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Abstract	We review the computation of Greeks for exponential Lévy models extending Lewis formula for the option value. This gives accurate approximations using fast Fourier transform. We present an exhaustive development of Greeks for call options. We provide error estimation in for all Greeks in the Black–Scholes model (where Greeks can be exactly computed) and consider other models used in the literature, such as the Merton and variance gamma models.	
Keywords (separated by "-")	Exponential Lévy models - Option pricing - Lévy processes - Derivatives - Greeks - Lewis pricing formula - Black Scholes - Merton model - Variance gamma model	

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Chapter 6
Computing Greeks for Lévy Models: The Fourier Transform Approach

Federico De Olivera and Ernesto Mordecki

Abstract We review the computation of Greeks for exponential Lévy models extending Lewis formula for the option value. This gives accurate approximations using fast Fourier transform. We present an exhaustive development of Greeks for call options. We provide error estimation in for all Greeks in the Black–Scholes model (where Greeks can be exactly computed) and consider other models used in the literature, such as the Merton and variance gamma models.

AQ1 **Keywords** Exponential Lévy models • Option pricing • Lévy processes • Derivatives • Greeks • Lewis pricing formula • Black Scholes • Merton model • Variance gamma model

6.1 Introduction

We consider a Lévy process $X = \{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbf{Q})$; a financial market model with two assets; a deterministic savings account $B = \{B_t\}_{t\geq 0}$, given by

$$B_t = B_0 e^{rt},$$

with $r \geq 0$ and $B_0 > 0$; and a stock $S = \{S_t\}_{t\geq 0}$, given by

$$S_t = S_0 e^{rt+X_t}, \tag{6.1}$$

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with $S_0 > 0$, where $X = \{X_t\}_{t\geq 0}$ is a Lévy process. When the process X has continuous paths, we obtain the classical Black–Scholes model (Merton 1973). For general reference on the subject, we refer to Kyprianou (2006) or Cont and Tankov (2004).

The aim of this paper is the computation of the price partial derivatives of a European option with general payoff with respect to any parameter of interest. These derivatives are usually named as "Greeks," and consequently, we use the term *Greek* to refer to any price partial derivative of the option (of any order and with respect to any parameter). Our approach departs from the subtle observation by Cont and Tankov (see Cont and Tankov 2004, p. 365):

Contrary to the classical Black–Scholes case, in exponential-Lévy models there are no explicit formulae for call option prices, because the probability density of a Lévy process is typically not known in closed form. However, the characteristic function of this density can be expressed in terms of elemen- elementary functions for the majority of Lévy processes discussed in the literature. This has led to the development of Fourier-based option pricing methods for exponential-Lévy models. In these methods, one needs to evaluate one Fourier transform numerically but since they simultaneously give option prices for a range of strikes and the Fourier transform can be efficiently computed using the FFT algorithm, the overall complexity of the algorithm per option price is comparable to that of evaluating the Black–Scholes formula.

In other words, in the need of computation of a range of option prices, from a practical point of view, the Lewis formula works as a closed formula, as it can be implemented and computed with approximately the same precision and in the same time as the Black–Scholes formula.

Some papers have addressed this problem. Eberlein et al. Eberlein et al. (2009) obtained a formula similar to the Lewis one and derived delta (Δ) and gamma (Γ), the price partial derivatives with respect to the initial value S_t of first and second order, for a European payoff function. The assumptions are similar to the ones we require. Takahashi and Yamazaki (2008) also obtain these Greeks in the case of call options, based on the Carr and Madan approach. The advantage of the Lewis formula is that it gives option prices for general European payoffs, while the Carr–Madan only price European vanilla options or some other type of strike-dependent options. It must be said that the Carr–Madan formula is applicable for general processes, not necessarily Lévy, but, as we are interested in the Lévy process, we heavily rely on Lewis' approach, being the main difference with the analysis in Lee (2004). Other works deal with the problem of Greek computation for more general payoff functions, including path-dependent options [e.g., see Boyarchenko and Levendorskiĭ (2009), Chen and Glasserman (2007), Glasserman and Liu (2007), Glasserman and Liu (2008), Jeannin and Pistorius (2010) and Kienitz (2008)]. These works are based on different techniques, such as simulation or finite differences introducing a method error that has to be analyzed, whereas our approach does not.

In the present paper, we obtain closed formulas for Greeks based on the Lewis formula that computes efficiently and with arbitrary precision [as exposed in Cont

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and Tankov (2004)], for arbitrary payoff European options in the Lévy models with respect to any parameter and arbitrary order. As an example, we analyze the case of call options.

6.2 Greeks for General European Options in Exponential Lévy Models

Denote by $\mathbf{Q}$ the risk-neutral pricing measure, i.e.,

$$\mathbf{E}\, e^{X_1} = 1, \tag{6.2}$$

where $\mathbf{E}$ denotes expectation under $\mathbf{Q}$. Furthermore, by the Lévy–Khintchine theorem, we obtain that $\mathbf{E}\, e^{izX_t} = e^{t\Psi(z)}$, where the characteristic exponent is

$$\Psi(z) = -iz(1-iz)\frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^{izy} - 1 - iz(e^y - 1)\right)\nu(dy), \tag{6.3}$$

with $\sigma \geq 0$ as the standard deviation of the Gaussian part of the Lévy process and ν its jump measure.

Regarding the payoff, following Lewis (2001), denote $s = \ln S_T$ and consider a payoff $w(s)$ and its Fourier transform $\hat{w}(z) = \int_{-\infty}^{\infty} \exp(isx)w(x)dx$. For instance, if K is a strike price, the call option payoff and its respective transform are

$$w(s) = (e^s - K)^+, \qquad \hat{w} = \frac{-K^{iz+1}}{z^2 - iz} \quad (\Im(z) > 1). \tag{6.4}$$

Then, the Lewis formula (Lewis 2001) for the European options, valued at time t, and denoting $\tau = T - t$ the time to maturity, is:

$$V_t = \frac{e^{-r\tau}}{2\pi} \int_{iv+\mathbb{R}} e^{-iz(\ln(S_t)+r\tau)} e^{\tau\Psi(-z)} \hat{w}(z)dz, \tag{6.5}$$

where $z \in S_V = \{u + iv : u \in \mathbb{R}\}$ and v must be chosen depending on the payoff function (Lewis 2001). In this context, it is simple to obtain some general formulas for the Greeks.

In order to differentiate under the integral sign, we present the following classical result.

Lemma 1. *Let $\Theta \subset \mathbb{R}$ an interval and $\mathscr{I} = iv + \mathbb{R}$. Let $h : \mathscr{I} \times \Theta \to \mathbb{C}$ and $g : \mathscr{I} \to \mathbb{C}$ such that*

- *$h(\cdot, \theta)g(\cdot)$ is integrable for all $\theta \in \Theta$ and g is integrable.*
- *$h(z, \cdot)$ is differentiable in Θ for all $z \in \mathscr{I}$ and $\frac{\partial h}{\partial \theta}$ is bounded.*

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Then, $\int_{\mathscr{I}} h(x,\theta)g(x)dx$ is differentiable and

$$\frac{\partial}{\partial\theta}\int_{\mathscr{I}} h(x,\theta)g(x)dx = \int_{\mathscr{I}} \frac{\partial h(x,\theta)}{\partial\theta}g(x)dx \qquad \forall\theta\in\Theta.$$

Proof. We observe that $\left|\frac{\partial h(z,\theta)g(z)}{\partial\theta}\right| \le C|g(z)|$ for all $z\in\mathscr{I}$, $\theta\in\Theta$. The result is obtained from Theorem 2.27 in Folland (1999).

In consequence, in the rest of the paper, we will always assume that the conditions in Lemma 1 are satisfied for the real part of the integrand because the price imaginary part integrate is zero.

6.2.1 First-Order Greeks

We introduce the auxiliary function

$$\vartheta(z) = e^{-iz(\ln(S_t)+r\tau)}e^{\tau\Psi(-z)}\hat{w}(z).$$

Departing from (6.5) and (6.3), by differentiation under the integral sign, we obtain

$$\begin{aligned}
\Delta_t &= \frac{\partial V_t}{\partial S_t} = -\frac{1}{S_t}\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} iz\vartheta(z)dz,\\
\rho_t &= \frac{\partial V_t}{\partial r} = -\tau\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}}(1+iz)\vartheta(z)dz,\\
\mathscr{V}_t &= \frac{\partial V_t}{\partial\sigma} = \tau\sigma\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} iz(1+iz)\vartheta(z)dz,\\
\Theta_t &= \frac{\partial V_t}{\partial\tau} = \frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}}(\Psi(-z)-(1+iz)r)\,\vartheta(z)dz.
\end{aligned}$$

Usually, the Lévy models used in the literature depend on a set of parameters that specify the jump measure. Therefore, we denote $\nu(dy)=\nu_\theta(dy)$ and $\Psi(z)=\Psi_\theta(z)$; then

$$\frac{\partial V_t}{\partial\theta} = \tau\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}}\frac{\partial\Psi_\theta(-z)}{\partial\theta}\vartheta(z)dz.$$

6.2.2 Second-Order Greeks

Similarly, we obtain

$$\Gamma_t = \frac{\partial^2 V_t}{\partial S_t^2} = \frac{1}{S_t^2}\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} iz(1+iz)\vartheta(z)dz,$$

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$$\text{Vanna}_t = \frac{\partial^2 V_t}{\partial\sigma\,\partial S_t} = \tau\sigma\frac{1}{S_t}\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} z^2(1+iz)\vartheta(z)dz,$$

$$\text{Vomma}_t = \frac{\partial^2 V_t}{\partial\sigma^2} = \tau\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} \left(1+\tau\sigma^2 iz(1+iz)\right) iz(iz+1)\vartheta(z)dz,$$

$$\text{Charm}_t = \frac{\partial^2 V_t}{\partial S_t\partial\tau} = -\frac{1}{S_t}\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} iz\left(\Psi(-z)-(1+iz)r\right)\vartheta(z)dz,$$

$$\text{Veta}_t = \frac{\partial^2 V_t}{\partial\sigma\,\partial\tau} = \sigma\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} iz(1+iz)\left(\tau\Psi(-z)-(iz+1)r\tau+1\right)\vartheta(z)dz,$$

$$\text{Vera}_t = \frac{\partial^2 V_t}{\partial\sigma\,\partial r} = -\tau^2\sigma\frac{e^{-r\tau}}{2\pi}\int_{iv+\mathbb{R}} iz(iz+1)^2\vartheta(z)dz.$$

Other derivatives can be obtained analogously. In the next section, we will focus in the case of call options. This allows to obtain more explicit formulas.

6.3 Greeks for Call Options in Exponential Lévy Models

In order to exploit the particular payoff function, we exhaustively develop the Greeks for call options. The put option corresponding formulas can be obtained immediately via put-call parity. For other payoffs, the procedure to obtain the Greeks is analogous.

When the strike K is fixed, $x = \ln(K/S_t) - r\tau$ is variable in terms of S_t, r, and τ. Then, we must consider this for the computation of Greeks Δ, Γ, ρ, and others.

Lemma 2. *Let X_τ be a Lévy process with triplet (γ,σ,ν) and characteristic exponent $\Psi(z)$ such that $\Psi(-i) = 0$ and $\int_{|y|>1} e^{vy}\nu(dy) < \infty$ with $v \geq 0$. Then, if $z \in iv + \mathbb{R}$,*

$$|\Psi_J(-z)| \leq (|z|^2+|z|)\frac{e^v}{2}\int_{|y|\leq 1} y^2\nu(dy) + 2\int_{|y|>1}(e^{vy}+1)\nu(dy)$$

and

$$|\Psi(-z)| \leq (|z|^2+|z|)\left(\frac{e^v}{2}\int_{|y|\leq 1} y^2\nu(dy) + \frac{\sigma^2}{2}\right) + 2\int_{|y|>1}(e^{vy}+1)\nu(dy), \tag{6.6}$$

where $\Psi_J(z) = \int_{\mathbb{R}}\left(e^{izy}-1-iz(e^y-1)\right)\nu(dy)$.

Proof. Let $I(z) = \int_{\mathbb{R}}\left(e^{izy}-1-izy\mathbf{1}_{\{|y|\leq 1\}}\right)\nu(dy)$. Applying Taylor's expansion with Lagrange error form at point $y = 0$, there exists θ_y with $|\theta_y| \leq |y|$ such that

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$$
\begin{aligned}
e^{izy} - 1 - izy\mathbf{1}_{\{|y|\le 1\}} =& izy\mathbf{1}_{\{|y|>1\}} - z^2y^2\frac{e^{iz\theta_y}}{2} \\
=& -z^2y^2\frac{e^{iz\theta_y}}{2}\mathbf{1}_{\{|y|\le 1\}} + \left(izy - z^2y^2\frac{e^{iz\theta_y}}{2}\right)\mathbf{1}_{\{|y|>1\}} \\
=& -z^2y^2\frac{e^{iz\theta_y}}{2}\mathbf{1}_{\{|y|\le 1\}} + (e^{izy}-1)\mathbf{1}_{\{|y|>1\}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
|I(-z)| \le& \int_{\mathbb{R}} \left| -z^2y^2\frac{e^{-iz\theta_y}}{2}\mathbf{1}_{\{|y|\le 1\}} + (e^{-izy}-1)\mathbf{1}_{\{|y|>1\}} \right| \nu(dy) \\
\le& |z|^2 \int_{|y|\le 1} \frac{e^{v\theta_y}}{2} y^2\nu(dy) + \int_{|y|>1} (e^{vy}+1)\nu(dy) \\
\le& |z|^2 \frac{e^v}{2} \int_{|y|\le 1} y^2\nu(dy) + \int_{|y|>1} (e^{vy}+1)\nu(dy). \qquad (6.7)
\end{aligned}
$$

Using (6.7), we have

$$
\begin{aligned}
|\Psi_J(-z)| =& |I(-z) + izI(-i)| \\
\le& (|z|^2 + |z|)\frac{e^v}{2} \int_{|y|\le 1} y^2\nu(dy) + 2\int_{|y|>1} (e^{vy}+1)\nu(dy).
\end{aligned}
$$

For the continuous part, let $\Psi_C(-z) = (iz - z^2)\frac{\sigma^2}{2}$; thus,

$$
\begin{aligned}
|\Psi(-z)| \le& |\Psi_C(-z)| + |\Psi_J(-z)| \\
\le& (|z|^2 + |z|)\left(\frac{e^v}{2}\int_{|y|\le 1} y^2\nu(dy) + \frac{\sigma^2}{2}\right) + 2\int_{|y|>1} (e^{vy}+1)\nu(dy).
\end{aligned}
$$

Lemma 3. *Let $\{X_\tau\}_{\tau\ge 0}$ be a Lévy process with triplet (γ,σ,ν) and characteristic exponent $\Psi(z)$, such that $\Psi(-i) = 0$ and $\mathbf{E}[e^{vX_\tau}] < \infty$ with $v > 0$.*

1. *If $\int_{iv+\mathbb{R}} |z|^{-1}|e^{\tau\Psi(-z)}|dz < \infty$, then*

$$
\mathbf{P}(X_\tau > x) = -\frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{izx}}{iz} e^{\tau\Psi(-z)}dz, \qquad (6.8)
$$

$$
\mathbf{E}(e^{X_\tau}\mathbf{1}_{\{X_\tau>x\}}) = -\frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{1+iz} e^{\tau\Psi(-z)}dz. \qquad (6.9)
$$

2. *If*

$$
\int_{iv+\mathbb{R}} |z^n e^{\tau\Psi(-z)}|dz < \infty, \quad \text{for some } n = 0, 1, \ldots, \qquad (6.10)
$$

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then X_τ has a density of class C^n and

$$\frac{\partial^n f(x)}{\partial x^n} = \frac{1}{2\pi}\int_{iv+\mathbb{R}} (iz)^n e^{izx} e^{\tau\Psi(-z)} dz.$$

Proof. For a call option, the Fourier transform of the payoff function is $\hat{w}(z) = \frac{e^{iz(\ln(S_t)+r\tau+x)}}{iz(1+iz)}$. *Then from the option value (6.5), we have, with* $x = \log(K/S_t) - r\tau$,

$$C_t(x) = S_t \frac{e^x}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{izx}e^{\tau\Psi(-z)}}{iz(iz+1)} dz. \tag{6.11}$$

Then, being $x \in [\alpha, \beta]$ *and* $C_1 = \max_{\alpha\le x\le\beta} e^{(1-v)x}$,

$$\left|\frac{\partial e^{(1+iz)x} e^{\tau\Psi(-z)}[iz(iz+1)]^{-1}}{\partial x}\right| \le C_1 |z^{-1}||e^{\tau\Psi(-z)}| \in L^1(iv+\mathbb{R}),$$

and by Theorem 2.27 in Folland (1999), we can differentiate under the integral sign. Therefore, with $S_t = 1$,

$$\begin{aligned}\mathbf{P}(X_\tau > x) &= -e^{-x}\frac{\partial}{\partial x}\int_x^\infty (e^s - e^x)F(ds)\\ &= -e^{-x}\frac{\partial C_t(x)}{\partial x} = -\frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{izx}}{iz} e^{\tau\Psi(-z)} dz.\end{aligned}$$

On the other hand, with $S_t = 1$,

$$\begin{aligned}\mathbf{E}(e^{X_\tau}\mathbf{1}_{\{X_\tau>x\}}) &= \int_x^\infty e^s F(ds) = -e^x\frac{\partial}{\partial x}\int_x^\infty (e^{s-x}-1)F(ds)\\ &= -e^x\frac{\partial e^{-x}C_t(x)}{\partial x} = C_t(x) - \frac{\partial C_t(x)}{\partial x}\\ &= \frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{iz(1+iz)} e^{\tau\Psi(-z)} dz - \frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{iz} e^{\tau\Psi(-z)} dz\\ &= -\frac{1}{2\pi}\int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{1+iz} e^{\tau\Psi(-z)} dz.\end{aligned}$$

For the second part, observe in (6.8) that if $x \in [\alpha, \beta]$ *and* $C_2 = \max_{\alpha\le x\le\beta} e^{-vx}$,

$$\left|\frac{\partial^{n+1}\frac{e^{izx}}{iz} e^{\tau\Psi(-z)}}{\partial x^{n+1}}\right| \le C_2|z|^n|e^{\tau\Psi(-z)}| \in L^1(iv+\mathbb{R}).$$

The result is obtained from Theorem 2.27 in Folland (1999).

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We conclude the preliminaries of this section with some notations. As (6.2) holds, the relation $d\tilde{\mathbf{Q}} = e^{X_T} d\,\mathbf{Q}$ defines a probability measure $\tilde{\mathbf{Q}}$, known as the *dual martingale measure*. A call option price with payoff (6.4), and log-forward moneyness[1] $x = \ln(K/S_0) - r\tau$, can be computed by

$$C_t(x) = S_t \left(\tilde{\mathbf{Q}}(X_\tau > x) - e^x \mathbf{Q}(X_\tau > x) \right).$$

Furthermore, if X_t has a density $f_t(x)$ under $\mathbf{Q}$, i.e., if condition (6.10) holds with $n = 0$, its density under $\tilde{\mathbf{Q}}$ is $\tilde{f}_t(s) = e^s f_t(s)$. In order to obtain Greeks in terms of the risk-neutral measure, we replace $\mathbf{P}$ by $\mathbf{Q}$ in (6.8), and consequently (6.9), (6.10), and (6.11) are related to the probability measure $\mathbf{Q}$.

6.3.1 First-Order Greeks for Call Options

In this section, we do not assume general requirements. We specify the requirements in each case.

6.3.1.1 Delta

Assume that $\int_{iv+\mathbb{R}} |z|^{-1} |e^{\tau\Psi(-z)}| dz < \infty$ and $S_t \in [A, B]$. From (6.11), we obtain

$$\begin{aligned}
\Delta_t^L &= \frac{\partial C_t(x(S_t))}{\partial S_t} \\
&= \frac{\partial}{\partial S_t} S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x(S_t)}}{iz(1+iz)} e^{\tau\Psi(-z)} dz \\
&= \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{iz(1+iz)} e^{\tau\Psi(-z)} dz - \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{iz} e^{\tau\Psi(-z)} dz \\
&= -\frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{1+iz} e^{\tau\Psi(-z)} dz = \tilde{\mathbf{Q}}(X_\tau > x).
\end{aligned}$$

6.3.1.2 Rho

Denote now $x = \ln(K/S_t) - r\tau$, that depends on the interest rate r. Assume that $\int_{iv+\mathbb{R}} |z|^{-1} |e^{\tau\Psi(-z)}| dz < \infty$ and $r \in [R_1, R_2]$. Then

[1] This seems to be the standard definition, although in Cont and Tankov (2004) it is defined as the opposite quantity.

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$$\rho_t^L = \frac{\partial C_t(x(r))}{\partial r} = S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} -\tau \frac{e^{(1+iz)x}}{iz} e^{\tau\Psi(-z)} dz = \tau S_t e^x \mathbf{Q}(X_\tau > x).$$

6.3.1.3 Vega

In Black–Scholes, vega shows the change in variance of the log price. In exp-Lévy models, the derivative of $C_t(x)$ w.r.t. σ does not give exactly the same information. We assume that X_τ has density f [see (6.10)] and $z \in iv + \mathbb{R}$. Let

$$h(z,\sigma) = e^{\tau iz(1+iz)\frac{\sigma^2}{2}}, \qquad g(z) = \frac{e^{izx+\tau\int_{\mathbb{R}}(e^{izy}-1-iz(e^y-1))\nu(dy)}}{iz(1+iz)}.$$

Thus, $\frac{\partial}{\partial\sigma}h(z,\sigma)$ is bounded. On the other hand, $\int_{iv+\mathbb{R}} |g(z)|dz < \infty$ because $|\mathbf{E}(e^{-i(iv+s)J_\tau})| \le \mathbf{E}(e^{vJ_\tau}) < \infty$, where J_τ is the jump part of X_τ. By Lemma 1, we can differentiate under the integral sign. Then,

$$\begin{aligned}\mathcal{V}_t^L &= \frac{\partial C_t(x)}{\partial\sigma} = S_t \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{izx} e^{\tau\Psi_\sigma(-z)}}{iz(iz+1)} \tau\sigma iz(1+iz)dz \\ &= S_t \tau\sigma e^x f_\tau(x).\end{aligned}$$

In order to complete the information provided by vega, we can calculate the derivative with respect to the jump intensity, when this intensity is finite, i.e., when $\nu(dy) = \lambda G(dy)$ for a probability distribution G, and assume the existence of a density (6.10). Denote

$$\begin{aligned}h(z,\lambda) &= \frac{e^{\tau\lambda\int_{\mathbb{R}}\left(e^{-izy}-1+iz(e^y-1)\right)G(dy)}}{\tau\int_{\mathbb{R}}\left(e^{-izy}-1+iz(e^y-1)\right)G(dy)}, \\ g(z) &= \frac{e^{(iz+1)x} e^{\tau iz(1+iz)\frac{\sigma^2}{2}}}{iz(iz+1)} \tau \int_{\mathbb{R}} \left(e^{-izy}-1+iz(e^y-1)\right) G(dy),\end{aligned}$$

where $\frac{\partial h(z,\lambda)}{\partial\lambda}$ is bounded and from Lemma 2 $\int_{iv+\mathbb{R}} |g(z)|dz < \infty$; then by Lemma 1, we obtain

$$\begin{aligned}\frac{\partial C_t(x)}{\partial\lambda} &= \frac{\partial}{\partial\lambda} S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau\Psi(-z)}}{iz(iz+1)} dz \\ &= \tau S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau\Psi(-z)}}{iz(iz+1)} \overline{\Psi}_J(-z)dz,\end{aligned}$$

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with $\overline{\Psi}_J(-z) = \int_{\mathbb{R}} \left(e^{-izy} - 1 + iz(e^y - 1)\right) G(dy)$. Using Fubini's theorem, we obtain

$$\begin{aligned}
\frac{\partial_\tau C_t(x)}{\partial\lambda} =& \frac{\partial}{\partial\lambda} S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau\Psi(-z)}}{iz(iz+1)} dz \\
=& \tau S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau\Psi(-z)}}{iz(iz+1)} \int_{\mathbb{R}} \left(e^{-izy} - 1 + iz(e^y - 1)\right) G(dy) dz \\
=& \tau S_t \Bigg[\int_{\mathbb{R}} \Bigg(e^y \frac{e^{x-y}}{2\pi} \int_{iv+\mathbb{R}} e^{iz(x-y)} \frac{e^{\tau\Psi(-z)}}{iz(1+iz)} dz \\
& - \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau\Psi(-z)}}{iz(1+iz)} dz \\
& + (e^y - 1) \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau\Psi(-z)}}{1+iz} dz \Bigg) G(dy) \Bigg] \\
=& \tau \left[\int_{\mathbb{R}} \Big(e^y C_t(x-y) - C_t(x) - S_t(e^y - 1) \tilde{\mathbf{Q}}(X_\tau > x) \Big) G(dy) \right].
\end{aligned}$$

The use of Fubini's theorem is justified by (6.7) and the additional hypothesis $\int_{iv+\mathbb{R}} |e^{\tau\Psi(-z)}| dz < \infty$.

6.3.1.4 Theta

Assume condition (6.10) holds with $n = 2$, and let

$$h(z,\tau) = e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}, \qquad g(z) = \frac{1}{iz(1+iz)}.$$

Then, $\int_{iv+\mathbb{R}} |g(z)| dz < \infty$; moreover, from (6.6) and $\int_{iv+\mathbb{R}} |z^2 e^{\tau\Psi(-z)}| dz < \infty$,

$$\frac{\partial h(z,\tau)}{\partial\tau} = e^{(iz+1)x_\tau} e^{\tau\Psi(-z)} \Big(-r(1+iz) + \Psi(-z) \Big)$$

is bounded, and by Lemma 1,

$$\begin{aligned}
\Theta_t^L =& \frac{\partial_\tau C_t(x_\tau)}{\partial\tau} = \frac{\partial}{\partial\tau} S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}}{iz(iz+1)} dz \\
=& S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}}{iz(1+iz)} \Big(\Psi(-z) - r(1+iz) \Big) dz.
\end{aligned}$$

Using Fubini's theorem, we obtain

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Author's Proof

$$
\begin{aligned}
\Theta_t^L =& S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}}{iz(1+iz)} \Big(\Psi(-z) - r(1+iz)\Big) dz \\
=& S_t \Big[-\frac{r}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}}{iz} dz \\
&+ \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}}{iz(iz+1)} \Big(iz(1+iz)\frac{\sigma^2}{2} \\
&+ \int_{\mathbb{R}} \big(e^{-izy} - 1 + iz(e^y - 1)\big)\, \nu(dy) \Big) dz \Big] \\
=& S_t \Big[re^{x_\tau}\, \mathbf{Q}(X_\tau > x_\tau) + \frac{\sigma^2}{2} e^{x_\tau} f_\tau(x_\tau) \\
&+ \int_{\mathbb{R}} \Big(e^y \frac{e^{x-y}}{2\pi} \int_{iv+\mathbb{R}} e^{iz(x-y)} \frac{e^{\tau\Psi(-z)}}{iz(1+iz)} dz - \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau\Psi(-z)}}{iz(1+iz)} dz \\
&+ (e^y - 1) \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau\Psi(-z)}}{1+iz} dz \Big) \nu(dy) \Big] \\
=& S_t \Big[re^{x_\tau}\, \mathbf{Q}(X_\tau > x_\tau) + \frac{\sigma^2}{2} e^{x_\tau} f_\tau(x_\tau) \Big] \\
&+ \int_{\mathbb{R}} \Big(e^y C_t(x_\tau - y) - C_t(x_\tau) - S_t(e^y - 1)\tilde{\mathbf{Q}}(X_\tau > x_\tau) \Big) \nu(dy).
\end{aligned}
$$

The use of Fubini's theorem is justified by (6.7) and the additional hypothesis $\int_{iv+\mathbb{R}} |z^2 e^{\tau\Psi(-z)}| dz < \infty$.

6.3.2 Second-Order Greeks for Call Options

6.3.2.1 Gamma

Once delta is obtained, we must only differentiate again with respect to S_t, to obtain gamma. We assume that X_τ has density f and $S_t \in [A, B]$, then

$$
\Gamma_t^L = \frac{\partial^2 C_t(x(S_t))}{\partial S_t^2} = \frac{\partial \tilde{\mathbf{Q}}(X_\tau > x(S_t))}{\partial S_t} = \frac{1}{S_t} \tilde{f}(x) = \frac{e^x}{S_t} f(x).
$$

6.3.2.2 Vanna

We assume that $\int_{iv+\mathbb{R}} |z e^{\tau\Psi(-z)}| dz < \infty$ and $0 < \Sigma_1 \le \sigma \le \Sigma_2$, then

$$
\begin{aligned}
\frac{\partial^2 C_t(x)}{\partial\sigma\, \partial S_t} =& \frac{\partial \mathcal{V}_t^L}{\partial S_t} = \tau\sigma e^{x(S_t)} f_t(x(S_t)) - \tau\sigma e^{x(S_t)} \Big(f_t(x(S_t)) + f_t'(x(S_t)) \Big) \\
=& -\tau\sigma e^x f_\tau'(x).
\end{aligned}
$$

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Author's Proof

6.3.2.3 Vomma

We assume that $\int_{iv+\mathbb{R}} |z^2 e^{\tau\Psi(-z)}|dz < \infty$ and $0 < \Sigma_1 \le \sigma \le \Sigma_2$, let $z \in iv + \mathbb{R}$, and denote

$$h(z,\sigma) = z^2 e^{\tau iz(1+iz)\frac{\sigma^2}{2}}, \quad g(z) = \frac{e^{izx+\tau\int_{\mathbb{R}}(e^{izy}-1-iz(e^y-1))\nu(dy)}}{z^2}.$$

Thus, $\frac{\partial h(z,\sigma)}{\partial\sigma}$ is bounded, and $\int_{iv+\mathbb{R}} |g(z)|dz < \infty$, because $|\mathbf{E}(e^{-izJ_\tau})| \le \mathbf{E}(e^{vJ_\tau}) < \infty$, where J_τ is the jump part of X_τ. By Lemma 1, we can differentiate under the integral sign

$$\begin{aligned}\frac{\partial^2 C_t(x)}{\partial\sigma^2} =& \frac{\partial \mathcal{V}_t^L}{\partial\sigma} = S_t\tau e^x f_\tau(x) + S_t\tau\sigma\frac{e^x}{2\pi}\int_{iv+\mathbb{R}} e^{izx}e^{\tau\Psi(-z)}\tau\sigma(iz-z^2)dz \\ =& S_t\tau e^x\Big(f_\tau(x) + \tau\sigma^2\left(f'_\tau(x) + f''_\tau(x)\right)\Big).\end{aligned}$$

6.3.2.4 Charm

Assume that (6.10) holds for $n = 3$, take $z \in iv + \mathbb{R}$, and let

$$h(z,\tau) = ze^{(iz+1)x_\tau}e^{\tau\Psi(-z)}, \qquad g(z) = \frac{1}{z(1+iz)}.$$

Then, $\int_{iv+\mathbb{R}} |g(x)|dx < \infty$, and by Lemma 2,

$$\frac{\partial h(z,\tau)}{\partial\tau} = ze^{(iz+1)x_\tau}e^{\tau\Psi(-z)}\Big(-r(1+iz)+\Psi(-z)\Big)$$

is bounded. By Lemma 1,

$$\begin{aligned}\frac{\partial^2 C_t(x)}{\partial\tau\partial S_t} =& \frac{\partial\tilde{\mathbf{Q}}(X_\tau > x_\tau)}{\partial\tau} = \frac{\partial}{\partial\tau}\frac{-1}{2\pi}\int_{iv+\mathbb{R}} e^{(iz+1)x_\tau}\frac{e^{\tau\Psi(-z)}}{1+iz}dz \\ =& \frac{1}{2\pi}\int_{iv+\mathbb{R}} e^{(iz+1)x_\tau}\frac{e^{\tau\Psi(-z)}}{1+iz}\Big(r(1+iz)-\Psi(-z)\Big)dz. \qquad (6.12)\end{aligned}$$

Using Fubini's theorem, we obtain

$$\begin{aligned}\frac{\partial^2 C_t(x)}{\partial\tau\partial S_t} =& \frac{1}{2\pi}\int_{iv+\mathbb{R}} e^{(iz+1)x_\tau}\frac{e^{\tau\Psi(-z)}}{1+iz}\Big(r(1+iz)-\Psi(-z)\Big)dz \\ =& re^{x_\tau}f_\tau(x_\tau) - \frac{\sigma^2}{2}e^{x_\tau}f'_\tau(x_\tau)\end{aligned}$$

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Author's Proof

$$
\begin{aligned}
&- \int_{\mathbb{R}} \Big[e^y \frac{e^{x_\tau - y}}{2\pi} \int_{iv+\mathbb{R}} e^{iz(x_\tau - y)} \frac{e^{\tau\Psi(-z)}}{1+iz} dz - \frac{e^{x_\tau}}{2\pi} \int_{iv+\mathbb{R}} e^{izx_\tau} \frac{e^{\tau\Psi(-z)}}{1+iz} dz \\
&+ (e^y - 1) \Big\{ \frac{e^{x_\tau}}{2\pi} \int_{iv+\mathbb{R}} e^{izx_\tau} e^{\tau\Psi(-z)} dz \\
&- \frac{e^{x_\tau}}{2\pi} \int_{iv+\mathbb{R}} e^{izx_\tau} \frac{e^{\tau\Psi(-z)}}{1+iz} dz \Big\} \Big] \nu(dy) \\
= &- r e^{x_\tau} f_\tau(x_\tau) + \frac{\sigma^2}{2} e^{x_\tau} f'_\tau(x_\tau) \\
&+ \int_{\mathbb{R}} \Big[- e^y \tilde{\mathbf{Q}}(X_\tau > x_\tau - y) + \tilde{\mathbf{Q}}(X_\tau > x_\tau) \\
&+ (e^y - 1) \big\{ e^x f_\tau(x) + \tilde{\mathbf{Q}}(X_\tau > x_\tau) \big\} \Big] \nu(dy) \\
= &\, r e^{x_\tau} f_\tau(x_\tau) - \frac{\sigma^2}{2} e^{x_\tau} f'_\tau(x_\tau) \\
&- \int_{\mathbb{R}} \Big[e^y \Big(\tilde{\mathbf{Q}}(X_\tau > x_\tau) - \tilde{\mathbf{Q}}(X_\tau > x_\tau - y) \Big) \\
&+ (e^y - 1) e^{x_\tau} f_\tau(x_\tau) \Big] \nu(dy). \qquad (6.13)
\end{aligned}
$$

The use of Fubini's theorem is justified by (6.7) and the additional hypothesis $\int_{iv+\mathbb{R}} |z^3 e^{\tau\Psi(-z)}| dz < \infty$.

6.3.2.5 Veta

We assume that $\int_{iv+\mathbb{R}} |z^4 e^{\tau\Psi(-z)}| dz < \infty$. Similar to Charm$_t$, we assume that $\tau \in [\mathscr{T}_1, \mathscr{T}_2]$ and $z \in iv + \mathbb{R}$ and denote

$$h(z,\tau) = z^2 e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}, \quad g(z) = \frac{1}{z^2}.$$

Then, $\int_{iv+\mathbb{R}} |g(z)| dz < \infty$, and by Lemma 2,

$$\frac{\partial h(z,\tau)}{\partial \tau} = z^2 e^{(iz+1)x_\tau} e^{\tau\Psi(-z)} \Big(- r(1+iz) + \Psi(-z) \Big)$$

is bounded. By Lemma 1, we can differentiate under the integral sign,

$$\frac{\partial^2 C_t(x_r)}{\partial\sigma\,\partial\tau} = \frac{\partial \mathscr{V}_t^L}{\partial\tau} = \frac{\partial S_t \tau\sigma e^{x_\tau} f_\tau(x_\tau)}{\partial\tau}$$

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Author's Proof

$$
\begin{aligned}
&= S_t\sigma\Big[e^{x_\tau}f_\tau(x_\tau) - r\tau e^{x_\tau}f_\tau(x_\tau) + \frac{\tau e^{x_\tau}}{2\pi}\int_{iv+\mathbb{R}} \frac{\partial}{\partial\tau}e^{izx_\tau}e^{\tau\Psi(-z)}dz\Big]\\
&= S_t\sigma\Big[e^{x_\tau}f_\tau(x_\tau) - r\tau e^{x_\tau}f_\tau(x_\tau)\\
&\quad + \frac{\tau e^{x_\tau}}{2\pi}\int_{iv+\mathbb{R}} e^{izx_\tau}e^{\tau\Psi(-z)}\Big(\Psi(-z) - riz\Big)dz\Big]. \qquad (6.14)
\end{aligned}
$$

Using Fubini's theorem, we obtain

$$
\begin{aligned}
\frac{\partial^2 C_t(x_r)}{\partial\sigma\,\partial\tau} &= S_t\sigma\Big[e^{x_\tau}f_\tau(x_\tau) - r\tau e^{x_\tau}f_\tau(x_\tau)\\
&\quad + \frac{\tau e^{x_\tau}}{2\pi}\int_{iv+\mathbb{R}} e^{izx_\tau}e^{\tau\Psi(-z)}\Big(\Psi(-z) - riz\Big)dz\\
&= S_t\sigma e^{x_\tau}\Big[f_\tau(x_\tau) - r\tau\left(f_\tau(x_\tau) + f'_\tau(x_\tau)\right)\\
&\quad + \frac{\tau}{2\pi}\int_{iv+\mathbb{R}} e^{izx_\tau}e^{\tau\Psi(-z)}\Big\{\frac{\sigma^2}{2}(iz - z^2)\\
&\quad + \int_{\mathbb{R}}\left(e^{-izy} - 1 + iz(e^y - 1)\right)\nu(dy)\Big\}dz\Big]\\
&= S_t\sigma e^{x_\tau}\Big[f_\tau(x_\tau) - r\tau\left(f_\tau(x_\tau) + f'_\tau(x_\tau)\right) + \tau\frac{\sigma^2}{2}\left(f'_\tau(x_\tau) + f''_\tau(\tau)\right)\\
&\quad + \tau\int_{\mathbb{R}}\Big(f_\tau(x_\tau - y) - f_\tau(x_\tau) + (e^y - 1)f'_\tau(x_\tau)\Big)\nu(dy)\Big]. \qquad (6.15)
\end{aligned}
$$

The use of Fubini's theorem is justified by (6.7) and the additional hypothesis $\int_{iv+\mathbb{R}} |z^4 e^{\tau\Psi(-z)}|dz < \infty$.

6.3.2.6 Vera

Assuming that $\int_{iv+\mathbb{R}} |ze^{\tau\Psi(-z)}|dz < \infty$ and $0 < \Sigma_1 \le \sigma \le \Sigma_2$,

$$
\begin{aligned}
\frac{\partial^2 C_t(x_r)}{\partial\sigma\,\partial r} &= \frac{\partial\mathscr{V}_t^L}{\partial r} = S_t\tau\sigma e^{x_r}\Big(-\tau f_\tau(x_r) - \tau f'_\tau(x_r)\Big)\\
&= -S_t\tau^2\sigma e^{x_r}\Big(f_\tau(x_r) + f'_\tau(x_r)\Big).
\end{aligned}
$$

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Author's Proof

6.3.3 *Third-Order Greeks for Call Options*

6.3.3.1 Color

We assume that $\int_{iv+\mathbb{R}} |z^4 e^{\tau\Psi(-z)}|dz < \infty$, $\tau \in [\mathcal{T}_1, \mathcal{T}_2]$. Let $z \in iv + \mathbb{R}$ and

$$h(z,\tau) = z^2 e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}$$
$$g(z) = \frac{1}{z^2}.$$

Then, $\int_{iv+\mathbb{R}} |g(x)|dx < \infty$, and by Lemma 2,

$$\frac{\partial h(z,\tau)}{\partial \tau} = z^2 e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}\Big(-r(1+iz) + \Psi(-z)\Big)$$

is bounded. By Lemma 1, we can differentiate under the integral sign.

Thus,

$$\frac{\partial^3 C_t(x)}{\partial S_t^2 \partial \tau} = \frac{\partial \Gamma_t^L}{\partial \tau} = \frac{1}{S_t 2\pi} \int_{iv+\mathbb{R}} e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}\Big(-r(iz+1) + \Psi(-z)\Big)dz. \tag{6.16}$$

Using Fubini's theorem, we obtain

$$\begin{aligned}
\frac{\partial^3 C_t(x)}{\partial S_t^2 \partial \tau} &= \frac{1}{S_t 2\pi} \int_{iv+\mathbb{R}} e^{(iz+1)x_\tau} e^{\tau\Psi(-z)}\Big(-r(iz+1) + \Psi(-z)\Big)dz \\
&= \frac{e^x}{S_t}\Big[-r\big(f(x) + f'(x)\big) + \frac{\sigma^2}{2}\big(f'(x) + f''(x)\big) \\
&\quad + \frac{1}{2\pi} \int_{iv+\mathbb{R}} e^{izx} e^{\tau\Psi(-z)} \int_{\mathbb{R}} e^{-izy} - 1 + iz(e^y - 1)\nu(dy)dz\Big] \\
&= -\frac{e^x}{S_t}\Big[r\Big(f_\tau(x) + f'_\tau(x)\Big) - \frac{\sigma^2}{2}\Big(f'_\tau(x) + f''_\tau(x)\Big) \\
&\quad + \int_{\mathbb{R}} \Big(f_\tau(x) - f_\tau(x-y) - (e^y - 1)f'_\tau(x)\Big)\nu(dy)\Big].
\end{aligned} \tag{6.17}$$

Fubini is justified by (6.7) and the hypothesis $\int_{iv+\mathbb{R}} |z^4 e^{\tau\Psi(-z)}|dz < \infty$.

6.3.3.2 Speed

Assuming that $\int_{iv+\mathbb{R}} |z e^{\tau\Psi(-z)}|dz < \infty$,

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$$\frac{\partial^3 C_t(x_r)}{\partial S_t^3} = \frac{\partial \Gamma_t^L}{\partial S_t} = \frac{e^{x(S_t)}\Big(-\frac{1}{S_t}f_\tau(x(S_t)) - \frac{1}{S_t}f'_\tau(x(S_t))\Big)S_t - e^{x(S_t)}f_\tau(x(S_t))}{S_t^2}$$
$$= -\frac{e^x}{S_t^2}\Big(2f_\tau(x) + f'_\tau(x)\Big).$$

6.3.3.3 Ultima

We assume that $\int_{iv+\mathbb{R}} |z^6 e^{\tau\Psi(-z)}|dz < \infty$. First, we calculate $\frac{\partial f_\tau^{(n)}(x)}{\partial\sigma}$ for $n = 0, 1, 2$.

AQ3 For $0 < \Sigma_1 \le \sigma \le \Sigma_2$ and $z \in iv + \mathbb{R}$, we denote

$$h_n(z,\sigma) = (iz)^{n+2}e^{\tau\Psi(-z)}, \quad g(z) = -\frac{e^{izx}}{z^2}.$$

Thus, $\int_{iv+\mathbb{R}} |g(z)|dz < \infty$ and $\frac{\partial h_n(z,\sigma)}{\partial\sigma}$ is bounded for $n = 0, 1, 2$. By Lemma 1, we can differentiate under the integral sign. Then,

$$\frac{\partial^n f_\tau(x)}{\partial\sigma} = \frac{\partial}{\partial\sigma}\frac{1}{2\pi}\int_{iv+\mathbb{R}} (iz)^n e^{izx}e^{\tau\Psi(-z)}dz$$
$$= \tau\sigma\frac{1}{2\pi}\int_{iv+\mathbb{R}} [(iz)^{n+1} - (iz)^{n+2}]e^{izx}e^{\tau\Psi(-z)}dz$$
$$= \tau\sigma\Big(f_\tau^{(n+1)}(x) + f_\tau^{(n+2)}(x)\Big). \tag{6.18}$$

Now, we have

$$\frac{\partial^3 C_t(x)}{\partial\sigma^3} = \frac{\partial S_t\tau e^x\Big(f_\tau(x) + \tau\sigma^2\left(f'_\tau(x) + f''_\tau(x)\right)\Big)}{\partial\sigma}$$
$$= S_t\tau^2\sigma e^x\Big(3\left(f'_\tau(x) + f''_\tau(x)\right) + \tau\sigma^2\left(f''_\tau(x) + 2f'''_\tau(x) + f_\tau^{(iv)}(x)\right)\Big).$$

6.3.3.4 Zomma

We assume that $\int_{iv+\mathbb{R}} |z^2 e^{\tau\Psi(-z)}|dz < \infty$ and $0 < \Sigma_1 \le \sigma \le \Sigma_2$. Then,

$$\frac{\partial^3 C_t(x_r)}{\partial S_t^2\partial\sigma} = \frac{\partial \text{Vanna}_t^L}{\partial S_t} = \frac{\partial \tau\sigma e^{x(S_t)}f'_\tau(x(S_t))}{\partial S_t}$$
$$= -\frac{\tau\sigma e^x}{S_t}\Big(f'_\tau(x) + f''_\tau(x)\Big).$$

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Author's Proof

6.4 Examples

6.4.1 The Black–Scholes Model

If we assume that the Gaussian distribution and density are exactly computed in R software, we can compare the Greeks for Black–Scholes model using Lewis representation. To approximate the Fourier transform, we cut the integral between $-A/2$ and $A/2$ and take a uniform partition of $[-A/2, A/2]$ of size N:

$$\int_{\mathbb{R}} e^{izx} g(z)dz \approx \int_{-A/2}^{A/2} e^{izx} g(z)dz \approx \frac{A}{N} \sum_{k=0}^{N-1} w_k e^{iz_k x} g(z_k),$$

where $z_k = -\frac{A}{2} + k\frac{A}{N-1}$ and w_k are weights that correspond to the integration numerical rule.

Table 6.1 shows the ℓ_∞-errors in the Black–Scholes model via Lewis representation and fast Fourier transform using $S_t = 1$, $r = 0.05$, $T = 1$, $\sigma = 0.1$, $A = 300$, and $N = 2^{22}$. Denoting by GL the value for the Greek given by our Lévy formula, and by G the direct computation, the ℓ_∞-errors are

$$\ell_\infty\text{-error}(GL) = \max_{x\in[-0.7,0.7]} |GL - G|, \tag{6.19}$$

for $x = \ln(K/S_t) - r\tau$.

Table 6.1 ℓ_∞-errors in the Black–Scholes model via Lewis representation and fast Fourier transform using $S_t = 1$, $r = 0.05$, $T = 1$, $\sigma = 0.1$, $A = 300$, and $N = 2^{22}$

Greek	Expression	ℓ_∞-error
Call	$C = S\,\mathbf{E}(e^{X_\tau} - e^x)^+$	1.2e-07
Delta	$\partial_S C(x)$	2.4e-07
Rho	$\partial_r C_t(x)$	1.9e-07
Vega	$\partial_\sigma C(x)$	9.5e-08
Theta	$\partial_\tau C(x)$	1.2e-08
Gamma	$\partial^2_{SS} C(x)$	9.5e-07
Vanna	$\partial^2_{\sigma S} C(x)$	6.3e-07
Vomma	$\partial^2_{\sigma\sigma} C(x)$	7.5e-07
Charm	$\partial^2_{S\tau} C(x)$	6.8e-08
Veta	$\partial^2_{\sigma\tau} C(x)$	8.9e-08
Vera	$\partial^2_{\sigma r} C(x)$	5.8e-07
Color	$\partial^3_{SS\tau} C(x)$	5.6e-07
Speed	$\partial^3_{SSS} C(x)$	6.3e-06
Ultima	$\partial^3_{\sigma\sigma\sigma} C(x)$	1.2e-05
Zomma	$\partial^3_{SS\sigma} C(x)$	9.5e-06

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6.4.2 The Merton Model

In this section, we show some results for the Merton model. The Merton model has four parameters $(\sigma, \mu_J, \sigma_J, \lambda)$ where σ is the diffusion parameter, λ is the jump intensity, and μ_J and σ_J are the mean and standard deviation of the jump which are Gaussianly distributed. The characteristic function for the Merton model is

$$\mathbf{E}(e^{izX_T}) = \exp\left\{iz\Big[\frac{\sigma^2}{2} - \lambda\big(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1\big)\Big] + z^2\frac{\sigma^2}{2} + \lambda\big(e^{iz\mu_J - z^2\frac{\sigma_J^2}{2}} - 1\big)\right\}. \tag{6.20}$$

All Greeks for *At The Money* ($K = S_0 e^{-rT}$) are shown in Table 6.2 following Sect. 6.3. To evaluate errors, we compute first a reasonable value *GL* and then increase the precision with the corresponding time cost, to obtain *G*, that we assume is the true value, and apply (6.19). Hence, we took $A = 500$, $N = 2^{20}$ and $A = 500$, $N = 2^{22}$; the error is in all Greeks lower than 10^{-5}. In Fig. 6.1, the curves are shown in terms of $x = \ln(K/S_0) - rT$ for all Greeks with the comparison of the Black–Scholes model with volatility equal to implied volatility *At The Money*.

The characteristic function in this case is (6.20). To compute sensitivities w.r.t. μ_J, σ_J, and λ, we only need to differentiate the characteristic exponent with respect to these parameters:

$$\frac{\partial \Psi(-z)}{\partial \mu_j} = \lambda iz\Big[e^{\mu_J + \sigma_J^2/2} - e^{-iz\mu_J - z^2\sigma_J^2/2}\Big],$$

Table 6.2 Greeks in Merton model with: $S_0 = 1$, $r = 0.05$, $x = 0$, $T = 1$, $\sigma = 0.1$, $\mu_J = -0.005$, $\sigma_J = 0.1$, $\lambda = 1$

		$A = 500, N = 2^{20}$	$A = 500, N = 2^{21}$	Error	
Call	C	0.0547129	0.0547129	2.6e-08	t5.1
Delta	$\partial_S C$	0.5273560	0.5273562	2.5e-07	t5.2
Rho	$\partial_r C$	0.4726431	0.4726433	2.2e-07	t5.3
Vega	$\partial_\sigma C$	0.3077754	0.3077755	1.5e-07	t5.4
Theta	$\partial_\tau C$	0.0524286	0.0524286	2.5e-08	t5.5
Gamma	$\partial^2_{SS} C$	3.0777536	3.0777550	1.5e-06	t5.6
Vanna	$\partial^2_{\sigma S} C$	0.1538877	0.1538878	7.3e-08	t5.7
Vomma	$\partial^2_{\sigma\sigma} C$	0.9091776	0.9091780	4.3e-07	t5.8
Charm	$\partial^2_{S\tau} C$	0.1682859	0.1682860	8.1e-08	t5.9
Veta	$\partial^2_{\sigma\tau} C$	0.1222075	0.1222076	5.8e-08	t5.10
Vera	$\partial^2_{\sigma r} C$	−0.1538877	−0.1538878	7.3e-08	t5.11
Color	$\partial^3_{SS\tau} C$	1.8556786	1.8556795	8.8e-07	t5.12
Speed	$\partial^3_{SSS} C$	−4.6166303	−4.6166325	2.2e-06	t5.13
Ultima	$\partial^3_{\sigma\sigma\sigma} C$	−11.5390901	−11.5390956	5.5e-06	t5.14
Zomma	$\partial^3_{SS\sigma} C$	−21.6857596	−21.6857699	1.0e-05	t5.15

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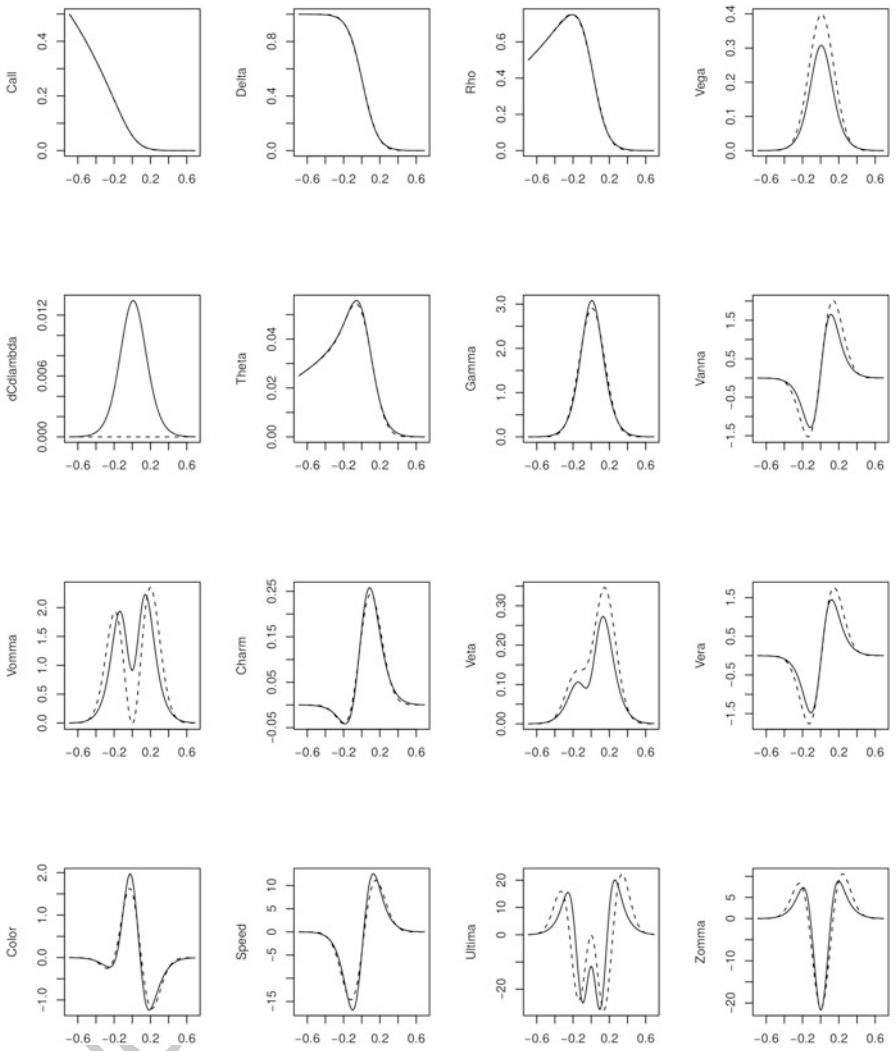


Fig. 6.1 Greeks in terms of $x = \ln(K/S_0) - rT$ for the Merton model with parameters equal to Table 6.2 (*continuous line*). *Discontinuous line*: Black–Scholes model with volatility equal to implied volatility in $x = 0$ ($\sigma_{\text{imp}}(0) \approx 0.137$)

$$\frac{\partial \Psi(-z)}{\partial \sigma_j} = \lambda \sigma_J \left[iz e^{\mu_J + \sigma_j^2/2} - z^2 e^{-iz\mu_J - z^2\sigma_j^2/2} \right],$$

$$\frac{\partial \Psi(-z)}{\partial \lambda_j} = iz \left[e^{\mu_J + \sigma_j^2/2} - 1 \right] + e^{-iz\mu_J - z^2\sigma_j^2/2} - 1,$$

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and for $\theta = \mu_J, \sigma_J, \lambda$,

$$\frac{\partial C_\theta(x)}{\partial \theta} = \tau S_t \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau \Psi_\theta(-z)}}{iz(1+iz)} \frac{\partial \Psi_\theta(-z)}{\partial \theta} dz.$$

The differentiation under the integral sign is justified as above.

Using the same parameters presented in Table 6.2, we obtain the sensitivities for ATM given in Table 6.3.

In Fig. 6.2, we show the Greeks in terms of $x = \ln(K/S_0) - rT$.

In Kienitz (2008) are shown some results for a digital option in the Merton model, which were obtained by applying finite difference approximations to the formula for the option prices in Madan et al. (1998). Now we will deduce delta, gamma, and vega for a digital option, and thus, we will compare the results.

A digital option has a payoff given by

$$\mathbf{1}_{\{S_\tau - K > 0\}} = \mathbf{1}_{\{X_\tau - x > 0\}}.$$

Using Lewis representation, the value for a digital option is given by (6.8) (with **Q** instead of **P**). If (6.10) holds with $n = 0$ and $n = 1$, respectively, differentiation leads to:

$$\frac{\partial D(x)}{\partial S_t} = \frac{1}{S_\tau} f_\tau(x), \tag{6.21}$$

Table 6.3 Sensitivities for the Merton model with: $S_0 = 1$, $r = 0.05$, $x = 0$, $T = 1$, $\sigma = 0.1$, $\mu_J = -0.005$, $\sigma_J = 0.1$, $\lambda = 1$

	$A = 500, N = 2^{20}$	$A = 1000, N = 2^{22}$	Error
μ_J-Sensitivity	0.006703850	0.006703855	4.7e-09
σ_J-Sensitivity	0.239001059	0.239001230	1.7e-07
λ-Sensitivity	0.013407701	0.013407711	9.6e-09

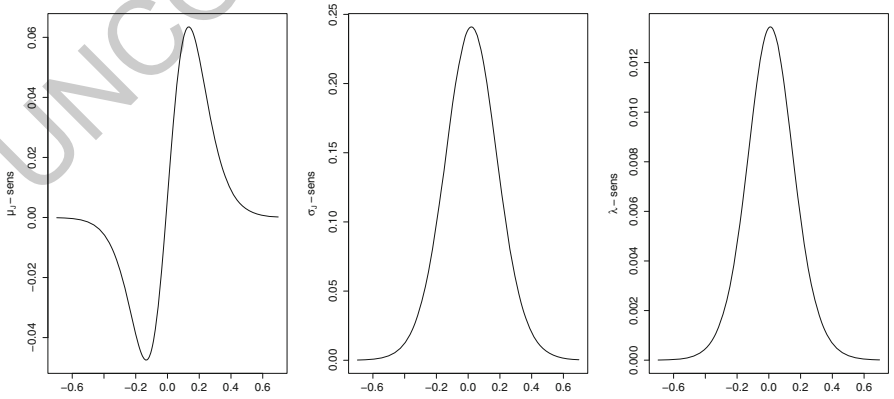


Fig. 6.2 Sensitivities in terms of $x = \ln(K/S_0) - rT$ for the Merton model with parameters equal to Table 6.3

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Table 6.4 Digital option and greeks in the Merton model with: $\delta = 0.01$, $S_0 = 100$, $K = 100$, $T = 1$, $r = 0.07$, $\sigma = 0.2$, $\mu_J = 0.05$, $\sigma_J = 0.15$, and $\lambda = 0.5$

	D	D-Delta	D-Gamma	D-Vega	
$N = 2^{20}$	0.531269863	0.016610445	-0.000280032	-0.560064360	t11.2
$N = 2^{22}$	0.531270245	0.016610457	-0.000280032	-0.560064763	t11.3
Error	3.8e-07	1.2e-08	2.0e-10	4.0e-07	t11.4

Table 6.5 Greeks and ρ-sensitivity for variance gamma model with $(\rho, \nu, \theta) = (0.2, 1, -0.15)$, $r = 0.05$, $T = 1$, $S_0 = K = 100$ $(x = -0.05)$

	Call	Delta	Gamma	$\frac{\partial \text{Call}}{\partial \rho}$	
$N = 2^{20}, \delta = 0.01$	11.26689113	0.72818427	0.01427437	23.04334371	t14.2
$N = 2^{22}, \delta = 0.01$	11.26689919	0.72818479	0.01427438	23.04336021	t14.3
err<	8.1e-06	5.2e-07	1.0e-08	1.6e-05	t14.4

$$\frac{\partial^2 D(x)}{\partial S_t^2} = -\frac{1}{S_\tau^2}\Big(f_\tau(x) + f'_\tau(x)\Big), \tag{6.22}$$

$$\frac{\partial D(x)}{\partial \sigma} = -\tau\sigma\Big(f_\tau(x) + f'_\tau(x)\Big). \tag{6.23}$$

In (6.23), differentiation under the integral sign is similar to (6.18).

Then, our results via FFT are shown in Table 6.4. In Kienitz (2008), these values are (by finite difference): D= 0.531270, D-Delta= 0.016610, D-Gamma= -2.800324×10^{-4}, D-Vega= -0.560070. To obtain a given strike, we define $\delta = 2\pi \frac{N-1}{NA}$.

6.4.3 The Variance Gamma Model

In this section, we will compare some results from the literature. As an example, in Glasserman and Liu (2007), some results are shown for the variance gamma model with parameters (ρ, ν, θ) where the characteristic function is:

$$\mathbf{E}[e^{izX_T}] = \exp\left\{\frac{T}{\nu}\left[iz\ln\left(1 - \theta\nu - \frac{\rho^2\nu}{2}\right) - \ln\left(1 - iz\theta\nu + \frac{z^2\rho^2\nu}{2}\right)\right]\right\}.$$

To obtain a given strike, we define $\delta = 2\pi\frac{N-1}{NA}$. Thus, in Table 6.5, we present two results for $N = 2^{20}$ and $N = 2^{22}$ with $\delta = 0.01$. The error shows the convergence of the complex integral. In Glasserman and Liu (2007), these results are obtained by applying finite difference approximations to the formula for the option prices in Madan et al. (1998), call= 11.2669, delta= 0.7282, and ρ-derivative = 23.0434, and in general with the LRM method, the error is worse than 10^{-2}.

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6.5 Conclusions

Greeks are an important input for market makers in risk management. A lot of options are path dependent and they do not have an explicit formula. However, for the European options in the exponential Lévy models, we have the Lewis formula, which allows us to obtain closed formulas for Greeks, many of which are only density dependent; others require integration. In general, all Greeks can be approximated with high accuracy because they are a simple integral, similar to the Black–Scholes model.

A large number of papers are dedicated to obtain Greeks for more complex payoff functions. However, in order to estimate the accuracy of their methods, Greek approximations are computed through the finite difference technique.

For a fix strike K, we consider $x = \ln(K/S_t) - r\tau$, with $\tau = T - \tau$ the time to maturity. Thus, the Greeks for call options can be calculated through Table 6.6.

We observe that if the density of X_τ is known, then many of the Greeks can be exactly obtained. Some examples of these are normal inverse Gaussian, variance gamma, generalized hyperbolic, Meixner, and others.

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Table 6.6 Greeks in exponential Lévy models in terms of $x = \ln(K/S) - r\tau$

First order		
Delta	$\partial C_S(x) = \tilde{\mathbf{Q}}(X_\tau > x)$	t17.1
Rho	$\partial C_r(x) = \tau S e^x \mathbf{Q}(X_\tau > x)$	t17.2
Vega	$\partial C_\sigma(x) = S\tau\sigma e^x f_\tau(x)$	t17.3
If $\nu = \lambda\bar{\nu}$	$\partial C_\lambda(x) = \tau\Big[\int_{\mathbb{R}} \Big(e^y C(x-y) - C(x)$	t17.4
	$-S(e^y - 1)\tilde{\mathbf{Q}}(X_\tau > x)\Big)G(dy)\Big]$	t17.5
Theta	$\partial C_\tau(x) = S\Big[re^x \mathbf{Q}(X_\tau > x) + \frac{\sigma^2}{2} e^x f_\tau(x)\Big] + \frac{\lambda}{\tau}\partial C_\lambda(x)$	t17.6
Second order		
Gamma	$\partial^2 C_{SS}(x) = S^{-1} e^x f_\tau(x)$	t17.7
Vanna	$\partial^2 C_{\sigma S}(x) = -\tau\sigma e^x f'_\tau(x)$	t17.8
Vomma	$\partial^2 C_{\sigma\sigma}(x) = S\tau e^x \Big(f_\tau(x) + \tau\sigma^2 \left(f'_\tau(x) + f''_\tau(x)\right)\Big)$	t17.9
Charm	$\partial^2 C_{S\tau}(x) =$ see (6.12) and (6.13)	t17.10
Veta	$\partial^2 C_{\sigma\tau}(x) =$ see (6.14) and (6.15)	t17.11
Vera	$\partial^2 C_{\sigma r}(x) = -S\tau^2\sigma e^x \Big(f_\tau(x) + f'_\tau(x)\Big)$	t17.12
Third order		
Color	$\partial^3 C_{SS\tau}(x) =$ see (6.16) and (6.17)	t17.13
Speed	$\partial^3 C_{SSS}(x) = -S^{-2} e^x \Big(2f_\tau(x) + f'_\tau(x)\Big)$	t17.14
Ultima	$\partial^3 C_{\sigma\sigma\sigma}(x) = S\tau^2\sigma e^x \Big(3\left(f'_\tau(x) + f''_\tau(x)\right)$	t17.15
	$+\tau\sigma^2 \left(f''_\tau(x) + 2f'''_\tau(x) + f^{iv}_\tau(x)\right)\Big)$	t17.16
Zomma	$\partial^3 C_{SS\sigma}(x) = -\tau\sigma S^{-1} e^x \Big(f'_\tau(x) + f''_\tau(x)\Big)$	t17.17

Author's Proof

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AQ1. Please check the inserted keywords from "Key Words.doc" and correct if necessary.
AQ2. Please check 'F. De Olivera' second affiliation taken from LOC.
AQ3. Please check if edit to sentence starting "For $0 < \Sigma_1 \leq \sigma \leq \Sigma_2$..." is okay.
AQ4. Please provide complete details for the reference Glasserman and Liu(2008).

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