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Rice formula for processes with jumps and applications

Federico Dalmao · Ernesto Mordecki

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Abstract We extend Rice Formula to a process which is the sum of two independent processes: a smooth process and a pure jump process with finitely many jumps. Formulas for the mean number of both continuous and discontinuous crossings through a fixed level on a compact time interval are obtained. We present examples in which we compute explicitly the mean number of crossings and compare which kind of crossings dominates for high levels. In one of the examples the leading term of the tail of the distribution function of the maximum of the process over a compact time interval as the level goes to infinity is obtained. We end giving a generalization, to the non-stationary case, of Borovkov-Last's Rice Formula for Piecewise Deterministic Markov Processes.

Keywords Rice formula · Level crossings · Process with jumps · Number of continuous and discontinuous crossings

AMS 2000 Subject Classifications Primary—60G10 stationary processes · 60J75 Jump processes; Secondary—60G70 Extreme value theory; extremal processes

F. Dalmao (✉)
Departamento de Matemática y Estadística del Litoral, Regional Norte, Universidad de la República, Rivera 1350, 50000, Salto, Uruguay
e-mail: fdalmao@unorte.edu.uy

E. Mordecki
Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, 11400, Montevideo, Uruguay
e-mail: mordecki@cmat.edu.uy

1 Introduction

In the present work, we are interested in the number of crossings through a fixed level by a class of stochastic processes having finitely many jumps on compact intervals and smooth stochastic evolution between the jumps. More precisely, we consider a process X which can be written in the form $\mathrm{X} = \mathrm{Z} + \mathrm{J}$ where Z is a process with continuously differentiable paths and J is a pure jump process, independent from Z. Thus, Z describes the continuous part of X and J describes the jumps of X.

Such a process can cross a fixed level u at a continuity point or at a jump one. Under general conditions, we obtain Rice-type formulas for the mean number of (both) continuous and discontinuous crossings through a fixed level $u \in \mathbb{R}$ on a compact time interval. Afterwards, we give explicit algebraic expressions for the mean number of crossings in two examples. Then, we compare which kind of crossing (continuous/discontinuous) dominates as the level u goes to infinity and find different behaviors on these examples. Besides, in the first example we derive second order Rice formulas for the continuous and discontinuous crossings and use them to obtain the exact asymptotic leading term of $\mathbb{P}(M(T) > u)$ as $u \to \infty$ where $M(T) = \max\{X(t) : t \in [0, T]\}$.

The paper is organized as follows. Section 2 gives some background on Rice formulas and compares the present work with similar previous ones. Section 3 starts with the definitions and then presents the main result (Theorem 1). Section 4 presents the examples. Section 5 moves a bit from the main line presenting a generalization of Borovkov-Last's Rice-type formula to the non-stationary case. Finally, our main result is proved in Section 6.

This work began under the initiative of Mario Wschebor, and is largely inspired by his fundamental contribution in the field, masterly exposed in Azaïs and Wschebor (2009).

2 Background

Counting the number of crossings through a fixed level by a stochastic process is a classical problem in Probability Theory. Nevertheless little is known, in general, about the distribution of the random variable *number of crossings*. Therefore, Rice Formulas, which give expressions for the moments of this r.v, are very important and useful.

Rice Formulas are named after S.O. Rice who obtained by informal arguments a simple formula for the mean number of crossings through a fixed level for stationary Gaussian processes in 1944 (Rice 1944, 1945). This formula was afterwards proved under successively weaker conditions, see Azaïs and Wschebor (2009), Kratz (2006), and Leadbetter and Spaniolo (2004) and references therein. The major part of the literature on the subject is devoted to the Gaussian case, in particular to the stationary one. Extensions for general Gaussian, non-Gaussian processes and fields were stated afterwards, see Adler and Samorodnitsky (1997), Azaïs and Wschebor (2009), Brillinger (1972), Kratz (2006), Marcus (1977), and Zähle (1984). Some recent extensions to non-Gaussian processes include the Shot Noise processes (Biermé

and Desolneux 2012a, b), the Generalized Hyperbolic Process (Alodat and Aluudat 2008), the Laplace Moving Average (Galtier 2011) and the Piecewise Deterministic Markov Processes (Borovkov and Last 2008, 2012).

The applications of Rice Formulas include telecommunications and signal processing (Rice 1944, 1945); reliability theory in engineering (Rychlik 2001); oceanography (the height of sea waves) (Azaïs et al. 2011; Longuett-Higgins 1957); physics and astronomy: random mechanics (Krée and Soize 1983), the Shot Noise processes (Biermé and Desolneux 2012a, b) and microlensing (Petters et al. 2009); random systems of polynomial equations (Cucker et al. 2012; Armentano and Wschebor 2009), among others.

The case of discontinuous processes has been treated only recently in the literature. Borovkov and Last (2008, 2012) consider the number of continuous crossings of a discontinuous process with random jumps and deterministic evolution between jumps. The continuous and discontinuous parts of the process are not independent, hence, our main result does not apply. Nevertheless, we extend the Rice formula given in Borovkov and Last (2008) in Section 5. Biermé and Desolneux (2012b), consider a process obtained by superposition of randomly scaled and translated copies of a given deterministic discontinuous function (Shot noise process). They are interested in the expectation of the total number of crossings for different levels for this particular process, and obtain the Fourier transform of this expectation with respect to these levels.

3 Preliminaries and Main Result

Let $f : [0, T] \to \mathbb{R}$ be càdlàg and C^1 between jumps. We say that f has a

- continuous crossing through the level u at $s \in (0, T)$ if f is continuous at s, $f(s) = u$ and $f'(s) \neq 0$;
- a discontinuous crossing if $(f(s^-) - u)(f(s) - u) < 0$.

If $f'(s) > 0$ in the continuous case, or $f(s^-) < u < f(s)$ in the discontinuous one, we say that f has an up-crossing at s, otherwise the crossing is a down-crossing.

Remark 1 For the processes considered here this definition coincides with that of Scheutzow (1994) and in the case of continuous crossings also with that of Leadbetter et al. (1983).

Now, we introduce the processes, all defined on $[0, T]$. Let $\mathrm{Z} = (Z(t))$ and $\mathrm{J} = (J(t))$ be independent stochastic processes and define $\mathrm{X} = (X(t))$ with $X(t) = Z(t) + J(t)$.

We assume some regularity conditions on the process Z, see Azaïs and Wschebor (2009), namely:

$A1$: the paths of Z are C^1, almost surely.

$A2$: the density $p_{Z(t)}(x)$ is jointly continuous for $t \in [0, T]$ and x in a neighborhood of u. Furthermore, assume that for every $t, t' \in [0, T]$ the joint

distribution of $(Z(t), \dot{Z}(t'))$ has a density $p_{Z(t),\dot{Z}(t')}(x, x')$ which is continuous w.r.t. t (t', x, x' fixed) and w.r.t. x at u (t, t', x' fixed).

$A3$: for every $t \in [0, T]$ there exists a continuous version of the conditional expectation $\mathbb{E}(\dot{Z}(t) \mid X(t) = x)$ for x in a neighborhood of u.

$A4$: the modulus of continuity of $\dot{Z}$ tends to 0 if $\delta \to 0$:

$$\sup_{0\le s<t\le T,\, |t-s|<\delta} |\dot{Z}(t) - \dot{Z}(s)| \underset{\delta\to 0}{\to} 0.$$

We also assume that

B : J is a pure jump process with almost surely finitely many jump epochs on any compact time interval.

The n-th jump epoch and the n-th jump magnitude of J are denoted respectively by τ_n and ξ_n. The process $(\tau_n, \xi_n)_{n\in\mathbb{N}}$ is a Marked Point Process, see Jacobsen (2006) for the general construction based on Markov kernels. Finally, for $\tau_n \le t < \tau_{n+1}$ let $\nu_t = n$. Hence

$$J(t) = \sum_{k=0}^{\nu_t} \xi_k.$$

We can identify the marked point process $\big(\tau_n, J(\tau_n)\big)_{n\in\mathbb{N}}$ with its associated random counting measure (RCM) $\mu := \sum_{n=0}^{\nu_T} \delta_{(\tau_n, J(\tau_n))}$ on $[0, T] \times \mathbb{R}$, where $\delta_{(t,x)}$ is the Dirac Delta measure concentrated at the point (t, x). Note that the RCM μ of a Borel set C in $[0, T] \times \mathbb{R}$ is just the number of points (jump epochs and marks) of the marked point process that lie in C.

There exists a random measure $L(dt, dy)$ on $[0, T] \times \mathbb{R}$, called the compensating measure of μ, such that the processes $t \mapsto L([0, t], A)$ are predictable for any Borel set A in $\mathbb{R}$ and, under quite general conditions (for example the absolute continuity of the kernels and the finiteness of the intensity of jumps), L can be written in terms of ordinary Lebesgue integrals. Furthermore

$$\mathbb{E} \int_{[0,T]\times\mathbb{R}} f\, d\mu = \mathbb{E} \int_{[0,T]\times\mathbb{R}} f\, L(dt, dy)$$

for any predictable function f, see Jacobsen (2006).

We can apply the definition of crossing to almost all of the paths. Denote N_u^c (resp. N_u^d) the random number of continuous (resp. discontinuous) crossings through the level u by the process X on the interval $[0, T]$ and by N_u the total number of crossings. For up-crossings we use the same notation with N replaced by U. It is clear that $N_u = N_u^c + N_u^d$, and consequently that $\mathbb{E}\, N_u = \mathbb{E}\, N_u^c + \mathbb{E}\, N_u^d$.

Theorem 1 *Let Z and J be two independent processes on $[0, T]$ verifying conditions $A1$-$A4$ and B respectively. Then, the mean number of continuous and discontinuous*

crossings through the level u by the process $X = Z + J$ *on the interval* $[0, T]$ *are given respectively by:*

$$\begin{aligned}\mathbb{E}\, N_u^c &= \int_{[0,T]} \mathbb{E}\left(|\dot{X}(t)| \mid X(t) = u\right) p_{X(t)}(u)dt \\ &= \int_{[0,T]} \mathbb{E}\left(|\dot{Z}(t)| \mid X(t) = u\right) p_{X(t)}(u)dt,\end{aligned}$$

and

$$\mathbb{E}\, N_u^d = \mathbb{E} \iint_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\} L(dt, dy),$$

where $\mathbf{1}A$ *is the indicator function of the set A and L is the compensating measure of the random counting measure generated on* $[0, T] \times \mathbb{R}$ *by the jump process J. Similar formulas hold for the number of up and down crossings.*

The proof is postponed to Section 6, let us say here that the continuous and the discontinuous crossings are treated separately and by different methods. An interesting feature of the approach we propose is the fact that Kac Formula (Kac 1943) counts only the continuous crossings. The discontinuous ones are counted using techniques from Point Processes Theory.

Some remarks are in order. First, note that when the jump process J vanishes, that is, $J(t) = 0$ almost surely for all $t \in [0, T]$, Theorem 1 reduces to Classical Rice Formula for the process Z.

Next, observe that the random variable $J(t)$ does not need to have a density for each t, but, in case it does we have a more explicit result.

Corollary 2 *If* $J(t)$ *has a continuous density* $p_{J(t)}(x)$ *for* $t \in [0, T]$ *and* $x \in \mathbb{R}$*, we can also write*

$$\mathbb{E}\, N_u^c = \int_{[0,T]} dt \int_{\mathbb{R}} \mathbb{E}\left(|\dot{Z}(t)| \mid Z(t) = v\right) p_{Z(t)}(v) p_{J(t)}(u - v)dv.$$

Finally, a careful analysis of the proof of Theorem 1 in Section 6 shows that the result in Theorem 1 holds true whenever the law of the process Z, restricted to the subintervals $[\tau_i, \tau_{i+1}]$, conditioned to the paths of the jump process verifies the hypothesis $A1$ - $A4$. Besides, $A3$ can be weakened assuming that the product of the conditional expectation and the density of $X(t)$ (or $Z(t)$ in Corollary 2) is continuous.

We end this section specializing these results to the case where Z is a Gaussian process, here, the hypothesis of continuity of the densities and of the conditional expectation may be released since they follow from the conditions of non-degeneracy of the distribution of $Z(t)$, $t \in [0, T]$, and on the regularity of the paths. Besides, the ingredients in the formulas are computable explicitly.

Corollary 3 *Let Z be a Gaussian process with* C^1 *paths such that for every t the distribution of* $Z(t)$ *is non-degenerated, assume further that J is a pure jump process independent from Z with finite intensity of jumps. Then the result of Theorem 1 holds true.*

Corollary 4 *Under the hypothesis of Corollary 3, if in addition Z is stationary, then the formula for the continuous crossings reduces to:*

$$\mathbb{E}\, N_u^c = \mathbb{E}\,|\dot{Z}(0)| \int_0^T p_{X(t)}(u) dt.$$

Remark 2 Let us define the density $p(u) := \frac{1}{T}\int_0^T p_{X(t)}(u)\, dt$. Then, we can write

$$\mathbb{E}\, N_u^c = T\, \mathbb{E}\,|\dot{Z}(0)|\, p(u).$$

Observe that p is a mixture of the density of $X(t)$ for $t \in [0, T]$. In particular, if X is a stationary process, the density p reduces to that of $X(0)$, just as in the original formula due to Rice.

4 Examples

In this section we present two examples. In both of them Z is a differentiable centered stationary Gaussian process. Let Γ be the covariance function of Z, i.e: $\Gamma(\tau) = \mathbb{E}\, Z(0)Z(\tau)$. Note that $-\Gamma''(0)$ is the variance of $\dot{Z}(0)$.

The jump epochs τ_n of J are such that for all n, $\tau_n - \tau_{n-1}$ are independent exponential random variables with intensity $\lambda < \infty$. The process $\nu = (\nu_t : t \geq 0)$ with $\nu_t = \max\{n : \tau_n \leq t\}$ is called a Simple Poisson Process.

For the sake of notational simplicity we consider only the up-crossings, the case of down-crossings is completely analogous.

4.1 Stationary processes with 1-dimensional Gaussian distribution

Assume that $\mathrm{Z} = (Z(t) : t \in [0, T])$ is a centered stationary Gaussian process with C^1 paths. Assume also that $\Gamma(0) = 1/2$.

Besides, let $\mathrm{J} = (J(t) : t \in [0, T])$ be constructed in the following way. Given a sequence of independent and identically distributed random variables $(\epsilon_n : n \in \mathbb{N})$ with common centered Gaussian distribution with variance $1/2$, and $\rho \in \mathbb{R}$ such that $|\rho| < 1$, define

$$A_0 = \epsilon_0, \text{ and for } n \geq 1 : A_n = \rho A_{n-1} + \sqrt{1 - \rho^2}\, \epsilon_n.$$

Consider also a Simple Poisson Process $\nu = (\nu_t : t > 0)$ with intensity $0 < \lambda < \infty$. Assume that ν is independent from $(\epsilon_n)_n$. Finally, let

$$J(t) = A_{\nu_t}.$$

We call such a process a Poisson-auto-regressive process and denote it by PAR(ρ, λ). Note that this process is similar, in certain sense, to the Random Telegraph Signal.

The following proposition is obtained by a direct computation conditioning on the number of jumps of J.

Proposition 5 *Let J be a PAR(ρ, λ) process, then J is wide-sense stationary and for $\tau \in [0, T]$ the r.v. $J(\tau)$ has centered Gaussian distribution with variance $1/2$ and the covariance between $J(0)$ and $J(\tau)$ is $e^{-\lambda(1-\rho)\tau}/2$.*

The next theorem gives the mean number of continuous and discontinuous crossings through the level u by the process X on the interval $[0, T]$. Let φ denote the standard Gaussian density function. Considering (X, Y) a two dimensional centered Gaussian vector with $\mathbb{E}\, X^2 = \mathbb{E}\, Y^2 = 1$ and $\mathbb{E}\, XY = (1+\rho)/2$, introduce

$$P_\rho(u) = \mathbb{P}\,(X < u, Y > u), \tag{1}$$

i.e. the probability that a two dimensional centered Gaussian vector with unit variances and covariance $(1+\rho)/2$ belongs to the set $(-\infty, u) \times (u, \infty)$.

Theorem 6 *Let $X = Z + J$ with Z, J independent processes such that Z is a centered stationary Gaussian process with $\Gamma(0) = 1/2$ and C^1 paths and J a PAR(ρ, λ) process. Then*

$$\mathbb{E}\, U_u^c = T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\,\varphi(u), \quad \mathbb{E}\, U_u^d = \lambda T\, P_\rho(u).$$

Proof We begin considering the mean number of continuous crossings. Since Z is Gaussian and stationary, we can apply Corollary 4. Besides, by Proposition 5, the density $p_{X(t)} = p_{X(0)} = \varphi$ for all t. Therefore,

$$\mathbb{E}\, U_u^c = T\, \mathbb{E}\, \dot{Z}^+(0)\, \varphi(u).$$

Routine computations show that $\mathbb{E}\, \dot{Z}^+(0) = \sqrt{-\Gamma''(0)/2\pi}$. This gives $\mathbb{E}\, U_u^c$.

Let us consider now the discontinuous up-crossings. The compensating measure of the point process $(\tau_n, \xi_n)_n$, $\xi_n := A_n - A_{n-1}$, is $\lambda dt\, F(dy)$, where F is the normal distribution centered at $(\rho-1)A_{\nu_{t^-}}$ and with variance $(1-\rho^2)/2$, see Jacobsen (2006, eq. 4.64). Hence

$$\begin{aligned}\mathbb{E}\, U_u^d &= \mathbb{E} \int_0^T \int_{\mathbb{R}} \mathbf{1}\{X(t^-) < u, X(t^-) + y > u\}\lambda dt\, F(dy) \\ &= \lambda \mathbb{E} \int_0^T dt \int_{\mathbb{R}} \mathbf{1}\{X(t^-) < u, X(t^-) + y > u\} F(dy).\end{aligned}$$

Actually F is the distribution of ξ_{ν_t} conditioned on the random vector $\left(Z(t), A_{\nu_{t^-}}\right)$. Thus

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \mathbb{E} \int_0^T \mathbb{P}\left(X(t^-) < u, X(t^-) + \xi_{\nu_t} > u \mid Z(t), A_{\nu_{t^-}}\right) dt \\ &= \lambda \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{E}\, \mathbb{P}\,(Z(t) + A_n < u, Z(t) + A_n + \xi_{n+1} > u \mid Z(t), A_n)\, dt \\ &= \lambda \int_0^T \sum_{n=0}^{\infty} p_{\nu_t}(n) \mathbb{P}\,(Z(t) + A_n < u, Z(t) + A_n + \xi_{n+1} > u)\, dt\end{aligned}$$

where we have conditioned on the number of jumps and used Fubini's theorem. Note that in this case a direct computation is possible. By the stationarity of Z and J, the last probability does not depend on t and n. Hence

$$\mathbb{E}\, U_u^d = \lambda T \mathbb{P}\,(Z(0) + A_0 < u,\, Z(0) + A_0 + \xi_1 > u) = \lambda T\, P_\rho(u).$$

This gives $\mathbb{E}\, U_u^d$ and concludes the proof. □

Now we can see that the continuous crossings dominate, in mean, for high levels.

Let Φ be the standard Gaussian distribution and $\overline{\Phi}(z) = 1 - \Phi(z)$. As usual, $f(u) = o(g(u))$ means that $\lim_{u\to\infty} f(u)/g(u) = 0$.

Corollary 7 *As $u \to \infty$, we have $\mathbb{E}\, U_u^d = o\left(\mathbb{E}\, U_u^c\right)$.*

Proof The result follows directly from Eq. 1:

$$P_\rho(u) \le \mathbb{P}\,(Y > u) \le \overline{\Phi}\,(u) = o\,(\varphi(u))\,.$$

□

Remark 3 The processes Z and J contribute in $1/2$ to the variance of X. More generally we can define

$$\mathrm{X} = \alpha \mathrm{Z} + \sqrt{1-\alpha^2}\mathrm{J}$$

for $\alpha \in [0, 1]$ and Z, J independent, centered, stationary with variance one. Note that $p_{X(t)} = \varphi$ for all α and $-\Gamma''_{\alpha Z(0)}(0) = -\alpha^2 \Gamma''_{Z(0)}(0)$. Hence

$$\mathbb{E}\, N_u^c(\mathrm{X}) = \alpha\, \mathbb{E}\, N_u^c(\mathrm{Z}).$$

Therefore, the mean number of continuous crossings of X is proportional to that of the continuous process Z, the constant of proportionality being α, that is, the standard deviation of the first summand αZ.

4.2 Stationary Gaussian continuous process plus CPP

In this example we consider a centered stationary Gaussian process Z with $\Gamma(0) = 1$ and C^1 paths and an independent Compound Poisson Process (CPP), J with finite intensity λ and standard Gaussian jumps.

The process J is defined as follows: let $(\xi_n)_{n\in\mathbb{N}}$ be independent standard Gaussian random variables and let $\nu = (\nu_t : t \ge 0)$ be a Simple Poisson Process independent from (ξ_n). Then, for $t \ge 0$ define $J(t) := \sum_{n=1}^{\nu_t} \xi_n$.

Thus, $J(t)$ is the sum of a random number of standard normal random variables. It is well known, see Jacobsen (2006) for instance, that the compensating measure of the CPP is $\lambda dt\, \Phi(dx)$. In particular, the compensating measure is deterministic.

The next theorem gives the mean number of up-crossings through u by X. Let φ_n denote the centered Gaussian density with variance n, in particular, $\varphi_1 = \varphi$. Note that $\varphi_n * \varphi_m = \varphi_{n+m}$, where $*$ stands for the convolution. Furthermore, let $p = \sum_{n=1}^{\infty} p_n \varphi_n$ with $p_n = \frac{1}{\lambda T}\mathbb{P}\,(\nu_T \ge n)$ and note that the density function p has expectation 0 and variance $\lambda T/2$.

Theorem 8 *For the process defined above we have*

$$\mathbb{E}\, U_u^c = T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\, p(u), \quad \mathbb{E}\, U_u^d = \lambda T \int_{-\infty}^{u} \overline{\Phi}(u-x)\, p(x)dx.$$

Observe that the mean number of continuous crossings has the usual form, see Remark 2.

Proof We begin with the continuous crossings. Using the formula in Corollary 2 we have

$$\begin{aligned}\mathbb{E}\, U_u^c &= \int_0^T dt \int_{-\infty}^{\infty} \mathbb{E}\left(\dot{Z}^+(t) \mid Z(t) = v\right) p_{Z(t)}(v) p_{J(t)}(u-v)dv \\ &= \mathbb{E}\, \dot{Z}^+(0) \int_0^T dt \int_{-\infty}^{\infty} p_{Z(0)}(v) p_{J(t)}(u-v)dv,\end{aligned}$$

where we used the stationarity of Z as in Corollary 4. Recall that $\mathbb{E}\, \dot{Z}^+(0) = \sqrt{-\Gamma''(0)/2\pi}$. Now, by Lemma 1 we decompose $p_{J(t)}$. Thus

$$\begin{aligned}\mathbb{E}\, U_u^c &= \sqrt{\frac{-\Gamma''(0)}{2\pi}} \int_0^T dt \sum_{n=0}^{\infty} p_{\nu_t}(n) \int_{-\infty}^{\infty} p_{Z(0)}(v)\varphi_n(u-v)dv \\ &= \sqrt{\frac{-\Gamma''(0)}{2\pi}} \int_0^T dt \sum_{n=0}^{\infty} p_{\nu_t}(n)(p_{Z(0)} * \varphi_n)(u) \\ &= \sqrt{\frac{-\Gamma''(0)}{2\pi}} \sum_{n=0}^{\infty} \varphi_{n+1}(u) \int_0^T p_{\nu_t}(n)dt.\end{aligned}$$

This integral is computed in the second item of Lemma 1. So we get

$$\mathbb{E}\, U_u^c = T\sqrt{\frac{-\Gamma''(0)}{2\pi}} \sum_{n=0}^{\infty} \frac{1}{\lambda T} \mathbb{P}\,(\nu_T \geq n+1)\, \varphi_{n+1}(u) = T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\, p(u).$$

Now we turn to the discontinuous crossings

$$\begin{aligned}\mathbb{E}\, U_u^d &= \mathbb{E} \int_0^T \int_{-\infty}^{\infty} \mathbf{1}\{X(t^-) < u;\, X(t^-) + y > u\} L(dt, dy) \\ &= \mathbb{E} \int_0^T \int_{-\infty}^{\infty} \mathbf{1}\{X(t^-) < u;\, X(t^-) + y > u\} \lambda dt\, \Phi(dy).\end{aligned}$$

Since the compensating measure is deterministic we have

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \int_0^T \int_{-\infty}^{\infty} \mathbb{P}\left(X(t^-) < u;\, X(t^-) + y > u\right) dt\, \Phi(dy) \\ &= \lambda \int_0^T \mathbb{P}\left(X(t^-) < u;\, X(t^-) + \xi > u\right) dt,\end{aligned}$$

being ξ a standard normal variable independent from $X(t^-)$. Now we condition on $X(t^-)$, since for fixed t, almost surely $J(t^-) = J(t)$ we may use the same computations as in the computation of $\mathbb{E}\, N_u^c$. Hence

$$\begin{aligned}\mathbb{E}\, U_u^d &= \lambda \int_0^T \int_{-\infty}^u \mathbb{P}\left(\xi > u - x \mid X(t^-) = x\right) p_{X(t^-)}(x) dx dt \\ &= \lambda \int_{-\infty}^u \overline{\Phi}(u-x) dx \int_0^T p_{X(t^-)}(x) dt \\ &= \lambda T \int_{-\infty}^u \overline{\Phi}(u-x) p(x) dx.\end{aligned}$$

This completes the proof. □

Next, we compare the mean numbers of continuous and discontinuous upcrossings through the level u by X as $u \to \infty$.

Corollary 9 *As $u \to \infty$ the mean numbers of continuous and discontinuous upcrossings through the level u are of the same order. More precisely,*

$$\lim_{u\to\infty} \frac{\lambda}{\sqrt{-\Gamma''(0)}}\, \mathbb{E}\, U_u^c \le \lim_{u\to\infty} \mathbb{E}\, U_u^d \le \lim_{u\to\infty} \lambda \sqrt{\frac{2\pi}{-\Gamma''(0)}}\, \mathbb{E}\, U_u^c.$$

Proof We bound $\mathbb{E}\, U_u^d$ from above and below. First, we have

$$\int_{-\infty}^u \overline{\Phi}(u-y) p(y) dy = \int_0^\infty \overline{\Phi}(y) p(u-y) dy \ge p(u) \int_0^{2u} \overline{\Phi}(y) dy,$$

where we used that p is even and decreasing on $[0, \infty)$. Furthermore,

$$\lim_{u\to\infty} \int_0^{2u} \overline{\Phi}(y) dy = \int_0^\infty \overline{\Phi}(y) dy = \mathbb{E}\, \xi^+ = \sqrt{\frac{2}{\pi}}.$$

Therefore,

$$\mathbb{E}\, U_u^d \ge \lambda T\, p(u) \int_0^\infty \overline{\Phi}(y) dy \underset{u\to\infty}{\sim} \frac{\lambda T}{\sqrt{2\pi}} p(u).$$

On the other hand,

$$\mathbb{E}\, U_u^d = \lambda T \sum_{n=1}^\infty p_n \int_0^\infty \overline{\Phi}(u-y) \varphi_n(y) dy \le \lambda T \sum_{n=1}^\infty p_n \int_{-\infty}^\infty \overline{\Phi}(u-y) \varphi_n(y) dy.$$

Note that this is a convolution formula for the tail of the distribution of two (independent) random variables, say $Z \sim \Phi$ and $V_n \sim \varphi_n$. Then the last integral equals $\mathbb{P}(Z + V_n > u)$. Furthermore, since Φ is the Gaussian standard distribution and φ_n is the Gaussian density with zero mean and variance n, this probability equals $\mathbb{P}(V_{n+1} > u) = \overline{\Phi}(u/\sqrt{n+1})$. Thus

$$\mathbb{E}\, U_u^d \le \lambda T \sum_{n=1}^\infty p_n \overline{\Phi}\left(\frac{u}{\sqrt{n+1}}\right) \le \lambda T \sum_{n=1}^\infty (n+1) p_n \varphi_{n+1}(u) = \lambda T (p(u) + \delta(u)),$$

where $\delta(u) := \sum_{n=1}^{\infty}(n+1)p_n\varphi_{n+1}(u) - p(u)$. In order to obtain the desired result, it suffices to prove that $\delta(u) \to 0$ as $u \to \infty$. In fact,

$$|\delta(u)| \leq \left|\sum_{n=1}^{\infty}\left[(n+1)p_n - p_{n+1}\right]\varphi_{n+1}(u)\right| + |p_1\varphi_1(u)|.$$

It is clear that the second term of the r.h.s tends to zero when $u \to \infty$. Let us look at the first term, note that $\sum_{n=1}^{\infty}(n+1)p_n$ is, roughly speaking, the second factorial moment of the Poisson distribution. In fact,

$$\sum_{n=1}^{\infty}(n+1)p_n = 1+\frac{1}{\lambda T}\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}np_{\nu_T}(k) = 1+\frac{1}{2\lambda T}\sum_{k=1}^{\infty}k(k-1)p_{\nu_T}(k) < \infty.$$

Thus, the first term is a mixture of Gaussian densities φ_n (times a constant), hence it tends to zero. Then $\delta(u) \to 0$ as $u \to \infty$ and

$$\mathbb{E}\,U_u^d \leq \lambda T(p(u)+\delta(u)) \underset{u\to\infty}{\sim} \lambda T p(u).$$

Now, the result follows putting together the two obtained bounds for $\mathbb{E}\,U_u^d$ and using Theorem 8. □

Remark 4 Note that in this case the discontinuous crossings through the level u are not negligible w.r.t. the continuous crossings when u tends to infinity, in contrast with the situation in the example of the previous section.

We end with an auxiliary lemma.

Lemma 1 *Let J be the CPP with standard Gaussian jumps, then*

1. *the density of the CPP can be written as: $p_{J(t)}(x) = \sum_{n=1}^{\infty} p_{\nu_t}(n)\varphi_n(x)$.*
2. *$\int_0^T p_{\nu_t}(n)dt = \frac{1}{\lambda}\mathbb{P}(\nu_T \geq n+1) = \frac{1}{\lambda}\mathbb{P}(\tau_{n+1} \leq T)$.*
3. *If $p_n = \frac{1}{\lambda T}\mathbb{P}(\nu_T \geq n)$, then $\sum_{n=1}^{\infty} p_n = 1$.*

Proof 1. Conditioning on the value of ν_t we have:

$$F_{J(t)}(x) = \mathbb{P}(J(t) \leq x) = \sum_{n=0}^{\infty} p_{\nu_t}(n)\mathbb{P}(J(t) \leq x \mid \nu_t = n).$$

Now, if $\nu_t = n$, $J(t)$ is the sum of n independent standard Gaussian random variables, hence, the conditional probability in the r.h.s. of the last equation is the distribution of centered normal random variable with variance n. The result follows taking derivatives on both sides.

2. By definition, $p_{\nu_t}(n)$, as a function of t, is equal to the density function of the Gamma distribution with parameters $(\lambda, n+1)$ divided by λ, it is well known that this is the distribution of τ_{n+1}, hence

$$\int_0^T p_{\nu_t}(n)dt = \frac{\mathbb{P}(\tau_{n+1} \leq T)}{\lambda}.$$

The result follows since the events $\{\tau_{n+1} \le T\}$ and $\{\nu_T \ge n+1\}$ coincide.

3. It follows directly from the facts that ν_T has Poisson distribution with mean λT and that for a non-negative integer valued random variable X: $\mathbb{E}\,X = \sum_{n=1}^{\infty} \mathbb{P}\,(X \ge n)$.

□

4.3 Application to the distribution of the maximum

Recall that

$$M(T) = \max\{X(s) : 0 \le s \le T\}.$$

We now use Rice Formula to get upper and lower bounds for $\mathbb{P}\,(M(T) > u)$ where X has differentiable stationary Gaussian continuous part and Poisson-auto-regressive jump part defined in Section 4.1.

The following bounds are based on the elementary relation

$$\{M(T) > u\} = \{X(0) > u\} \biguplus \{X(0) < u, U_u \ge 1\},$$

where $\biguplus$ denotes the disjoint union. It follows that

$$\mathbb{P}\,(M(T) > u) \le \mathbb{P}\,(X(0) > u) + \mathbb{P}\,(U_u \ge 1) \le \mathbb{P}\,(X(0) > u) + \mathbb{E}\,U_u, \qquad (2)$$

and

$$\begin{aligned}\mathbb{P}\,(M(T) > u) &= \mathbb{P}\,(X(0) > u) + \mathbb{P}\,(U_u \ge 1) - \mathbb{P}\,(U_u \ge 1, X(0) > u)\\ &\ge \mathbb{P}\,(X(0) > u) + \mathbb{E}\,U_u - \frac{1}{2}\mathbb{E}\,U_{u.[2]} - \mathbb{P}\,(U_u \ge 1, X(0) > u),\end{aligned} \qquad (3)$$

where $a_{[2]} = a(a-1)$ is the Pochammer symbol and $U_{u.[2]} = (U_u)_{[2]}$

The following theorem contains the upper bound for the tail of the distribution of the maximum of X on the interval $[0, T]$.

Theorem 10 *As $u \to \infty$, the tail of the distribution of the maximum verifies*

$$\mathbb{P}\,(M(T) > u) \le 1 - \Phi(u) + T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\varphi(u) + \lambda T\, P_\rho(u), \qquad (4)$$

where $P_\rho(u)$ is defined in Eq. 1.

Proof The proof is a direct consequence of Theorem 6 and formula (2). □

Remark 5 Furthermore, if we denote the r.h.s. of Eq. 4 by $rhs(u)$, by Corollary 7 we have:

$$rhs(u) \underset{u\to\infty}{\sim} T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\,\varphi(u).$$

The goal of the rest of this section is to show that this upper bound is sharp. In order to do that, we use the lower bound given in Eq. 3 for the tail of the distribution of the maximum $M(T)$. So we have to deal with the second moment of the number up-crossings.

Theorem 11 *Let the processes Z, J and X be as in Corollary 4. If in addition J is a PAR(ρ, λ) process, and Z verifies that $\Gamma(\tau) \neq \pm 1/2$ for all $\tau > 0$ and the Geman condition:*

$$\int_0^{\eta} \frac{\theta'(\tau)}{\tau^2} dt \text{ converges for some } \eta > 0, \tag{5}$$

where θ is defined by $\Gamma(\tau) = 1 + \Gamma''(0)\tau^2/2 + \theta(\tau)$. Then

$$\mathbb{P}(M(t) > u) = 1 - \Phi(u) + T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\varphi(u) + o(\varphi(u))$$

Proof It suffices to show that the additional terms in Eq. 3, w.r.t. (2), are $o(\varphi(u))$. Since

$$U_{u,[2]} = U_u(U_u - 1) = U^c_{u,[2]} + U^d_{u,[2]} + 2\,U^c_u\,U^d_u,$$

the proof is divided into several steps, considering separately each of the resulting terms.

Claim 1 We have $\mathbb{E}\,U^c_{u,[2]} = o(\varphi(u))$. □

Step 1. We begin with an upper bound for the second moment of U_u.

$$\begin{aligned}
&\mathbb{E}\,U^c_{u.[2]} \\
&\leq \int_0^T \int_0^T \sum_{m,n=0}^{\infty} \mathbb{E}\left[\dot{Z}^+(s)\dot{Z}^+(t) \mid X(s) = X(t) = u, \nu_s = m, \nu_{t-s} = n\right] \\
&\qquad \cdot p_{X(s),X(t),\nu_s,\nu_{t-s}}(u,u,m,n)dsdt. \qquad (6)
\end{aligned}$$

This bound is enough for our current purposes, but one can easily obtain the equality by the method of approximation by polygonals, see Dalmao (2013).

We adapt the proof of the Rice formula for the factorial moments in Azaïs and Wschebor (2009, Theorem 3.2). Let C_u be the set of continuous up-crossings of X in $[0, T]$, $C^2_u = C_u \times C_u$ and for any Borel set J in $[0, T]^2$ let $\mu(J) = \#(C^2_u \cap J)$. It follows that $U^c_{u.[2]} = \mu([0, T]^2 \setminus \Delta)$, where Δ is the diagonal, that is, $\Delta = \{(s, t) \in [0, T]^2 : s = t\}$.

Take J_1 and J_2 disjoint intervals in $[0, T]$ and let $J = J_1 \times J_2 (\subset [0, T]^2 \setminus \Delta)$. Then

$$\begin{aligned}
&\mu(J) = U_u(J_1) \cdot U_u(J_2) \\
&= \lim_{\delta\to 0} \frac{1}{(2\delta)^2} \int_{J_1} \dot{Z}^+(s)\mathbf{1}\{|X(s)-u| < \delta\}ds \cdot \int_{J_2} \dot{Z}^+(t)\mathbf{1}\{|X(t)-u| < \delta\}dt \\
&= \lim_{\delta\to 0} \frac{1}{(2\delta)^2} \int\int_J \dot{Z}^+(s)\dot{Z}^+(t)\mathbf{1}\{|X(s)-u| < \delta, |X(t)-u| < \delta\}dsdt,
\end{aligned}$$

where we applied Kac Formula on each interval, and noted that for δ small enough the quantity in the limit becomes constant, so we can use the same mute variable δ in both limits.

Now, we take expectation on both sides and apply Fatou's Lemma and Fubini's Theorem to pass the expectation inside the integral sign. Then, we condition on the number of jumps of the process in the intervals $[0, s]$, $[0, t]$ and on the values of the process X at these points, that is (if $s < t$):

$$\mathbb{E}\,\dot{Z}^+(s)\dot{Z}^+(t)\mathbf{1}\{|X(s)-u|<\delta, |X(t)-u|<\delta\} =$$
$$\int_{u-\delta}^{u+\delta}\int_{u-\delta}^{u+\delta}\sum_{m,n=0}^{\infty}\mathbb{E}\left[\dot{Z}^+(s)\dot{Z}^+(t)\mid X(s)=x, X(t)=y, \nu_s=m, \nu_t=m+n\right]$$
$$p_{X(s),X(t)|\nu_s=m,\nu_t-\nu_s=n}(x,y)\,p_{\nu_s}(m)\,p_{\nu_t}(m+n)\,dx\,dy$$

Observe that, under these conditions $X(s) = Z(s)+A_m$ and $X(t) = Z(t)+A_{m+n}$ (A_m is the basic auto-regressive sequence defined at the beginning of Section 4.1) are jointly Gaussian, thus the conditional expectation is well defined (and may be computed) via regression. Besides, this fact yields the regularity conditions needed for the integrand. In fact, the conditional expectation and the joint density function are continuous for $t \in [0, T]$ and x, y in a neighborhood of u. Hence, we can pass to the limit w.r.t. u inside the integral sign.

So far we have stated that $\mathbb{E}\,\mu(J)$ is bounded by the integral in the r.h.s. of Eq. 6 for any interval $J \subset [0, T]^2 \setminus \Delta$. As both sides represent measures on $[0, T]^2 \setminus \Delta$, the result follows by the standard arguments of Measure Theory.

Step 2. Actually we can take inequality (6) a little further:

$$\mathbb{E}\,U^c_{u.[2]} \le 2\int_0^T (T-\tau)\sum_{n=0}^{\infty}\mathbb{E}\left[\dot{Z}_0^+\dot{Z}_\tau^+ \mid Y_0(0)=Y_n(\tau)=u\right]\cdot$$
$$\cdot\, p_{Y_0(0),Y_n(\tau)}(u,u)\,p_{\nu_\tau}(n)\,d\tau,$$

where we set $Y_k(t) = Z(t) + A_k$, for $t \in [0, T]$ and $k \in \mathbb{N}$.

In fact, conditioned on $\nu_s = m$, $\nu_{t-s} = n$ (if $s < t$; similarly on the other case) we have $X(s) = Z(s) + A_m =: Y_m(s)$ and $X(t) = Z(t) + A_{m+n} =: Y_{m+n}(t)$. It is easy to see that the vector $(Y_m(s), Y_{m+n}(t))$ is independent from ν and has centered normal distribution with variances 1 and covariance $\Gamma(t-s) + \rho^n/2$, in particular, this law does not depend on s, t but on the difference $t - s$, neither does it depend on m. Therefore, the conditional expectation in inequality (6) reduces to that in the r.h.s. of the claimed bound.

Then, we factorize the density function as:

$$\begin{aligned} p_{X(s),X(t),\nu_s,\nu_{t-s}}(u,u,m,n) &= p_{X(s),X(t)|\nu_s,\nu_{t-s}}(u,u)\,p_{\nu_s}(m)\,p_{\nu_{t-s}}(n)\\ &= p_{Y_m(s),Y_{m+n}(t)}(u,u)\,p_{\nu_s}(m)\,p_{\nu_{t-s}}(n)\\ &= p_{Y_0(0),Y_n(\tau)}(u,u)\,p_{\nu_s}(m)\,p_{\nu_\tau}(n). \end{aligned}$$

Clearly $\sum_m p_{\nu_s}(m) = 1$. Finally we make the change of variables $(s, t) \mapsto (s, \tau = t - s)$, and obtain the desired inequality.

In the next two steps we bound each factor in the integrand.

Step 3. Ordinary computations, similar to those in Proposition 4.2 of Azaïs and Wschebor (2009) show that

$$\mathbb{E}\,(\dot{Z}^+(0)\dot{Z}^+(\tau) \mid X_0(0) = X_n(\tau) = u) \le -\Gamma''(0) - \frac{\Gamma'(\tau)^2}{1-(\Gamma(\tau)+\rho^n/2)^2}.$$

Note that for $n \ge 1$, as $\rho < 1$, there is no problem when $\tau \to 0$. In fact, $1-(\Gamma(\tau)+\rho^n/2) \to 1/2 - \rho^n/2 > 0$.

Step 4. The vector $(Y_0(0), Y_n(\tau))$ is normally distributed with variances 1 and covariance $\Gamma(\tau)+\rho^n/2$. Therefore, the exponential in the density is

$$\exp\left\{-\frac{u^2}{1+\Gamma(\tau)+\rho^n/2}\right\}.$$

The case $n = 0$, when there are no jumps in $[0, \tau]$, is treated as in Azaïs and Wschebor (2009, Proposition 4.2), in particular, we need Geman condition to ensure the convergence of the integral.

For $n \ge 1$ we can bound $\rho^n \le |\rho| < 1$. Hence

$$\exp\left\{-\frac{u^2}{1+\Gamma(\tau)+\rho^n/2}\right\} < \exp\left\{-\frac{u^2}{3/2+\Gamma(\tau)}\right\} = o(\varphi(u)).$$

Then, replacing this in inequality (6) of Step 1, we have $\mathbb{E}\,U^c_{u,[2]} = o(\varphi(u))$.

Claim 2 We have $\mathbb{E}\,U^d_{u,[2]} = o(\varphi(u))$.

By the arguments in Corollary 7 it suffices to show that $\mathbb{E}\,U^d_{u,[2]} \le c\,P_{|\rho|}(u)$, for some constant c and u large enough. Recall that $P_{|\rho|}(u)$ is defined in Eq. 1. We can write

$$U^d_u = \sum_{n=0}^{\nu_T}\sum_{k=0}^{n} \mathbf{1}\{X(\tau_k^-) < u, X(\tau_k) > u\}.$$

Hence, making the product and taking expectation, we have:

$$\begin{aligned}\mathbb{E}\,U^d_{u,[2]} &= \sum_{n=2}^{\infty}\sum_{1=k<\ell}^{n-1} p_{\nu_T}(n)\\ &\cdot \mathbb{P}\,(Z(\tau_k)+A_{k-1} < u; Z(\tau_k)+A_k > u; Z(\tau_\ell)+A_{\ell-1} < u; Z(\tau_\ell)+A_\ell > u)\\ &\le \sum_{n=2}^{\infty}\sum_{k<\ell=1}^{n-1} p_{\nu_T}(n)\mathbb{P}\,(Z(\tau_k)+A_{k-1} < u; Z(\tau_\ell)+A_\ell > u). \quad (7)\end{aligned}$$

Besides,

$$\begin{aligned}&\mathbb{P}\,(Z(\tau_k)+A_{k-1} < u; Z(\tau_\ell)+A_\ell > u)\\ &\quad = \int_0^T ds \int_s^T dt\, p_{\tau_k,\tau_\ell|\nu_T=n}(s,t)\mathbb{P}\,(Z(s)+A_{k-1} < u; Z(t)+A_\ell > u)\\ &\quad = \int_0^T ds \int_s^T dt\, p_{\tau_k,\tau_\ell|\nu_T=n}(s,t)\mathbb{P}\,(Z(0)+A_{k-1} < u; Z(t-s)+A_\ell > u),\end{aligned}$$

in the last equality we used the stationarity of the process Z. The vector $(Z(0) + A_{k-1}; Z(t-s) + A_\ell)$ is centered Gaussian with variances 1 and covariance $\Gamma(\tau) + \rho^{\ell-k+1}/2$. It is easy to check that $-\Gamma(\tau) - \rho^{\ell-k+1}/2 < 1/2(1+|\rho|)$ which is the covariance of the vectors $Z + S$ and $Z + |\rho|S + \sqrt{1-\rho^2}V$. Therefore, by the Plackett-Slepian inequality, see Azaïs and Wschebor (2009, Section 2.1) we have

$$\mathbb{P}\left(Z(0)+A_{k-1} < u;\, Z(t-s)+A_\ell > u\right) \le \mathbb{P}\left(Z+S < u;\, Z+|\rho|S+\sqrt{1-\rho^2}V > u\right).$$

This bound does not depend on s, t. Hence

$$\begin{aligned}
&\mathbb{P}\left(Z(\tau_k) + A_{k-1} < u;\, Z(\tau_\ell) + A_\ell > u\right) \\
&\qquad \le \mathbb{P}\left(Z+S<u;\, Z+|\rho|S+\sqrt{1-\rho^2}V>u\right) \int_0^T ds \int_s^T dt\, p_{\tau_k,\tau_\ell|\nu_T=n}(s,t) \\
&\qquad = \mathbb{P}\left(Z+S<u;\, Z+|\rho|S+\sqrt{1-\rho^2}V>u\right) \\
&\qquad = P_{|\rho|}(u),
\end{aligned}$$

since $\tau_k, \tau_\ell \mid \nu_T = n$ is concentrated on $[0,T]^2$. Finally, replacing in the Eq. 7 we have

$$\mathbb{E}\, U^d_{u,[2]} \le \sum_{n=2}^{\infty} \sum_{1=k<\ell}^{n-1} P_{|\rho|}(u) = \frac{(\lambda T)^2}{2} P_{|\rho|}(u),$$

and the result follows.

Claim 3 We have $\mathbb{E}\, U_{u,[2]} = o(\varphi(u))$.

In fact

$$\begin{aligned}
\mathbb{E}\, U_{u,[2]} &= \mathbb{E}\left(U^c_u + U^d_u\right)\left(U^c_u + U^d_u - 1\right) \\
&= \mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^d_{u,[2]} + 2\mathbb{E}\, U^c_u U^d_u \\
&\le \mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^d_{u,[2]} + 2\sqrt{\mathbb{E}\left(U^c_u\right)^2 \mathbb{E}\left(U^d_u\right)^2} \\
&= \mathbb{E}\, U^c_{u,[2]} + \mathbb{E}\, U^d_{u,[2]} + 2\sqrt{\left[\mathbb{E}\left(U^c_{u,[2]}\right) + \mathbb{E}\left(U^c_u\right)\right]\left[\mathbb{E}\left(U^d_{u,[2]}\right) + \mathbb{E}\left(U^d_u\right)\right]},
\end{aligned}$$

where we used Cauchy-Schwarz inequality. The first two terms in the r.h.s. are treated in the previous lemmas, under the square root sign the first factor is equivalent to $\varphi(u)$ and the second one is $o(\varphi(u))$. Thus, $\mathbb{E}\, U_{u,[2]} = o(\varphi(u))$.

Claim 4 We have $\mathbb{P}\left(X(0) > u, U_u \ge 1\right) = o(\varphi(u))$.

We follow the proof of the analogue assertion in Azaïs and Wschebor (2009, Proposition 4.2). The key fact is that the distribution of the process J remains unchanged under time reversal $t \mapsto T - t$.

In fact, let us condition on the number of jumps $\nu_T = n$. Then, it is easy to check that the (conditional) distribution of $(A_1, A_2, \ldots, A_n) \mid \nu_T = n$ is the same as the distribution of $(A_n, A_{n-1}, \ldots, A_1) \mid \nu_T = n$. Besides, the distribution of

$\tau_1, \tau_2, \ldots, \tau_n \mid \nu_T = n$ is that of a uniform (ordered) sample of size n, so it looks the same from 0 and from T. Since the construction of the process J depends on these elements and there is no difference if we start at 0 or at T the claim follows.

In conclusion,

$$\begin{aligned}\mathbb{P}(M(t) > u) &\geq 1 - \Phi(u) + T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\,\varphi(u) + \lambda T\, P_\rho(u) + O(\varphi((1+\delta)u)) \\ &= 1 - \Phi(u) + T\sqrt{\frac{-\Gamma''(0)}{2\pi}}\,\varphi(u) + o(\varphi(u)).\end{aligned}$$

Taking into account (4), this completes the proof.

5 Generalization of Borovkov-Last's formula

Borovkov and Last (2008) are interested in the continuous crossings through a level u by a stationary Piecewise Deterministic Markov Process. A process X of this kind, starts at a random position, then jumps a random quantity at random times but moves deterministically between jumps.

Such a process is described by a general point process $(\tau_n, \xi_n)_n$, and a (non-random) continuous rate function $r : \mathbb{R} \to \mathbb{R}$. More precisely, the process X has jumps at the points (τ_n), the magnitude of the jump is ξ_n and on the interval $[\tau_n, \tau_{n+1})$, $X(t)$ follows the integral curve of r (that is $\dot{X}(t) = r(X(t))$) with initial condition $X(\tau_n) = X(\tau_n^-) + \xi_n$.

Note that the jump part of the process is not independent from the continuous one. Actually, observe that if $r(u) > 0$ (resp. $<$) the continuous crossings through the level u can only be up-crossings (resp. down-crossings). Hence, between two consecutive discontinuous crossings there can be only one continuous crossing. Therefore, the number of continuous crossings is obtained from the discontinuous up and down-crossings.

The next theorem extends Borovkov-Last Formula to the non-stationary case. Let $D_r = \{u : r(u) = 0\}$.

Theorem 12 *Let $u \notin D_r$ and assume that r and $p_{X(t)}$ are continuous w.r.t. x in a neighborhood of u and $t \in [0, T]$. Then:*

$$\mathbb{E}\, N_u^c = |r(u)| \int_0^T p_{X(t)}(u)dt$$

Proof For the levels $u \notin D_r$ we can apply Kac counting formula pathwise for almost all paths of X. In fact, the continuity of r implies that the paths are of class C^1 between the jumps and that $\dot{X}(t) = r(X(t))$ for almost all $t \in [0, T]$. Since $X(t)$ has a density, the value u is not taken at the extremes of the interval neither at the jump points almost surely. Furthermore, by the continuity of this density, there are not tangencies at level u.

Now we take expectation on both sides of Kac Counting Formula and observe that the number of continuous crossings of the level $u \notin D_r$ is bounded by the number

of jumps $+1$ of X in $[0, T]$. In fact, the sign of $r(u)$ determines the direction of the continuous crossings of u, so, between two continuous crossings there must be a discontinuous one in the opposite direction, thus there is at most one continuous crossing at each of the intervals of the partition $\tau_0, \ldots, \tau_{\nu_T}, T$. Then, since ν_T in integrable, we may pass to the limit under the expectation sign:

$$\mathbb{E}\, N_u^c = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^T \mathbb{E}\left[|\dot{X}(t)| \mathbf{1}_{\{|X(t)-u|<\delta\}} \right] dt.$$

Now, $\dot{X}(t)$ is a deterministic function of $X(t)$, namely $\dot{X}(t) = r(X(t))$, so the integrand is simply the expectation of a function of $X(t)$. Therefore,

$$\mathbb{E}\, N_u^c = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \int_0^T \int_{u-\delta}^{u+\delta} |r(x)| p_{X(t)}(x) dx dt.$$

By the continuity of the integrand and the compactness of the domain we can pass the limit inside the integral w.r.t. t. Then, the result follows by the mean value theorem. □

As a corollary, when X is stationary we obtain Borovkov-Last's Formula.

Corollary 13 *If in addition to the conditions of Theorem 12 the process X is stationary, then*

$$\mathbb{E}\, N_u^c = |r(u)| p_{X(0)}.$$

6 Proof of theorem 1

We need the following relations. Let T, X, Y, Z be random variables. Then

$$\mathbb{E}\,(T \mid X = x, Y = y) = \int_{-\infty}^{\infty} \mathbb{E}\,(T \mid X = x, Y = y, Z = z) p_{Z|X=x,Y=y}(z) dz.$$

$$\mathbb{E}\,(T \mid X = x, Z = z) p_{X|Z=Z}(x) = \int_{-\infty}^{\infty} \mathbb{E}\,(T \mid X = x, Y = y, Z = z) p_{X,Y|Z=z}(x, y) dy.$$

Proof of Theorem 1 Let us start with the formula for the mean number of continuous crossings. We condition on the number of jumps, $\nu_T = n$ and on the jump epochs, $\tau_k = t_k$. Thus

$$\mathbb{E}\, N_u^c = \mathbb{E}\left(\mathbb{E}\left[N_u^c\,(\mathrm{X}, [0, T]) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t} \right] \right) = \sum_{n=0}^{\infty} p_{\nu_T}(n) \int_{[0,T]^n} p_{\boldsymbol{\tau}}(\mathbf{t})\, \mathbb{E}\left[N_u^c\,(\mathrm{X}, [0, T]) \mid \nu_T = n; \boldsymbol{\tau} = \mathbf{t} \right] d\mathbf{t},$$

where we set $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and $\mathbf{t} = (t_1, \ldots, t_n)$.

Now we look at the integrand and since the number of crossings is additive w.r.t. the interval, we split the interval $[0, T]$ as the union of the intervals $\mathcal{I}_k := [t_{k-1}, t_k)$.

Then

$$\mathbb{E}\left[N_u^c(\mathrm{X},[0,T]) \mid \nu_T = n;\boldsymbol{\tau}=\mathbf{t}\right] = \sum_{k=1}^{n} \mathbb{E}\left[N_u^c(\mathrm{X},\mathcal{I}_k) \mid \nu_T = n;\boldsymbol{\tau}=\mathbf{t}\right].$$

Each term can be written as

$$\int_{-\infty}^{\infty} \mathbb{E}\left[N_u^c(\mathrm{X},\mathcal{I}_k) \mid \nu_T = n;\boldsymbol{\tau}=\mathbf{t};J(t_{k-1})=y\right] p_{J(t_{k-1})|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(y)dy.$$

Now, conditionally on $\nu_T = n;\boldsymbol{\tau}=\mathbf{t}$ and $J(\tau_{k-1}) = y$, the process X can be written as $\mathrm{Z}+y$ on $\mathcal{I}_k$. Since Z verifies the conditions $A1$, $A2$, $A3$ and $A4$ on $\mathcal{I}_k$ so does the process $\mathrm{Z}+y$. Therefore, we may apply Rice Formula, see Azaïs and Wschebor (2009), on each interval under these conditions to obtain

$$\begin{aligned}
&\mathbb{E}\left[N_u^c(\mathrm{X},\mathcal{I}_k) \mid \nu_T = n;\boldsymbol{\tau}=\mathbf{t};J(t_{k-1})=y\right]\\
&\quad=\int_{\tau_{k-1}}^{\tau_k} \mathbb{E}\left[|\dot{Z}(t)| \mid X(t)=u,\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_{k-1})=y\right]\\
&\quad\cdot p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_{k-1})=y}(u)dt.
\end{aligned}$$

Finally, we have to integrate (the conditions), for notational simplicity, let us write $g(n,\mathbf{t},y) = \mathbb{E}\left[|\dot{Z}(t)| \mid X(t)=u,\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_{k-1})=y\right]$, $g(n,\mathbf{t}) = \mathbb{E}\left[|\dot{Z}(t)| \mid X(t)=u,\nu_T=n;\boldsymbol{\tau}=\mathbf{t}\right]$ and $g(n) = \mathbb{E}\left[|\dot{Z}(t)| \mid X(t)=u,\nu_T=n\right]$. Let us perform the integrals one by one, starting w.r.t. y, (use Fubini). Then

$$\begin{aligned}
&\int_{\mathbb{R}} g(n,\mathbf{t},y)p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t},J(\tau_k)=y}(u)p_{J(t_k)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(y)dy\\
&\quad=\int_{\mathbb{R}} g(n,\mathbf{t},y)p_{(X(t),J(\tau_k))|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u,y)dy\\
&\quad=g(n,\mathbf{t})p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u).
\end{aligned}$$

Now, we sum the integrals over k and integrate w.r.t. $\mathbf{t}$:

$$\begin{aligned}
&\int_0^T dt \int_{[0,T]^n} g(n,\mathbf{t})p_{X(t)|\nu_T=n;\boldsymbol{\tau}=\mathbf{t}}(u)p_{\boldsymbol{\tau}|\nu_T=n}(\mathbf{t})d\mathbf{t}\\
&\quad=\int_0^T g(n)p_{X(t)|\nu_T=n}(u)dt.
\end{aligned}$$

By the same arguments one can remove the condition on ν_T. The result follows.

Now we proceed to the formula for the mean number of discontinuous crossings through level u. For the definitions used below see Jacobsen (2006).

Clearly, X only can have a discontinuous crossing through u at the points $\tau_n; n = 1,\ldots,\nu_T$, the jump of X at each of these points is due to the jump of J. Hence, we consider the Marked Point Process $((\tau_k,\xi_k) : k \geq 0)$ associated to J on $[0,\infty)\times\mathbb{R}$,

which defines a Random Counting Measure $\mu(dt, dy)$, in terms of which we can write:

$$\begin{aligned} N_u^d &= \sum_{k=1}^{\nu_T} \mathbf{1}\left\{(X(\tau_k^-) - u)(X(\tau_k^-) + \Delta X(\tau_k) - u) < 0\right\} \\ &= \sum_{0\le t\le T} \mathbf{1}\left\{(X(t^-) - u)(X(t^-) + \xi_{\nu_t} - u) < 0\right\} \\ &= \int_{[0,T]\times\mathbb{R}} \mathbf{1}\left\{(X(t^-) - u)(X(t^-) + y - u) < 0\right\} \mu(dt, dy). \end{aligned}$$

It is easy to see that this RCM has a compensating measure denoted by $L(dt, dy)$, see Jacobsen (2006) again. Taking expectations on both sides we have:

$$\begin{aligned} \mathbb{E}\, N_u^d &= \mathbb{E}\left[\mathbb{E}\,(N_u^d \mid \mathrm{Z})\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\}\mu(dt, dy) \mid \mathrm{Z}\right]\right] \\ &= \mathbb{E}\int_{[0,T]\times\mathbb{R}} \mathbf{1}\{(X(t^-) - u)(X(t^-) + y - u) < 0\}L(dt, dy) \end{aligned}$$

where we used that conditioned on Z the integral is done w.r.t. the RCM μ associated with J which coincides with the integral w.r.t. the compensating measure $L(dt, dy)$ since the integrand is predictable, in fact it is a function of t^-. Finally we integrate with respect to Z and the result follows. □

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