad \sigma = 0.4971.$$

This comparison helps us to verify that the variation in the values of the parameters is not significant (perhaps the most important variation is registered in σ), and for this reason in what follows and in this first approach to the problem, we will use all the available time series. In Fig. 17.2 we observe the price curve corresponding to August 13, 2013, given by the model in both estimations carried out and also the bond prices issued by the CBU.

17.5.2.3 Estimation in G2++ Model

As we mentioned above, we use two bond price time series, with expiration in 2025 and 2033, respectively. We take weekly prices corresponding to Wednesdays. The obtained values for the parameters in (17.3), with the ML estimation method described above, are:

$$a = 0.1300; \quad b = 0.3526; \quad \sigma = 0.2062; \quad \eta = 0.4892; \quad \rho = -0.99.$$

Using these values with the corresponding bond prices, we can obtain the daily time series corresponding to the short rate. This allows us to compute the bond prices for arbitrary maturities in each one of the days. We show the bond price curve for the following maturities: 1, 2, 3, 6 and 9 months and 1, 2, 3, 5, 7, 10, 15, 20 and 25 years.

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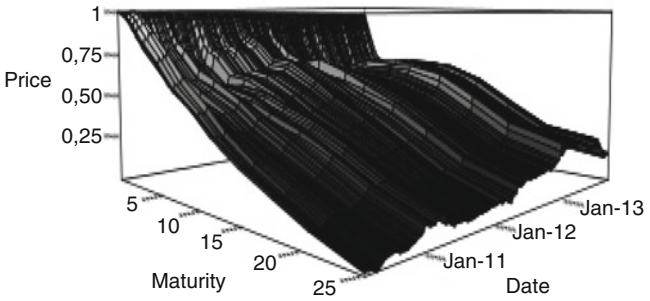


Fig. 17.3 Surface of zero-coupon bond prices using the G2++ model

In Fig. 17.3 we see the daily zero-coupon bond price curve corresponding to the analysed time period issued by the Uruguayan government, as a result of the application of the G2++ model.

If we analyse the data in Fig. 17.3 for the second semester of 2012, the model gives a curve with non-negative slope for some maturities. This fact introduces arbitrage possibilities, as it gives cheaper bonds with smaller maturities than others with larger maturities. More precisely, our model adjustment gives parameters that result in increasing bond prices in some intervals of the maturity T. Our parameter estimations include a correlation very close to -1 and instances of negative values of the factors $x(t)$ and $y(t)$. We will return to this fact in more detail in the next subsection.

17.5.2.4 On Arbitrage Possibilities in G2++ Model

We analyse how formula (17.6) for a zero-coupon bond price in G2++ model can give arbitrage opportunities. Equation (17.5) can be written as

$$V(t,T) = \frac{\sigma^2}{a^2}\int_t^T (1-e^{-a(T-u)})^2 du + \frac{\eta^2}{b^2}\int_t^T (1-e^{-b(T-u)})^2 du + \frac{2\rho\sigma\eta}{ab}\int_t^T (1-e^{-a(T-u)})(1-e^{-b(T-u)})du.$$

We then take logarithms and differentiate formula (17.6) w.r.t T, to obtain

$$\frac{\partial}{\partial T}\log P(t,T) = \frac{\partial}{\partial T}\log P^M(0,T) + \frac{\sigma^2}{2a^2}\Big((1-e^{-a(T-t)})^2 - (1-e^{-aT})^2\Big) + \frac{\eta^2}{2b^2}\Big((1-e^{-b(T-t)})^2 - (1-e^{-bT})^2\Big)$$

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$$+\frac{\rho\sigma\eta}{ab}\Big((1-e^{-a(T-t)})(1-e^{-b(T-t)})-(1-e^{-aT})(1-e^{-bT})\Big)$$
$$-e^{-a(T-t)}x(t)-e^{-b(T-t)}y(t).$$

In this formula the first three addends are negative. The fourth has the opposite sign of ρ, and the last two depend on $x(t)$ and $y(t)$. In our application ρ is close to -1, which is associated with the fact that $x(t)$ and $y(t)$ take opposite signs; therefore, in some time intervals, the derivative is positive, as shown in Fig. 17.3. Nevertheless the model gives general valuable information about the structure of the debt.

AQ9

17.5.3 Results for Forward-Rate Models

17.5.3.1 Data

In this analysis we use weekly prices corresponding to the whole year 2014, of 10 USD nominated bonds issued by the CBU. The maximum expiration date corresponds to the *BE451120F* bond, in November 2045. To obtain an approximation of the initial interest rate, we used the reference curve *CUD-BEVSA* daily issued by BEVSA for three months, and, to adjust the initial forward-rate curve, we used the yield curve corresponding to January 5, 2013.

17.5.3.2 Results in Ho–Lee Model and Hull–White Model

Calibrating the Ho–Lee model, we obtain the parameter σ for each day. In Fig. 17.4 we observe then the daily variation of σ. The mean value of the estimation is 0.0232, with a standard deviation of 0.0094.

For the Hull–White model, the same procedure is carried out. In Fig. 17.5 (above), we observe the daily values of the parameter a and in Fig. 17.5 (below) the values of the parameter σ. The mean value for a is 0.0693 with a standard deviation of 0.0257. The corresponding mean and standard deviation for σ are 0.0177 and 0.0079, respectively.

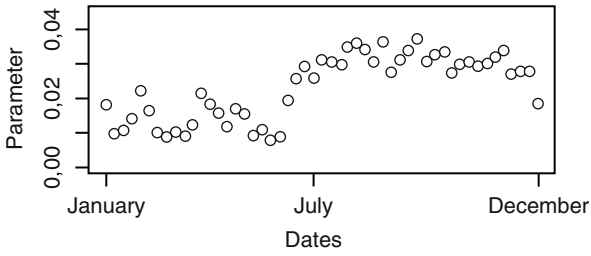


Fig. 17.4 Calibrated Ho–Lee σ parameter over time

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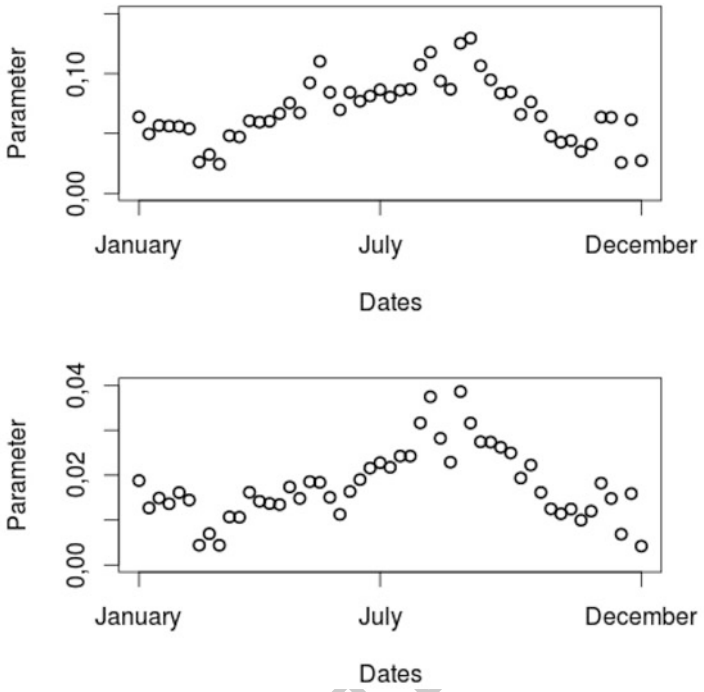


Fig. 17.5 Calibrated Hull–White parameters over time. *Above* a, *below* σ

The calibration of the Ho–Lee model corresponding to Wednesday, June 25, 2014, gives $\sigma = 0.3071$ and is shown in Fig. 17.6 (above). In Fig. 17.6 (below), we see the calibration for the same day corresponding to the Hull–White model. The parameters in this case are $a = 0.0813$ and $\sigma = 0.0215$.

Both in the Ho–Lee and Hull–White models, the daily calibration allows us to construct term structure of interest rates for the corresponding day. The result is shown in Fig. 17.7.

17.6 Conclusions

The objective of the present work is to present an arbitrage-free model to price the Uruguayan debt nominated in USD. A second purpose is to provide an instrument capable of pricing derivatives on bonds, as interest rate swaps, that are beginning to be used in the Uruguayan market. Based on data from the Uruguayan market for coupon bonds, we adjust four different Gaussian models.

This information is of valuable interest to financial practitioners and policymakers alike. Policymakers monitor expectations of future monetary policy to gauge the effectiveness of their strategy. For practitioners, the availability of accurate interest

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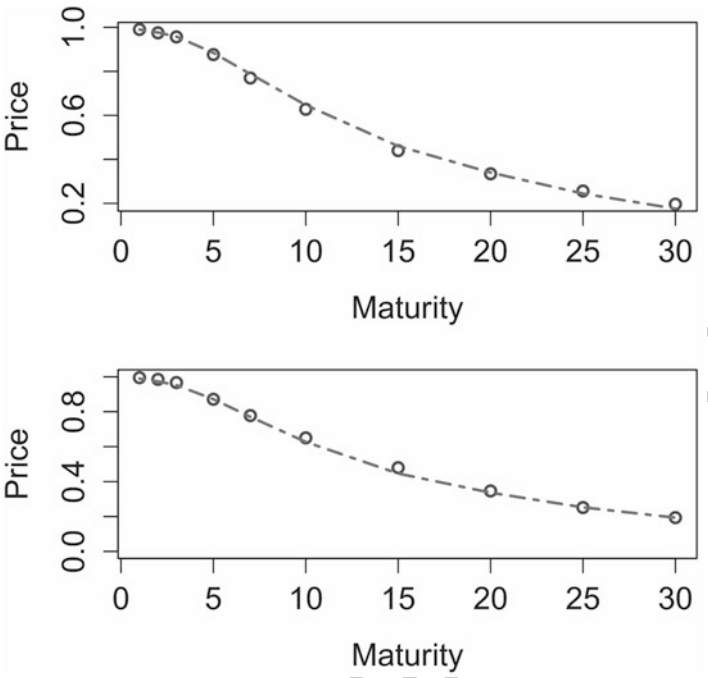


Fig. 17.6 Bond price curve adjusted to market data. *Above* Ho–Lee, *below* Hull–White. In both cases the *dots* represent the market prices

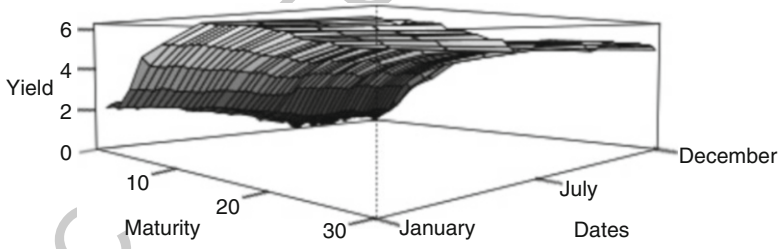


Fig. 17.7 Term structure in Uruguayan market via Hull–White model

rate forecasts can be the key to a successful trading strategy. We hope that these modelling exercises would enrich our understanding of market expectations and improve the understanding of the characteristic behaviour of the term structure of interest rate.

First, we adjust the Vasicek and the G2++ models for interest rates, departing from data from one bond and two bonds, respectively. We follow the methodology proposed by Chen and Scott (1993) that departs from historical data to estimate the parameters of the model through maximum likelihood. The second model is more flexible than the first one, as it involves more parameters, and also adjusts the initial

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interest rate curve avoiding the possibility of arbitrage. Nevertheless, as we show, when high values of correlation are obtained (and this is our case), the model can give arbitrage situations.

Second, we model the debt through the HJM models. We choose two different situations, assuming first that the volatility is constant, giving rise to the Ho–Lee model, the simplest HJM model, and second we assume an exponentially decreasing volatility, giving rise to the two-parameter Hull–White model. In this case the adjustment seems to be better, as no-arbitrage possibilities appear (the HJM model uses the initial curve and also works under a risk-neutral measure), and the different values for different days in both cases show certain stability. We conclude that HJM models fit better to the data and are capable of describing the whole market structure with information of a series of then more traded bonds.

AQ10 Further work includes several issues, such as the comparison of the two methods employed (maximum likelihood and calibration) in both interest rate and forward-rate models; the analysis of the effect of the lack of liquidity, including adjustment with respect to transaction volumes in the least-square minimization; the use of more data with the help of the methodology proposed by Chen and Scott (1993); and the consideration of more sophisticated HJM models.

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AQ1. Please check if edit to sentence starting “We model bond...” is okay.
AQ2. Please check the inserted keywords from “Key Words.doc” and correct if necessary.
AQ3. Please check if edit to sentence starting “There are essentially...” is okay.
AQ4. Please check if edit to sentence starting “Although empirical data...” is okay.
AQ5. Please check if edit to the sentences “For this reason...” and “as a result...” is okay.
AQ6. Please check if “;” should be changed to “,”.
AQ7. Please check if edit to sentence starting “This can lead...” is okay.
AQ8. Please check if edit to sentence starting “Model prices are...” is okay.
AQ9. Please check if edit to sentence starting “In our application...” is okay.
AQ10. Please check if edit to sentence starting “Further work includes...” is okay.
AQ11. Please provide volume and page range for the reference Di Francesco (2012).

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