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Robustness of Cucker–Smale flocking model

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Abstract: Consider a system of autonomous interacting agents moving in space, adjusting each own velocity as a weighted mean of the relative velocities of the other agents. In order to test the robustness of the model, the authors assume that each pair of agents, at each time step, can fail to connect with certain probability, the failure rate. This is a modification of the (deterministic) Flocking model introduced by Cucker and Smale (2007). They prove that, if this random failures are independent in time and space, and have linear or sub-linear distance dependent rate of decay, the characteristic behaviour of flocking exhibited by the original deterministic model, also holds true under random failures, for all failure rates.

1 Introduction

A central question in the collective motion of a group of autonomous interacting agents moving in space is the study of its long-term behaviour. If for large times all agent velocities become equal with fixed relative positions we say that 'flocking' occurs. Under the denomination of 'flocking' we find works in several scientific areas, ranging from natural sciences (from where the term of the field originated), mechanics, robotics, linguistics and other fields. Regarding the methods of study we have experimental works, simulation studies and theoretical approaches. In these respects we recommend the recent review by Vicsek and Zafeiris [1] and its complete list of references.

From the rigorous point of view, the complexity of the situation of interest typically requires simplifications, as very complex models do not admit reasonable mathematical treatment, leading to simulations or other type of experimental verification. In this direction, the seminal proposal by Cucker and Smale [2] is a landmark in the comprehension of collective motion since it gives the possibility of a mathematical treatment of the question of the asymptotic behaviour of the system and under reasonable dynamical laws it exhibits cases of flocking and also non-flocking situations. More precisely, the interaction between each pair of agents is present through a coefficient, that decays with the distance between them, and the emergence of flocking is guaranteed by the verification of simple conditions depending only on the parameters of the model and on the initial configuration of the system [2]. Previous rigorous approaches in some cases required conditions on the whole trajectory of the system (compare with Jadbabaie *et al.* [3]).

In the present work, we are concerned with the applicability of the results obtained by Cucker and Smale [2]. Our proposal consists in relaxing the model assumptions in the following way: we assume that each interaction is subject to random failure at each time step. The results obtained under this natural way of relaxing the interactions is understood as a 'robustness' property of the model.

Cucker–Smale model has been extensively studied. It has been extended to the hierarchical case with free-will leader in Shen [4]; to the rooted hierarchical case in Li and Xue [5] and to the rooted hierarchical case with free-will leader in Li and Xue [6]; to systems with large number of individuals (swarming) in Carrillo *et al.* [7] and Ha and Liu [8]. In Caponigro *et al.* [9] the authors introduce external intervention in Cucker–Smale model. All these models are deterministic.

There is an increasing interest in the introduction of some kind of randomness in the models describing the collective motion of a multi-agent group since this provides a way of dealing with the perturbations or bruits that are present in the applications. For instance, in Cucker and Mordecki [10], Ha *et al.* [11], Shang [12] and in Ton *et al.* [13] the authors consider an additive random noise in Cucker–Smale model; in Ahn and Ha [14] the authors consider a multiplicative noise in Cucker–Smale model. In the case of 'hierarchical' Cucker–Smale flocking model (where each agent only takes into account its superiors in a prescribed hierarchy in order to update its position and velocity), a similar approach to the present one was considered by Dalmao and Mordecki in [15, 16]. Finally, in Martin *et al.* [17] the authors introduce randomness in the radius of interaction between agents (the coefficients a_{ij} being 1 or 0 according to the distance

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between agents being less or more than the random radius) in Viscek model.

Finally, let us say that flocking models can also be seen as control laws. Therefore it is natural to use them for actual systems of autonomous interacting agents without a central direction. For instance, one of the main applications of Cucker–Smale model is in the Darwin Mission [18]. Besides, in Ahn *et al.* [19] the authors apply Cucker–Smale model for modelling the stochastic volatility in a market.

2 Mathematical model

Consider a system of k agents with positions and velocities denoted, respectively, by $X = (X_1, \dots, X_k)$ and $V = (V_1, \dots, V_k)$. All individual positions and velocities are vectors in $\mathbb{R}^3$ and we refer to X and V as the position and velocity of the system, respectively. Assume that the system evolves following the discrete-time dynamic

$$\begin{cases} X_i(t+h) = X_i(t) + hV_i(t) \\ V_i(t+h) = V_i(t) + h\sum_{j=1}^{k} a_{ij}(V_j(t) - V_i(t)) \end{cases} \tag{1}$$

where $i = 1, \dots, k$; $h > 0$ is the time step and $a_{ij}(t)$ are the weighting coefficients. Cucker and Smale [2] propose for $\alpha \geq 0$

$$a_{ij} = a_{ij}^{(\mathrm{cs})}(t) = \frac{1}{\left(1 + \|X_i(t) - X_j(t)\|\right)^{\alpha}} \tag{2}$$

One of the characteristics of this proposal is that it lets aside some questions, as the volume of the agents, or its difference in weights, and it also assumes that the communication between agents is perfect.

In the present paper we address the third issue. Departing from the Cucker–Smale model given by (1) and (2) we analyse the situation in which each interaction is subject to random failure. More precisely, we introduce the possibility that at each time step each pair of agents (i,j) can fail to connect (i.e. agents i and j fail to see each other). These failures are assumed to be random, independent and with a fixed failure rate probability $\lambda \in (0,1)$. Therefore we modify the weighting coefficients in (2) in the following way: for each pair of agents $i \neq j$ let

$$a_{ij} = a_{ij}(t) = \zeta_t^{ij} \cdot \frac{1}{\left(1 + \|X_i(t) - X_j(t)\|\right)^{\alpha}} \tag{3}$$

where ζ_t^{ij} $(t \geq 0)$ are independent and identically distributed Bernoulli random variables with success probability $1 - \lambda$, that is

$$\mathbf{P}(\zeta_t^{ij} = 1) = 1 - \lambda \quad \mathbf{P}(\zeta_t^{ij} = 0) = \lambda$$

Through the article **P** and **E** denote probability and expectation, respectively; the (random) event $\zeta_t^{ij} = 0$ (resp. $= 1$) means that the pair of agents (i,j) fails to (resp. does) connect (see each other) at time step t.

Our main result states that the flocking result obtained in [2] holds in our framework with probability one (i.e. almost surely) in the case of agent interactions with linear or sublinear distance-dependent decay rate (i.e. $\alpha \leq 1$).

3 Main result and proof

Theorem 1: Consider the dynamical system governed by equations (1) with time step $h \leq 1/k$ and coefficients subject to random failure as defined in (3). Then

(i) Under the condition $\alpha < 1$ all velocities tend to a common velocity that is the mean initial velocity $\bar{V}(0)$, that is

$$(V_1(th), \dots, V_k(th)) \to (\bar{V}(0), \dots, \bar{V}(0)) \quad \text{as } t \to \infty \tag{4}$$

Furthermore, there exists a limiting configuration $\hat{\boldsymbol{x}} = (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k)$ that is the limit of the relative positions of the system with respect to the centre of mass defined in (7). More precisely

$$(X_1(th) - \bar{X}(th), \dots, X_k(th) - \bar{X}(th)) \to (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k) \quad \text{as } t \to \infty \tag{5}$$

(ii) Under the condition $\alpha = 1$ there exists a critical initial velocity $\boldsymbol{v}^*$ (that depends only on the failure rate), such that if

$$\|(V_1(0), \dots, V_k(0)) - (\bar{V}(0), \dots, \bar{V}(0))\| < \boldsymbol{v}^*$$

the statements of (i) hold.

Remark 1: The critical velocity $\boldsymbol{v}^*$ in (ii) above is, in fact, the expectation of the (random) Fiedler number of the non-coloured graph associated with the system, see Section 3.1.1. In particular, we have $\boldsymbol{v}^* = 0$ (no convergence) when $\lambda = 1$ (complete failure), and $\boldsymbol{v}^* = k$ (maximum connectivity) when $\lambda = 0$ (non-failure).

Remark 2: The case $\lambda = 0$ corresponds to the non-failure system, that is to the Cucker–Smale model introduced in [2]. In case $\alpha = 1$ we obtain the condition

$$\|V(0)\| \leq \boldsymbol{v}^* = k \leq \frac{1}{h}$$

according to the assumptions of Theorem 1. This is similar to the condition in Theorem 3, statement (ii) in [2]. (Case $\beta = 1/2$ in [2] corresponds to case $\alpha = 1$ in our framework.)

3.1 Some preliminaries

An equivalent way of writing the second equation in (1) is

$$V_i(t+h) = \left(1 - h\sum_{j=1}^{k} a_{ij}\right) V_i(t) + h\sum_{j=1}^{k} a_{ij} V_j(t) \tag{6}$$

that, under the assumed condition $0 < h \leq 1/k$ [that makes the first coefficient in the r.h.s. of (6) always positive], implies that the velocity at $t + h$ is a linear convex combination of the system velocities at time t. From this follows that the velocity is decreasing in the following sense

$$\max_{1 \leq i \leq k} \|V_i(t+h)\| \leq \max_{1 \leq i \leq k} \|V_i(t)\| \leq \max_{1 \leq i \leq k} \|V_i(0)\|$$

where $\|V\|$ is the Euclidean norm of the vector V in $\mathbb{R}^3$.

It is interesting to observe (as was done in [8, 10] for continuous time) that the centre of mass of the system travels

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with constant velocity that happens to be the initial mean velocity of the flock. Define

$$\bar{X}(t) = \frac{1}{k}\sum_{i=1}^{k} X_i(t), \quad \bar{V}(t) = \frac{1}{k}\sum_{i=1}^{k} V_i(t)$$

In view of (1) (because of the symmetry of the coefficients $a_{ij} = a_{ji}$), we obtain that $\bar{V}(t+h) = \bar{V}(t)$. We conclude that

$$\bar{X}(th) = \bar{X}(0) + ht\bar{V}(0), \quad \bar{V}(th) = \bar{V}(0), \quad t = 1, 2, \ldots \tag{7}$$

3.1.1 Connectivity of the associated graph: A key feature in order to obtain flocking is the connectedness of the graph induced by the agents of the flock. That is, the coloured (weighted) graph with vertices $1, 2, \ldots, k$ corresponding to the agents and edges $a_{ij}(t)$.

Within the several ways of quantifying the connectivity of a graph, Cucker and Smale [2] show that a key concept in this situation is the 'algebraic connectivity', also known as the 'Fiedler number' (see [20]), that we denote by ϕ. This number is defined as the second smallest eigenvalue of the Laplacian matrix associated with the graph (the first eigenvalue always vanishes). The larger the Fiedler number is, the 'more connected' the graph is. A relevant role will be played also by the Fiedler number of the non-coloured graph induced by the interactions of the system, that is defined to have an edge if and only if $a_{ij} \neq 0$. This second Fiedler number is denoted by φ. In particular, when $\varphi = 0$ the graph is not connected, and the complete graph with k vertices has $\varphi = k$.

Observe that since the edges of our graph are random both Fiedler's numbers also become random quantities. The statistical control of their behaviour in time gives us the possibility of establishing our results.

3.1.2 Change of coordinates: We now write the equations that govern the system in matrix form. Denote by Id the $k \times k$ identity matrix. Let $L(t)$ the Laplacian matrix corresponding to the induced graph defined as $L(t) = D(t) - A(t)$ with $A(t) = (a_{ij}(t))$ the incidence matrix of the graph and $D(t)$ the diagonal matrix with $d_{ii}(t) = \sum_j a_{ij}(t)$. With this notation, the matrix form of (1) are

$$\begin{aligned} X(t+h) &= X(t) + hV(t) \\ V(t+h) &= (\mathrm{Id} - hL(t))V(t) \end{aligned} \tag{8}$$

where the notation AV means that the matrix A is acting on $(\mathbb{R}^3)^k$ by mapping the vector $(V_1, \ldots, V_k)$ into the vector $(a_{i1}V_1 + \cdots + a_{ik}V_k)_{1 \le i \le k}$.

As we have seen in (7), the centre of mass of the system has constant velocity. It is useful then to consider coordinates with respect to this point, introducing the relative position and velocity of the flock by

$$\begin{aligned} \boldsymbol{x}(t) &= (\boldsymbol{x}_1(t), \ldots, \boldsymbol{x}_k(t)) = (X_1(t) - \bar{X}(t), \ldots, X_k(t) - \bar{X}(t)) \\ \boldsymbol{v}(t) &= (\boldsymbol{v}_1(t), \ldots, \boldsymbol{v}_k(t)) = (V_1(t) - \bar{V}(0), \ldots, V_k(t) - \bar{V}(0)) \end{aligned}$$

This change of coordinates is equivalent to the projection on the diagonal space in velocities introduced by Cucker and Smale in [2]. A further simplification in the notation is to write $\boldsymbol{x}[t]$ and $\boldsymbol{v}[t]$ instead of $\boldsymbol{x}(th)$ and $\boldsymbol{v}(th)$, respectively (and similarly for other time-dependent quantities).

With this change of coordinates and notation, the statements of Theorem 1 are

$$(\boldsymbol{v}_1[t], \ldots, \boldsymbol{v}_k[t]) \to (0, \ldots, 0) \quad \text{as } t \to \infty$$

and

$$(\boldsymbol{x}_1[t], \ldots, \boldsymbol{x}_k[t]) \to (\hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_k) \quad \text{as } t \to \infty$$

It is easy to verify that the relative positions $\boldsymbol{x}[t]$ and velocities $\boldsymbol{v}[t]$ just introduced follow the same dynamic than the original ones, that is, the system (8) holds for $\boldsymbol{x}$ and $\boldsymbol{v}$ instead of X and V, respectively

$$\begin{aligned} \boldsymbol{x}[t+1] &= \boldsymbol{x}[t] + h\boldsymbol{v}[t] \\ \boldsymbol{v}[t+1] &= (\mathrm{Id} - hL[t])\boldsymbol{v}[t] \end{aligned}$$

3.2 Proof of the theorem

We introduce the norm of a vector $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$ in $(\mathbb{R}^3)^k$ by

$$\|\boldsymbol{x}\| = \sum_{i=1}^{k}\sum_{\ell=1}^{3} x_{i\ell}^2$$

With this notation the statements (4) and (5) reduce to

$$\|\boldsymbol{v}[t]\| \to 0 \quad \text{and} \quad \|\boldsymbol{x}[t] - \hat{\boldsymbol{x}}\| \to 0 \quad \text{as } t \to \infty$$

We further consider the usual operator norm for a matrix A acting as described above by

$$\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$$

As the Fiedler number of A is the second smallest eigenvalue of A, and the smallest is associated with the eigenvector $(1, \ldots, 1)$ in $(\mathbb{R}^3)^k$ (and our vectors are orthogonal with respect to this diagonal vector), from the velocity equation in (8) we obtain that

$$\|\boldsymbol{v}[t+1]\| \le (1 - h\phi[t])\|\boldsymbol{v}[t]\| \tag{9}$$

where $\phi[t]$ is the Fiedler number of the (random) matrix $\mathrm{Id} - hL[t]$.

First, we observe that the Fiedler number of a coloured graph satisfies

$$\phi \le \frac{k}{k-1}\min\{d(v) : v \in V(G)\}$$

where $d(v)$ is the degree of the vertex v (see Theorem 2.2 in [21]). From this, as $a_{ij} \le 1$ for all pairs i, j, we obtain $d(v) \le k - 1$ for all v, that gives $\phi \le k$ (the same bound that holds for non-coloured graphs). This means, as $h \le 1/k$ that $0 \le 1 - h\phi \le 1$.

Second, we obtain that the norm of the relative velocity of the flock is decreasing, giving a linear bound for the norm of the relative position

$$\begin{aligned} \|\boldsymbol{x}[t]\| &= \|\boldsymbol{x}[0] + h(\boldsymbol{v}[0] + \cdots + \boldsymbol{v}[t-1])\| \\ &\le \|\boldsymbol{x}[0]\| + th\|\boldsymbol{v}[0]\| \end{aligned}$$

Furthermore, the iteration of (9) gives

$$\|\boldsymbol{v}[\tau+1]\| \le \prod_{i=0}^{\tau}(1 - h\phi[i])\|\boldsymbol{v}[0]\|$$

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that using the equation of the position in (8) gives us the bound

$$\|\boldsymbol{x}[\tau]\| \leq \|\boldsymbol{x}[0]\| + h\left(\|\boldsymbol{v}[0]\| + \sum_{j=1}^{\tau-1}\|\boldsymbol{v}[j]\|\right)$$
$$\leq \|\boldsymbol{x}[0]\| + h\|\boldsymbol{v}[0]\|\left(1 + \sum_{j=1}^{\tau-1}\prod_{i=0}^{j-1}(1 - h\phi[i])\right)$$

It is crucial then to study the convergence of the series

$$S[\tau] = \sum_{j=1}^{\tau-1}\prod_{i=0}^{j-1}(1 - h\phi[i]) \tag{10}$$

to obtain an upper bound of the position, that, in its turn, will give us a lower bound on the Fiedler number.

An important observation is that the connectedness of the coloured graph with incidence matrix $(a_{ij}[t])$ coincides with the one of the 0–1 graph with incidence matrix (ζ_t^{ij}) (because both have the same zero and non-zero entries). This means that the connectedness of the graph is independent of the position and velocity of the system.

Let $\varphi[t]$ be the Fiedler number of the non-coloured graphs generated by (ζ_t^{ij}). Then, by Proposition 2 in [2] we have

$$\phi[t] \geq \varphi[t]\mu[t]$$

with

$$\mu[t] = \min\{a_{ij}[t]\colon a_{ij}[t] > 0, i \neq j\}$$

Note that when the graph is not connected we have $\phi[t] = \varphi[t] = 0$.

We now prove that $\mu[t] \geq [A/(B + t^\alpha)]$ for some constants A and B. For this, we rely on the inequality $(a + b)^\alpha \leq a^\alpha + b^\alpha$ that holds for $a \geq 0$, $b \geq 0$ and $0 \leq \alpha \leq 1$. Applying this inequality with $a = 1 + \|x[0]\|$ and $b = h\|v[0]\|t$ we obtain

$$\mu[t] \geq \frac{1/(h\|v[0]\|)^\alpha}{([1 + \|x[0]\|]/[h\|v[0]\|])^\alpha + t^\alpha} =: \frac{A}{B + t^\alpha}$$

for all t. Hence

$$\phi[t] \geq \varphi[t]\frac{A}{B + t^\alpha} =: \nu[t]$$

With this result we obtain the following bound for our sum in (10). Denote $\bar{\varphi} = \mathbf{E}\varphi[t]$ and note that $\bar{\varphi} > 0$ since φ is a non-negative and not identically zero random variable. As the quantities $\{\nu[t]\}$ form a sequence of independent (non-identically distributed) random variables, and

$$\mathbf{E}(1 - h\nu[t]) = 1 - h\bar{\varphi}\frac{A}{B + t^\alpha}$$

Furthermore

$$\mathbf{E}S[\tau] \leq \sum_{j=1}^{\tau-1}\prod_{i=1}^{j}\mathbf{E}(1 - h\nu[i]) = \sum_{j=1}^{\tau-1}\prod_{i=1}^{j}\left(1 - \bar{\varphi}h\frac{A}{B + i^\alpha}\right)$$

Assume $\alpha < 1$. For each summand above, since $\log(1 - x) \leq -x$, we have

$$\prod_{i=1}^{j}\left(1 - \bar{\varphi}h\frac{A}{B + i^\alpha}\right) = \exp\left(\sum_{i=1}^{j}\log\left(1 - \bar{\varphi}h\frac{A}{B + i^\alpha}\right)\right)$$
$$\leq \exp\left(-\sum_{i=1}^{j}\bar{\varphi}h\frac{A}{B + i^\alpha}\right)$$
$$\leq \exp\left(-\bar{\varphi}hA\sum_{i=1}^{j}\frac{1}{i^\alpha}\right)$$
$$\leq \exp\left(-\gamma j^{1-\alpha}\right) \tag{11}$$

where $\gamma = (\bar{\varphi}hA)/(1 - \alpha)$. Finally we observe that a series with general term given by (11) is convergent, and this implies the convergence of $\mathbf{E}(S[\tau])$ as $\tau \to \infty$ to a finite limit. As the series $S[\tau]$ itself is increasing with τ, we obtain that there exists an almost sure finite limit

$$S = \lim_{\tau\to\infty} S[\tau] < \infty \tag{12}$$

The case $\alpha = 1$ is treated separately. We have to modify the final bound in (11), in this case we have

$$\prod_{i=1}^{j}\left(1 - \bar{\varphi}h\frac{A}{B + i}\right) = \exp\left(\sum_{i=1}^{j}\log\left(1 - \bar{\varphi}h\frac{A}{B + i}\right)\right)$$
$$\leq \exp\left(-\bar{\varphi}hA\sum_{i=1}^{j}\frac{1}{B + i}\right)$$
$$\leq \exp\left(-\bar{\varphi}hA\log\left(\frac{B + j + 1}{B + 1}\right)\right)$$
$$= \left(\frac{B + 1}{B + j + 1}\right)^{\bar{\varphi}hA}$$

Now, a series with this general term converges when $\bar{\varphi}hA > 1$, that is $\|v[0]\| < \bar{\varphi}$. Under this condition we obtain (12).

From the convergence obtained in (12) (in both cases $\alpha < 1$ and $\alpha = 1$) for the series defined in (10), we obtain that the series of the velocities norm is convergent, that is

$$\sum_{j=0}^{\tau-1}\|v[j]\| \leq \|v[0]\|(1 + S)$$

This implies the convergence stated in (4). It also implies that the series of the velocities converges itself, that is, there exists

$$\hat{\boldsymbol{v}} = \sum_{j=0}^{\infty}\boldsymbol{v}[j]$$

This fact gives the convergence for the positions of the system

$$\boldsymbol{x}[\tau] = \boldsymbol{x}[0] + \sum_{j=0}^{\tau-1}(\boldsymbol{x}[j + 1] - \boldsymbol{x}[j]) = \boldsymbol{x}[0] + h\sum_{j=0}^{\tau-1}\boldsymbol{v}[j]$$
$$\to \boldsymbol{x}[0] + h\hat{\boldsymbol{v}} =: \hat{\boldsymbol{x}}$$

concluding the proof of the theorem.

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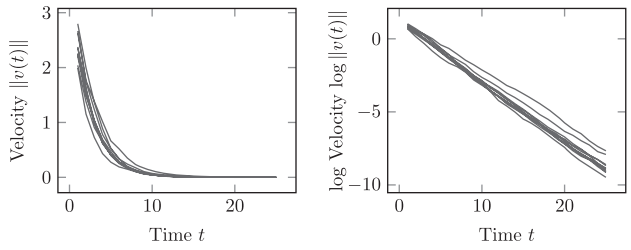


Fig. 1 *Evolution of velocity for* $\alpha = 0.5$ *and* $\lambda = 0.25$

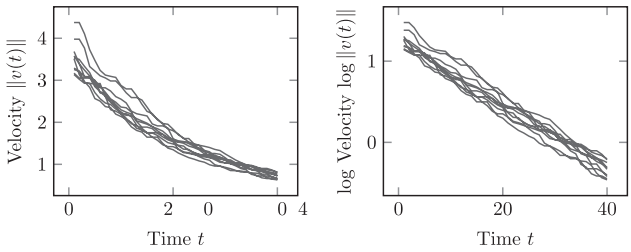


Fig. 2 *Evolution of velocity for* $\alpha = 0.5$ *and* $\lambda = 0.9$

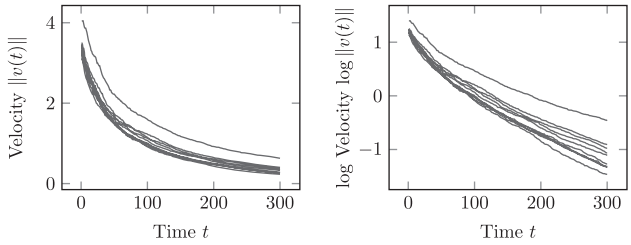


Fig. 3 *Evolution of velocity for* $\alpha = 1$ *and* $\lambda = 0.9$

4 Numerical simulations

We perform some simulations of the evolution of the system defined by (1) and Figs. 1–3 correspond to that evolution for some different values of the parameters. The graphics show the norm of the system relative velocity, that is the maximum of the two norms of the velocity of each agent relative to v_0, and the logarithm of that relative velocity.

The initial positions and velocities are sampled from standard normal distribution in $\mathbb{R}^3$, and we take $h = 1/k$. It can be observed that the decay seems to be exponential in the sub-linear cases ($\alpha < 1$), as in the hierarchical case (see [16]) although here our techniques do not provide this rate of convergence. The simulation results in the linear case ($\alpha = 1$) do not provide clear information about the rate of decay.

5 Conclusions

In this paper, we intend to make some contributions from the rigorous point of view in the study of the 'collective motion' of a group of autonomous interacting agents. This highly multidisciplinary field of research is inspired by diverse phenomena as the movement of a flock of bird, a school of fish, swarming of large number of locusts etc. and include works on data collection (observation); on describing the governing laws, on simulations and so on.

We relax the assumption of perfect communication between agents in Cucker–Smale model for the collective motion of a group of autonomous interacting agents. This well-known model permits the mathematical treatment of the asymptotic convergence to a common final velocity, known as 'flocking', requiring some natural conditions.

We prove that if the agents fail to connect randomly, with failures being independent in time and space and with fixed probability, the emergence of a consensus (flocking) still holds true with probability one in the critical and sub-critical regimes ($\alpha < 1$ and $\alpha = 1$ resp.).

Besides, we perform numeric simulations which suggest a (better) linear rate of decay of the velocity of the system w.r.t. the time in the case $\alpha < 1$.

For future work, the case $\alpha > 1$ remains open since our method does not provide good conditions to ensure flocking. Furthermore, our results involve the discrete setting. Hence, it is natural to consider the question of passing to the limit to the continuous case.

Besides, it is quite reasonable to consider other kinds of randomness. Also it should be interesting to address the relaxation of other assumptions of the model.

6 References

1 Vicsek, T., Zafeiris, A.: 'Collective motion', *Phys. Rep.*, 2012, **517**, pp. 71–140
2 Cucker, F., Smale, S.: 'Emergent behavior in flocks', *IEEE Trans. Autom. Control*, 2007, **52**, pp. 852–862
3 Jadbabaie, A., Lin, J., Morse, A.S.: 'Coordination of groups of mobile autonomous agents using nearest neighbor rules', *IEEE Trans. Autom. Control*, **48**, 2003, pp. 988–1001
4 Shen, J. 'Cucker–Smale flocking under hierarchical leadership', *SIAM J. Appl. Math.*, 2007/08, **68**, (3), pp. 694–719
5 Li, Z. (PRC-HIT), Xue, X. (PRC-HIT): 'Cucker–Smale flocking under rooted leadership with fixed and switching topologies. (English summary)'. *SIAM J. Appl. Math.*, 2010, **70**, (8), pp. 3156–3174
6 Li, Z., Xue, X.: 'Cucker–Smale flocking under rooted leadership with free-will agents', *Physica A*, 2014, **410**, pp. 205–217
7 Carrillo, J.A., Fornasier, M., Rosado, J., Toscani, G.: 'Asymptotic flocking dynamics for the kinetic Cucker–Smale model', *SIAM J. Math. Anal.*, 2010, **42**, (1), pp. 218–236
8 Ha, S.Y., Liu, J.G.: 'A simple proof of the Cucker–Smale flocking dynamics and mean-field limit', *Commun. Math. Sci.*, 2009, **7**, (2), pp. 297–325
9 Caponigro, M., Fornasier, M., Piccoli, B., Trélat, E.: 'Sparse stabilization and optimal control of the Cucker–Smale model', *Math. Control Relat. Fields*, 2013, **3**, (4), pp. 447–466
10 Cucker, F., Mordecki, E.: 'Flocking in noisy environments', *J. Math. Pures Appl.*, 2008, **89**, (3), pp. 278–296
11 Ha, S.-Y., Lee, K., Levy, D.: 'Emergence of time-asymptotic flocking in a stochastic Cucker–Smale system', *Commun. Math. Sci.*, 2009, **7**, (2), pp. 453–469
12 Shang, Y.: 'Emergence in random noisy environment', *Int. J. Math. Anal.*, 2010, **4**, (25–28), pp. 1205–1215
13 Ton, T.V., Linh, N.T.H., Yagi, A.: 'Flocking and non-flocking behavior in a stochastic Cucker–Smale system', *Anal. Appl. (Singap.)*, 2014, **12**, (1), pp. 63–73
14 Ahn, S.M., Ha, S.-Y.: 'Stochastic flocking dynamics of the Cucker–Smale model with multiplicative white noises', *J. Math. Phys.*, 2010, **51**, (10), 103301-1–103301-17
15 Dalmao, F., Mordecki, E.: 'Cucker–Smale flocking under hierarchical leadership and random interactions', *SIAM J. Appl. Math.*, 2011, **71**, (4), pp. 1307–1316
16 Dalmao, F., Mordecki, E.: 'Hierarchical Cucker–Smale model subject to random failure', *IEEE Trans. Autom. Control*, 2012, **57**, (7), pp. 1789–1793
17 Martin, S., Fazeli, A., Jadbabaie, A., Girard, A.: 'Multi-agent flocking with random communication radius'. American Control Conf., 2012, pp. 3871–3876
18 Perea, L., Gomez, G., Elosegui, P.: 'Extension of the Cucker–Smale control law to space flight formations', *J. Guid. Control Dyn.*, 2009, **32**, (2), pp. 526–536
19 Ahn, S., Bae, H.-O., Ha, S.-Y., Kim, Y., Lim, H.: 'Application of flocking mechanism to the modeling of stochastic volatility', *Math. Models Methods Appl. Sci.*, 2013, **23**, (9), pp. 1603–1628
20 Abreu, N.M.M.: 'Old and new results on algebraic connectivity of graphs', *Linear Algebra Appl.*, 2007, **423**, (1), pp. 53–73
21 Mohar, B.: 'The Laplacian spectrum of graphs'. Proc. Int. Conf. Theory and Applications of Graphs, Michigan, USA, 1991, pp. 871–898

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