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Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions

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Consider the optimal stopping problem of a one-dimensional diffusion with positive discount. Based on Dynkin's characterization of the value as the minimal excessive majorant of the reward and considering its Riesz representation, we give an explicit equation to find the optimal stopping threshold for problems with one-sided stopping regions, and an explicit formula for the value function of the problem. This representation also gives light on the validity of the smooth-fit (SF) principle. The results are illustrated by solving some classical problems, and also through the solution of: optimal stopping of the skew Brownian motion and optimal stopping of the sticky Brownian motion, including cases in which the SF principle fails.

Keywords: optimal stopping; diffusion processes; explicit solutions

AMS Subject Classification: 60G40; 62L15

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1. Introduction and problem formulation

Consider a non-terminating and regular one-dimensional (or linear) diffusion $X = \{X_t : t \geq 0\}$, in the sense of Itô and McKean [9] (see also Borodin and Salminen [2]). The state space of X is denoted by $\mathcal{I}$, an interval of the real line $\mathbb{R}$ with left endpoint $\ell = \inf \mathcal{I}$ and right endpoint $r = \sup \mathcal{I}$, where $-\infty \leq \ell < r \leq \infty$. We exclude the possibility of absorbing and killing boundaries; if some of the boundaries belong to $\mathcal{I}$ we assume it to be both entrance and exit (i.e. reflecting boundary). Denote by $\mathbf{P}_x$ the probability measure associated with X when starting from x, and by $\mathbf{E}_x$ the corresponding mathematical expectation. Denote by $\mathbf{T}$ the set of all stopping times with respect to $\{\mathcal{F}_t : t \geq 0\}$, the usual augmentation of the natural filtration generated by X (see I.14 in [2]).

Given a non-negative lower semi-continuous reward function $g : \mathcal{I} \to \mathbb{R}$ and a discount factor $\alpha > 0$, consider the optimal stopping problem (OSP) consisting in finding a function V_α and a stopping time $\tau^* \in \mathbf{T}$, such that

$$V_\alpha(x) = \mathbf{E}_x\left(\mathrm{e}^{-\alpha\tau^*} g(X_{\tau^*})\right) = \sup_{\tau \in \mathbf{T}} \mathbf{E}_x\left(\mathrm{e}^{-\alpha\tau^*} g(X_{\tau^*})\right). \tag{1}$$

The *value function* V_α and the *optimal stopping time* τ^* are the solution of the problem.

The first problems in optimal stopping appeared in the framework of statistics, more precisely, in the context of sequential analysis (see the book by Wald [30]). For continuous time processes a relevant reference is the book of Shiryaev [28] that also has applications to statistics. A new impulse to these problems is related to

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mathematical finance, where arbitrage considerations give that in order to price an American option one has to solve an OSP. The first results in this direction were provided by McKean [15] in 1965 and Merton [16] in 1973, who, respectively, solved the perpetual put and call option pricing problem, by solving the corresponding OSPs in the context of the Black and Scholes model [1]. For the state of the art in the subject see the book by Peskir and Shiryaev [21] and the references therein. A new approach for solving one-dimensional OSPs for very general reward functions is provided in the recent paper [20].

When considering OSPs we typically find two classes of results. The first one consists in the explicit solution to a concrete OSP (1). Usually in this case one has to – somehow – guess the solution and prove that this guess in fact solves the optimization problem; we call this approach 'verification'. For example we can consider the papers [15,16,26,29]. The second class consists of general results, for wide classes of processes and reward functions. We call this the 'theoretical' approach. It typically includes results about properties of the solution. In this class we mention for example [3,6,8]. But these two classes not always meet, as frequently in concrete problems the assumptions of the theoretical studies are not fulfilled, and, what is more important, many of these theoretical studies do not provide concrete ways to find solutions. In what concerns the first approach, a usual procedure is to apply the *principle of smooth fit* (SF), which generally leads to the solution of two equations: the *continuous fit* equation and the SF equation. Once these equations are solved, a verification procedure is needed in order to prove that the candidate is the effective solution of the problem (see Chapter IV in [21]). This approach, when an explicit solution can be found, is very effective. In what concerns the second approach, maybe the most important result is Dynkin's characterization of the solution of the value function V_α as the least α-excessive (or α-superharmonic) majorant of the pay-off function g [3]. Other ways of classifying approaches in order to study OSPs include the martingale–Markovian dichotomy as exposed in [21].

In this study we provide an explicit solution of a *right-sided* OSP for a one-dimensional diffusion process, i.e. when the optimal stopping time has the form

$$\tau^* = \inf\{t \geq 0 : X_t \geq x^*\} \tag{2}$$

for some optimal threshold $x^* \in \mathcal{I}$, under mild regularity conditions. Right-sided problems are also known as *call-type* OSPs. Analogous results are valid for left-sided problems.

An important by-product of our results has to do with the SF principle. Our results are independent of this principle, but they give sufficient conditions in order to guarantee it.

In Section 2 some necessary definitions and preliminary results are given. Our main results are presented in Section 3. In Section 4 we discuss the consequences of our results related to the SF principle. Finally, in Section 5 we present some examples, including the optimal stopping of the skew Brownian motion (SBM) and of the sticky Brownian motion (suggested by Paavo Salminen), where particular attention to the SF principle is given.

2. Definitions and preliminary results

Denote by L the *infinitesimal generator* of diffusion X, and by $\mathcal{D}_L$ its domain. For any stopping time τ and for any $f \in \mathcal{D}_L$ the following discounted version of Dynkin's formula

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holds:

$$f(x) = \mathbf{E}_x\left(\int_0^\tau \mathrm{e}^{-\alpha t}(\alpha - L)f(X_t)\mathrm{d}t\right) + \mathbf{E}_x\left(\mathrm{e}^{-\alpha\tau}f(X_\tau)\right). \tag{3}$$

The *resolvent* of the process X is the operator R_α defined by

$$R_\alpha u(x) = \int_0^\infty \mathrm{e}^{-\alpha t}\mathbf{E}_x(u(X_t))\mathrm{d}t,$$

applied to a function $u \in \mathcal{C}_b(\mathcal{I}) = \{u : \mathcal{I} \to \mathbb{R},\ u \text{ is continuous and bounded}\}$. The range of operator R_α is independent of $\alpha > 0$ and coincides with the domain of the infinitesimal generator $\mathcal{D}_L$. Moreover, for any $f \in \mathcal{D}_L$, $R_\alpha(\alpha - L)f = f$, and for any $u \in \mathcal{C}_b(\mathcal{I})$, $(\alpha - L)R_\alpha u = u$. In other terms, R_α and $(\alpha - L)$ are inverse operators. Denoting by s and m the scale function and the speed measure of diffusion X, respectively, we have that, for any $f \in \mathcal{D}_L$, the lateral derivatives with respect to the scale function exist for every $x \in (\ell, r)$. Furthermore, they satisfy

$$\frac{\partial^+ f}{\partial s}(x) - \frac{\partial^- f}{\partial s}(x) = m(\{x\})Lf(x), \tag{4}$$

and the following identity holds for $z > y$:

$$\frac{\partial^+ f}{\partial s}(z) - \frac{\partial^+ f}{\partial s}(y) = \int_{(y,z]} Lf(x)m(\mathrm{d}x). \tag{5}$$

This last formula allows us to compute the infinitesimal generator of f at $x \in (\ell, r)$ as Feller's differential operator [7]

$$Lf(x) = \frac{\partial}{\partial m}\frac{\partial^+}{\partial s}f(x). \tag{6}$$

The infinitesimal generator at ℓ and r (if they belong to $\mathcal{I}$) can be computed as $Lf(\ell) = \lim_{x\to\ell^+} Lf(x)$ and $Lf(r) = \lim_{x\to r^-} Lf(x)$, respectively.

Given a function $u : \mathcal{I} \to \mathbb{R}$, and $x \in (\ell, r)$ we give to $Lu(x)$ the meaning given in (6) if it makes sense. We also define, if $\ell \in \mathcal{I}$, $Lu(\ell) = \lim_{x\to\ell^+} Lu(x)$ and if $r \in \mathcal{I}$, $Lu(r) = \lim_{x\to r^-} Lu(x)$, if the limit exists.

There exist two continuous functions $\varphi_\alpha : \mathcal{I} \mapsto \mathbb{R}^+$ decreasing, and $\psi_\alpha : \mathcal{I} \mapsto \mathbb{R}^+$ increasing, solutions of $\alpha u = Lu$, such that any other continuous function u is a solution of the differential equation if and only if $u = a\varphi_\alpha + b\psi_\alpha$, with a, b in $\mathbb{R}$. Denoting by $\tau_z = \inf\{t : X_t = z\}$ the hitting time of level $z \in \mathcal{I}$, we have

$$\mathbf{E}_x(\mathrm{e}^{-\alpha\tau_z}) = \begin{cases} \dfrac{\psi_\alpha(x)}{\psi_\alpha(z)}, & x \le z, \\ \dfrac{\varphi_\alpha(x)}{\varphi_\alpha(z)}, & x \ge z. \end{cases} \tag{7}$$

Functions φ_α and ψ_α, though not necessarily in $\mathcal{D}_L$, also satisfy (4) for all $x \in (\ell, r)$, which allow us to conclude that in case $m(\{x\}) = 0$, the derivative at x of both functions with respect to the scale exists. From (5) applied to ψ_α, and taking into account $\alpha\psi_\alpha = L\psi_\alpha$

we obtain for $z > y$

$$\frac{\partial^+ \psi_\alpha}{\partial s}(z) - \frac{\partial^+ \psi_\alpha}{\partial s}(y) = \int_{(y,z]} \alpha\psi_\alpha(x) m\,(\mathrm{d}x);$$

the right-hand side is strictly positive since $\alpha\psi_\alpha$ is positive and m charge every open set. We conclude that $\partial^+ \psi_\alpha/\partial s$ is strictly increasing. In an analogous way it can be proven that $\partial^+ \varphi_\alpha/\partial s$ is increasing as well. The previous consideration, together with the fact that $(\partial^+ \psi_\alpha/\partial s)(x) \geq 0$ and $(\partial^+ \varphi_\alpha/\partial s)(x) \leq 0$, allows us to conclude for $x \in (\ell, r)$:

$$-\infty < \frac{\partial^- \varphi_\alpha}{\partial s}(x) \leq \frac{\partial^+ \varphi_\alpha}{\partial s}(x) < 0 < \frac{\partial^- \psi_\alpha}{\partial s}(x) \leq \frac{\partial^+ \psi_\alpha}{\partial s}(x) < \infty.$$

The *Green function* of process X with discount factor α is defined by

$$G_\alpha(x, y) = \int_0^\infty \mathrm{e}^{-\alpha t}\, p(t; x, y)\mathrm{d}t,$$

where $p(t; x, y)$ is the transition density of the diffusion with respect to the speed measure $m\,(\mathrm{d}x)$ (this density always exists, see [9] or [2]). The Green function may be expressed in terms of φ_α and ψ_α as follows:

$$G_\alpha(x, y) = \begin{cases} w_\alpha^{-1} \psi_\alpha(x)\varphi_\alpha(y), & x \leq y, \\ w_\alpha^{-1} \psi_\alpha(y)\varphi_\alpha(x), & x \geq y, \end{cases} \tag{8}$$

where w_α is the *Wronskian*, given by

$$w_\alpha = \frac{\partial \psi_\alpha^+}{\partial s}(x)\varphi_\alpha(x) - \psi_\alpha(x)\frac{\partial \varphi_\alpha^+}{\partial s}(x).$$

Observe that the Wronskian is positive and independent of x [2,9]. Given $u : \mathcal{I} \to \mathbb{R}$, under the condition $\int_{\mathcal{I}} G_\alpha(x, y)|u(y)|m(\mathrm{d}y) < \infty$, an application of Fubini's Theorem gives

$$R_\alpha u(x) = \int_{\mathcal{I}} G_\alpha(x, y)u(y)m(\mathrm{d}y). \tag{9}$$

A non-negative Borel function $u : \mathcal{I} \to \mathbb{R}$ is called *α-excessive* for process X if $\mathrm{e}^{-\alpha t}\,\mathbf{E}_x(u(X_t)) \leq u(x)$ for all $x \in \mathcal{I}$ and $t \geq 0$, and $\lim_{t\to 0} \mathbf{E}_x\,(u(X_t)) = u(x)$ for all $x \in \mathcal{I}$. A 0-excessive function is said to be *excessive*.

Consider the process killed at an independent exponential time of parameter α, i.e.

$$Y_t = \begin{cases} X_t, & t < e_\alpha, \\ \Delta, & \text{else} \end{cases}$$

with e_α a random variable with exponential distribution of parameter α independent of X, and Δ the cemetery state, at which any function is defined to be zero. It is easy to see that the Green function G^Y of the process $Y = \{Y_t : t \geq 0\}$ coincides with G_α; a Borel function $u : \mathcal{I} \mapsto \mathbb{R}$ is excessive for Y if and only if it is α-excessive for X. In fact, the non-discounted OSP for process Y has the very same solution (value function and optimal stopping time) as the α-discounted OSP for X.

For general reference on diffusions and Markov processes see [2,5,9,11,22].

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3. Main results

Our departing point, inscribed in the Markovian approach, is Dynkin's characterization of the OSP solution. Dynkin's characterization [3] states that, if the reward function is lower semi-continuous, V is the value function of the non-discounted OSP with reward g if and only if V is the least excessive function such that $V(x) \geq g(x)$ for all $x \in \mathcal{I}$. Applying this result for the killed process Y, and taking into account the relation between X and Y, we obtain that V_α, the value function of the problem with discount α, is characterized as the least α-excessive majorant of g.

The second step uses Riesz's decomposition of an α-excessive function. We recall this decomposition in our context (see [4,12,13]). A function $u : \mathcal{I} \to \mathbb{R}$ is α-excessive if and only if there exist a non-negative Radon measure μ and an α-harmonic function such that

$$u(x) = \int_{(\ell,r)} G_\alpha(x,y)\mu(\mathrm{d}y) + (\alpha\text{-harmonic function}). \tag{10}$$

Furthermore, the previous representation is unique. The measure μ is called the representing measure of u.

The third step is based on the fact that the resolvent and the infinitesimal generator of a Markov process are inverse operators. Suppose that we can write

$$V_\alpha(x) = \int_{\mathcal{I}} G_\alpha(x,y)(\alpha - L)V_\alpha(y)m(\mathrm{d}y), \tag{11}$$

where L is the infinitesimal generator and $m(\mathrm{d}y)$ is the speed measure of the diffusion. Assuming that the stopping region has the form $\mathcal{I} \cap \{x \geq x^*\}$, and taking into account that V_α is α-harmonic in the continuation region and $V_\alpha = g$ in the stopping region, we obtain as a suitable candidate to be the representing measure

$$\mu(\mathrm{d}y) = \begin{cases} 0, & \text{if } y < x^*, \\ k\delta_{x^*}(\mathrm{d}y), & \text{if } y = x^*, \\ (\alpha - L)g(y)m(\mathrm{d}y), & \text{if } y > x^*. \end{cases} \tag{12}$$

This approach was initiated by Salminen [23] (see also [17]). According to Salminen's approach, once the excessive function is represented as an integral with respect to the Martin kernel $M_\alpha(x,y)$,

$$V_\alpha(x) = \int_{\mathcal{I}} M_\alpha(x,y)\kappa(\mathrm{d}y), \tag{13}$$

one has to find the representing measure κ. Martin and Green kernels are related by $M_\alpha(x,y) = G_\alpha(x,y)/G_\alpha(x_0,y)$, where x_0 is a reference point. Therefore, Riesz's representation of V_α is related with the one in (13) by considering $\nu(\mathrm{d}y) = \kappa(\mathrm{d}y)/G_\alpha(x_0,y)$ and $M_\alpha(x,\ell)\kappa(\{\ell\}) + M_\alpha(x,r)\kappa(\{r\})$ as the α-harmonic function.

It is useful to observe that when the OSP (1) is right sided with optimal threshold x^*, it has a value function V_α of the form

$$V_\alpha(x) = \begin{cases} \mathbf{E}_x(\mathrm{e}^{-\alpha\tau_{x^*}})g(x^*), & x < x^*, \\ g(x), & x \geq x^*. \end{cases}$$

Furthermore, $V_\alpha(x) \geq g(x)$ for all $x \in \mathcal{I}$ and, in virtue of Equation (7), we have

$$V_\alpha(x) = \begin{cases} \dfrac{g(x^*)}{\psi_\alpha(x^*)}\psi_\alpha(x), & x < x^*, \\ g(x), & x \geq x^*. \end{cases} \tag{14}$$

The state space of the process can include or not the left endpoint ℓ and the right endpoint r. In order to simplify, with a slight abuse of notation, we write $[\ell, x]$, $[\ell, x)$, $[x, r]$, $(x, r]$ to denote, respectively, $\mathcal{I} \cap \{y \leq x\}$, $\mathcal{I} \cap \{y < x\}$, $\mathcal{I} \cap \{y \geq x\}$, $\mathcal{I} \cap \{y > x\}$.

We say that function $g : \mathcal{I} \mapsto \mathbb{R}$ satisfies the *right regularity condition* (RRC) if there exist a point $x_1 \in \mathcal{I}$ and a function $\tilde{g} : \mathcal{I} \to \mathbb{R}$ (not necessarily non-negative) such that $\tilde{g}(x) = g(x)$ for $x \geq x_1$ and

$$\tilde{g}(x) = \int_{\mathcal{I}} G_\alpha(x, y)(\alpha - L)\tilde{g}(y)m(\mathrm{d}y) \quad (x \in \mathcal{I}). \tag{15}$$

Proposition 3.4 gives conditions in order to verify the inversion formula (15). Informally speaking, the RRC is fulfilled by functions g that satisfy all the local conditions – regularity conditions – to belong to $\mathcal{D}_L$ for $x \geq x_1$, and does not increase as quick as ψ_α does when approaching r (in the case $r \notin \mathcal{I}$). Observe that if g satisfies the RRC for certain x_1, it also satisfies it for any greater value; and of course, if g itself satisfies (15) then it satisfies the RRC for all x_1 in $\mathcal{I}$. To take full advantage of the following result it is desirable to find the least x_1 such that the RRC holds.

The main result follows.

THEOREM 3.1. Consider a diffusion X and a reward function g that satisfies the RRC for some x_1. The OSP is right sided with optimal threshold $x^* \geq x_1$ if and only if

$$g(x^*) \geq w_\alpha^{-1}\psi_\alpha(x^*)\int_{(x^*, r]} \varphi_\alpha(y)(\alpha - L)g(y)m(\mathrm{d}y), \tag{16}$$

$$(\alpha - L)g(x) \geq 0, \quad x \in \mathcal{I} : x > x^* \tag{17}$$

and

$$\frac{g(x^*)}{\psi_\alpha(x^*)}\psi_\alpha(x) \geq g(x), \quad x \in \mathcal{I} : x < x^*. \tag{18}$$

Furthermore, in the previous situation:

- Riesz's representation of the value function V_α has representing measure as given in (12) with
$$k = \frac{g(x^*) - w_\alpha^{-1}\psi_\alpha(x^*)\int_{(x^*, r]} \varphi_\alpha(y)(\alpha - L)g(y)m(\mathrm{d}y)}{w_\alpha^{-1}\psi_\alpha(x^*)\varphi_\alpha(x^*)},$$
while the α-harmonic part vanishes;
- if $x^* > x_1$ and inequality (16) is strict, then x^* is the smallest number satisfying this strict inequality and (17), in particular
$$g(x^*) \leq w_\alpha^{-1}\psi_\alpha(x^*)\int_{[x^*, r]} \varphi_\alpha(y)(\alpha - L)g(y)m(\mathrm{d}y); \tag{19}$$
which implies that $k \leq (\alpha - L)g(x^*)m(\{x^*\})$.

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Remark 1. From this theorem we obtain an algorithm to solve right-sided OSPs which works in most cases: (i) find the largest root x^* of the equation

$$g(x^*) = w_\alpha^{-1}\psi_\alpha(x^*)\int_{(x^*,r]}\varphi_\alpha(y)(\alpha - L)g(y)m(\mathrm{d}y), \tag{20}$$

(ii) verify $(\alpha - L)g(y) \geq 0$ for $x \geq x^*$ and (iii) verify $g(x) \leq (g(x^*)/\psi_\alpha(x^*))\psi_\alpha(x)$. If these steps are fulfilled, the problem is right sided with optimal threshold x^*. Observe that if $m(\{x^*\}) = 0$, then inequalities (16) and (19) are equalities.

Proof. We start by observing that if the problem is right sided with threshold x^* then (17) holds. In general $(\alpha - L)g$ is non-negative in the stopping region (this can be seen with the help of the Dynkin's operator, see ex. 3.17 p. 310 in [22], see also Equation (10.1.35) in [18]). Under the made assumption the value function V_α is given by (14), which implies (18) since the value function dominates the reward. To finish the proof of the 'only-if' part it remains to prove (16). Consider $W_\alpha : \mathcal{I} \mapsto \mathbb{R}$ defined by

$$W_\alpha(x) := \int_{(x^*,r]} G_\alpha(x,y)(\alpha - L)g(y)m(\mathrm{d}y),$$

observe that W_α is α-excessive in virtue of (17) and Riesz's representation. Let $\tilde{V}_\alpha : \mathcal{I} \mapsto \mathbb{R}$ be defined by

$$\tilde{V}_\alpha(x) := W_\alpha(x) + kG_\alpha(x,x^*),$$

where k is such that $\tilde{V}_\alpha(x^*) = g(x^*)$, i.e. $k = (g(x^*) - W_\alpha(x^*))/G_\alpha(x^*,x^*)$. Observe that, by (8), $W_\alpha(x^*)$ is the right-hand side of (16). In fact, (16) holds if and only if $k \geq 0$. By the definition of $\tilde{V}_\alpha$ and the representation (8) of G_α we get for $x \leq x^*$

$$\tilde{V}_\alpha(x) = \frac{\psi_\alpha(x)}{\psi_\alpha(x^*)}\tilde{V}_\alpha(x^*) = \frac{\psi_\alpha(x)}{\psi_\alpha(x^*)}g(x^*) = V_\alpha(x).$$

Let us compute $\tilde{V}_\alpha(x) - g(x)$ for $x \geq x^*$. In this region we have $g = \tilde{g}$, where $\tilde{g}$ is the extension given by the RRC. For $\tilde{g}$ can use the inversion formula (15). Denoting by $\nu_{\tilde{g}}(\mathrm{d}y) = (\alpha - L)\tilde{g}(y)m(\mathrm{d}y)$, we have

$$\begin{aligned}
\tilde{V}_\alpha(x) - g(x) &= kG_\alpha(x,x^*) - \int_{[\ell,x^*]} G_\alpha(x,y)\nu_{\tilde{g}}(\mathrm{d}y) \\
&= kw_\alpha^{-1}\varphi_\alpha(x)\psi_\alpha(x^*) - w_\alpha^{-1}\varphi_\alpha(x)\int_{[\ell,x^*]}\psi_\alpha(y)\nu_{\tilde{g}}(\mathrm{d}y) \\
&= \frac{\varphi_\alpha(x)}{\varphi_\alpha(x^*)}(\tilde{V}_\alpha(x^*) - g(x^*)) = 0,
\end{aligned}$$

because $\tilde{V}_\alpha(x^*) = g(x^*)$. So far, we have proved that $\tilde{V}_\alpha(x) = V_\alpha(x)$ for all $x \in \mathcal{I}$. We are ready to prove that $k \geq 0$, based on the uniqueness of Riesz's decomposition: the α-excessive function W_α has Riesz's representation given by its definition, and, if $k < 0$ then

$$W_\alpha(x) = -kG_\alpha(x,x^*) + V_\alpha(x)$$

would give another Riesz's representation (the representing measure being $-k\delta_{\{x^*\}}(\mathrm{d}x) + \mu(\mathrm{d}x)$, where μ is the representing measure of V_α). An easy way of

verifying that the measures are not the same is to observe that the former does not charge $\{x*\}$, while the latter does.

To prove the 'if' statement observe that, assuming (16)–(18), function $\tilde{V}_\alpha$, already defined, is α-excessive (by Riesz's representation, bearing in mind that $k \geq 0$) and dominates g. By Dynkin's characterization, the value function V_α is the minimal α-excessive function that dominates g. Therefore $\tilde{V}_\alpha \geq V_\alpha$. Since $\tilde{V}_\alpha$ satisfies (14) (we have proved this in the first part of this proof), it follows that $\tilde{V}_\alpha$ is the expected reward associated with the hitting time of the set $[x^*, r]$, then

$$\tilde{V}_\alpha(x) \leq V_\alpha(x) = \sup_\tau \mathbf{E}_x\big(\mathrm{e}^{-\alpha\tau} g(X_\tau)\big),$$

concluding that the problem is right sided with threshold x^*.

The consideration about Riesz's representation of V_α stated in the 'furthermore' part is a direct consequence of the made proof.

To prove the minimality of x^*, suppose that there exists x^{**} such that $x_1 < x^{**} < x^*$ satisfying the strict inequality in (16) and (17). Let us check $V_\alpha(x^{**}) - g(x^{**}) < 0$, in contradiction with the fact that V_α is a majorant of g. Considering the extension $\tilde{g}$ given by the RRC and denoting $\nu_{\tilde{g}}(\mathrm{d}y) = (\alpha - L)\tilde{g}(y)m(\mathrm{d}y)$ we have

$$V_\alpha(x^{**}) - g(x^{**}) = -\int_{[\ell, x^*]} G_\alpha(x^{**}, y)\nu_{\tilde{g}}(\mathrm{d}y) + kG_\alpha(x^{**}, x^*) = s_1 + s_2 + s_3 + s_4,$$

where

$$\begin{aligned}
s_1 &= -\int_{[\ell, x^{**}]} G_\alpha(x^{**}, y)\nu_{\tilde{g}}(\mathrm{d}y),\\
s_2 &= -w_\alpha^{-1}\psi_\alpha(x^{**})\int_{(x^{**}, x^*]} \varphi_\alpha(y)\nu_{\tilde{g}}(\mathrm{d}y),\\
s_3 &= \frac{\psi_\alpha(x^{**})}{\psi_\alpha(x^*)}\frac{\varphi_\alpha(x^*)}{\varphi_\alpha(x^{**})}\int_{[\ell, x^{**}]} G_\alpha(x^{**}, y)\nu_{\tilde{g}}(\mathrm{d}y),\\
s_4 &= \frac{\psi_\alpha(x^{**})}{\psi_\alpha(x^*)}\int_{(x^{**}, x^*]} G_\alpha(x^*, y)\nu_{\tilde{g}}(\mathrm{d}y).
\end{aligned}$$

To check that $s_3 + s_4 = kG_\alpha(x^{**}, x^*)$, use

$$k = \frac{1}{G_\alpha(x^*, x^*)}\int_{[\ell, x^*]} G_\alpha(x^*, y)\nu_{\tilde{g}}(\mathrm{d}y) \quad \text{and} \quad \frac{G_\alpha(x^{**}, x^*)}{G_\alpha(x^*, x^*)} = \frac{\psi_\alpha(x^{**})}{\psi_\alpha(x^*)}.$$

Finally, observe that

$$s_1 + s_3 = \left(\frac{\psi_\alpha(x^{**})}{\psi_\alpha(x^*)}\frac{\varphi_\alpha(x^*)}{\varphi_\alpha(x^{**})} - 1\right)\int_{[\ell, x^{**}]} G_\alpha(x^{**}, y)\nu_{\tilde{g}}(\mathrm{d}y) < 0,$$

because the first factor is negative and the second factor is positive, by the assumption about x^{**}, and

$$s_4 + s_2 = \frac{w_\alpha^{-1}\psi_\alpha(x^{**})}{\psi_\alpha(x^*)}\int_{(x^{**}, x^*]} \big(\psi_\alpha(y)\varphi_\alpha(x^*) - \psi_\alpha(x^*)\varphi_\alpha(y)\big)\nu_{\tilde{g}}(\mathrm{d}y) < 0,$$

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because the measure $\nu_{\tilde{g}}(\mathrm{d}y)$ is positive and the integrand is non-positive (it is increasing and vanishes in $y = x^*$). We have obtained that $V_\alpha(x^{**}) - g(x^{**}) = s_1 + s_2 + s_3 + s_4 < 0$, concluding the proof. □

The previous theorem gives necessary and sufficient conditions for problem (1) to be right sided under the RRC. The following result gives simpler sufficient conditions for the problem to be right sided.

THEOREM 3.2. Consider a diffusion X and a reward function g such that (15) is fulfilled for $\tilde{g} = g$. Suppose that there exists a root $c \in (\ell, r)$ of the equation $(\alpha - L)g(x) = 0$, such that $(\alpha - L)g(x) < 0$ if $x < c$ and $(\alpha - L)g(x) > 0$ if $x > c$, and that $\int_{\mathcal{I}} \psi_\alpha(y)(\alpha - L)g(y)m(\mathrm{d}y) \in (0, \infty]$. Then the OSP (1) is right sided, with optimal threshold

$$x^* = \min\{x : b(x) \geq 0\}, \tag{21}$$

where

$$b(x) = \int_{[\ell, x]} \psi_\alpha(y)(\alpha - L)g(y)m(\mathrm{d}y) \quad (x \in \mathcal{I}).$$

Proof. The idea is to apply Theorem 3.1, with x^* defined in (21). By the assumptions on $(\alpha - L)g$ and the fact that $m(\mathrm{d}y)$ is strictly positive in any open set, we obtain that the function $b(x)$ is decreasing in $[\ell, c)$ and increasing in (c, r). Moreover $b(x) < 0$ if $\ell < x \leq c$. Since b is right continuous and increasing in (c, r), the set $\{x : b(x) \geq 0\} = [x^*, r)$ with $x^* > c$. Observe that by (15) and (8) we get

$$g(x) = w_\alpha^{-1}\varphi_\alpha(x)b(x) + \int_{(x,r]} G_\alpha(x, y)(\alpha - L)g(y)m(\mathrm{d}y),$$

and $b(x^*) \geq 0$ is equivalent to (16). Since $x^* \geq c$ we have $(\alpha - L)g(y) > 0$ for $x > x^*$. It only remains to verify (18). By definition of x^* there exists a signed measure $\sigma_\ell(\mathrm{d}y)$ whose support is contained in $[\ell, x^*]$, and $\sigma_\ell(\mathrm{d}y) = (\alpha - L)g(y)m(\mathrm{d}y)$ for $y < x^*$ and such that

$$\int_{[\ell, x^*]} \psi_\alpha(y)\sigma_\ell(\mathrm{d}y) = 0.$$

Furthermore $\sigma_r(\mathrm{d}y) = (\alpha - L)g(y)m(\mathrm{d}y) - \sigma_\ell(\mathrm{d}y)$ is a positive measure supported in $[x^*, r]$. Using the inversion formula for g and (8), we have for $x < x^*$

$$g(x) - \frac{\psi_\alpha(x)}{\psi_\alpha(x^*)}g(x^*) = \int_{[\ell, x^*]} G_\alpha(x, y)\sigma_\ell(\mathrm{d}y) \leq \frac{G_\alpha(x, c)}{\psi_\alpha(c)} \int_{[\ell, x^*]} \psi_\alpha(y)\sigma_\ell(\mathrm{d}y) = 0,$$

where the inequality follows from the following facts: if $y < c$ then $\sigma_\ell(\mathrm{d}y) \leq 0$ and

$$\psi_\alpha(y)\frac{G_\alpha(x, c)}{\psi_\alpha(c)} \leq G_\alpha(x, y),$$

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while if $y > c$ then $\sigma_\ell(\mathrm{d}y) \geq 0$

$$\psi_\alpha(y)\frac{G_\alpha(x,c)}{\psi_\alpha(c)} \geq G_\alpha(x,y).$$

We can now apply Theorem 3.1 completing the proof. □

3.1 On the RRC

In order to apply the previous results it is necessary to verify the inversion formula (15). As we have seen in the preliminaries, if $f \in \mathcal{D}_L$ we have $R_\alpha(\alpha - L)f = f$, and if Equation (9) holds for $(\alpha - L)f$, we have (15). This is the content of the following result.

LEMMA 3.3. Assume $f \in \mathcal{D}_L$, and

$$\int_{\mathcal{I}} G_\alpha(x,y)|(\alpha - L)f(y)|m(\mathrm{d}y) < \infty \quad \text{for all } x \in \mathcal{I}.$$

Then f satisfies Equation (15).

The conditions of the previous lemma are very restrictive in order to solve concrete problems, as reward functions typically satisfy $\lim_{x\to r} g(x) = \infty$. The following result extends the previous result to unbounded functions.

PROPOSITION 3.4. Consider the case $r \notin \mathcal{I}$. Suppose $u : \mathcal{I} \mapsto \mathbb{R}$ is such that $Lu(x)$ in (6) can be defined for all $x \in \mathcal{I}$. Assume

$$\int_{\mathcal{I}} G_\alpha(x,y)|(\alpha - L)u(y)|m(\mathrm{d}y) < \infty, \tag{22}$$

and

$$\lim_{z\to r^-} \frac{u(z)}{\psi_\alpha(z)} = 0. \tag{23}$$

Suppose also that for each $y \in \mathcal{I}$ there exist a function $u_y \in \mathcal{D}_L$ such that $u_y(x) = u(x)$ for $x \leq y$. Then u satisfies (15).

Proof. By (9) we have $\int_{\mathcal{I}} G_\alpha(x,y)(\alpha - L)u(y)m(\mathrm{d}y) = R_\alpha(\alpha - L)u(x)$. Consider a strictly increasing sequence $r_n \to r$ $(n \to \infty)$ and denote by u_n the function $u_{r_{n+1}} \in \mathcal{D}_L$ of the hypothesis. By the continuity of the sample paths, by our assumptions on the right boundary r, we have $\tau_{r_n} \to \infty$ $(n \to \infty)$. Applying formula (3) to u_n and the stopping time τ_{r_n} we obtain, for $x < r_n$,

$$u_n(x) = \mathbf{E}_x\left(\int_0^{\tau_{r_n}} \mathrm{e}^{-\alpha t}(\alpha - L)u_n(X_t)\mathrm{d}t\right) + \mathbf{E}_x(\mathrm{e}^{-\alpha\tau_{r_n}})u_n(r_n),$$

using $u_n(x) = u(x)$ and $(\alpha - L)u(x) = (\alpha - L)u_n(x)$ for $x < r_{n+1}$ we have

$$u(x) = \mathbf{E}_x\left(\int_0^{\tau_{r_n}} \mathrm{e}^{-\alpha t}(\alpha - L)u(X_t)\mathrm{d}t\right) + \mathbf{E}_x(\mathrm{e}^{-\alpha\tau_{r_n}})u(r_n).$$

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Taking limits when $n \to \infty$, by (7) and (23) we have

$$\mathbf{E}_x(\mathrm{e}^{-\alpha\tau_{r_n}})u(r_n) = \frac{\psi_\alpha(x)}{\psi_\alpha(r_n)}u(r_n) \to 0.$$

To compute the limit of the first term above we use dominated convergence theorem and (22). The result is

$$u(x) = \int_0^\infty \mathbf{E}_x\left(\mathrm{e}^{-\alpha t}(\alpha - L)u(X_t)\right)\mathrm{d}t = \int_{\mathcal{I}} G_\alpha(x,y)(\alpha - L)u(y)m(\mathrm{d}y),$$

concluding the proof. □

4. On the principle of SF

The principle of SF holds when condition $V'(x^*) = g'(x^*)$ is satisfied, being a helpful tool to find candidate solutions to OSPs. Salminen [23] proposes an alternative version of this principle, considering derivatives with respect to the scale function. We say that there is *scale smooth fit* (SSF) when the value function has derivative at x^* with respect to the scale function. Note that if g also has derivative with respect to the scale function they coincide, since $g = V_\alpha$ in $[x^*, r]$. Peskir [19] presents two interesting examples: one of them consists of the OSP of a regular diffusion with a differentiable pay-off function in which the principle of SF does not hold, but the alternative principle of SSF does; while in the other the principle of SF holds but the principle of SSF fails. Later, Samee [25] analysed the validity of the principle of SF for killed diffusions and introduced other alternative principles considering derivatives of g/ψ_α and g/φ_α with respect to the scale function s. See also the paper by Jacka [10] for a study of the principle of SF related to the Snell envelope.

We now analyse the relation between Riesz's representation of V_α, stated in the previous section, and the principle of SF. We start by proving that $k = \nu(\{x^*\}) = 0$ in (12) implies that the reward function has derivatives with respect to function ψ_α. Then we follow by stating some corollary results.

THEOREM 4.1. Given a diffusion X and a reward function g, if the value function associated with problem (1) satisfies

$$V_\alpha(x) = \int_{(x^*,r]} G_\alpha(x,y)(\alpha - L)g(y)m(\mathrm{d}y),$$

for $x^* \in (\ell, r)$, then V_α is differentiable at x^* with respect to ψ_α.

Proof. For $x \leq x^*$

$$V_\alpha(x) = w_\alpha^{-1}\psi_\alpha(x)\int_{(x^*,r]} \varphi_\alpha(y)\nu_g(\mathrm{d}y),$$

and the left derivative of V_α with respect to ψ_α in x^* is

$$\frac{\partial V_\alpha^-}{\partial \psi_\alpha}(x^*) = w_\alpha^{-1}\int_{(x^*,r]} \varphi_\alpha(y)\nu_g(\mathrm{d}y).$$

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For $x > x^*$

$$V_\alpha(x) = \varphi_\alpha(x) w_\alpha^{-1} \int_{(x^*,x)} \psi_\alpha(y)\nu_g(\mathrm{d}y) + \psi_\alpha(x) w_\alpha^{-1} \int_{[x,r]} \varphi_\alpha(y)\nu_g(\mathrm{d}y).$$

Computing the difference between $V_\alpha(x)$ and $V_\alpha(x^*)$ we obtain

$$\begin{aligned} V_\alpha(x) - V_\alpha(x^*) = {} & w_\alpha^{-1}\big(\psi_\alpha(x) - \psi_\alpha(x^*)\big) \int_{[x,r]} \varphi_\alpha(y)\nu_g(\mathrm{d}y) \\ & + w_\alpha^{-1} \int_{(x^*,x)} \big(\varphi_\alpha(x)\psi_\alpha(y) - \psi_\alpha(x^*)\varphi_\alpha(y)\big)\nu_g(\mathrm{d}y). \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial V_\alpha^+}{\partial \psi_\alpha}(x^*) &= \lim_{x \to x^{*+}} \frac{V_\alpha(x) - V_\alpha(x^*)}{\psi_\alpha(x) - \psi_\alpha(x^*)} \\ &= w_\alpha^{-1} \lim_{x \to x^{*+}} \int_{[x,r]} \varphi_\alpha(y)\nu_g(\mathrm{d}y) \\ &\quad + w_\alpha^{-1} \lim_{x \to x^{*+}} \frac{\int_{(x^*,x)} \varphi_\alpha(x)\psi_\alpha(y) - \psi_\alpha(x^*)\varphi_\alpha(y)\nu_g(\mathrm{d}y)}{\psi_\alpha(x) - \psi_\alpha(x^*)}. \end{aligned}$$

If the last limit vanishes, we obtain that the right derivative exists, and

$$\frac{\partial V_\alpha^+}{\partial \psi_\alpha}(x^*) = \frac{\partial V_\alpha^-}{\partial \psi_\alpha}(x^*) = w_\alpha^{-1} \int_{(x,r]} \varphi_\alpha(y)\nu_g(\mathrm{d}y).$$

This means that we have to prove

$$\lim_{x \to x^{*+}} \int_{(x^*,x)} \frac{\varphi_\alpha(x)\psi_\alpha(y) - \psi_\alpha(x^*)\varphi_\alpha(y)}{\psi_\alpha(x) - \psi_\alpha(x^*)} \nu_g(\mathrm{d}y) = 0. \tag{24}$$

Denoting by $f(y)$ the numerator of the integrand in (24), observe that

$$f(y) = \varphi_\alpha(x)(\psi_\alpha(y) - \psi_\alpha(x^*)) + \psi_\alpha(x^*)(\varphi_\alpha(x) - \varphi_\alpha(y)).$$

For the first term, we have (observe that $x^* < y < x$)

$$0 \le \varphi_\alpha(x)(\psi_\alpha(y) - \psi_\alpha(x^*)) \le \varphi_\alpha(x)(\psi_\alpha(x) - \psi_\alpha(x^*)),$$

while for the second

$$0 \ge \psi_\alpha(x^*)(\varphi_\alpha(x) - \varphi_\alpha(y)) \ge \psi_\alpha(x^*)(\varphi_\alpha(x) - \varphi_\alpha(x^*)).$$

We conclude that

$$\psi_\alpha(x^*)(\varphi_\alpha(x) - \varphi_\alpha(x^*)) \le f(y) \le \varphi_\alpha(x)(\psi_\alpha(x) - \psi_\alpha(x^*)).$$

Dividing by $\psi_\alpha(x) - \psi_\alpha(x^*)$ we see that the integrand has a lower bound $b(x)$ given by

$$b(x) := \psi_\alpha(x^*) \frac{\varphi_\alpha(x) - \varphi_\alpha(x^*)}{\psi_\alpha(x) - \psi_\alpha(x^*)} = \psi_\alpha(x^*) \frac{\varphi_\alpha(x) - \varphi_\alpha(x^*)}{s(x) - s(x^*)} \frac{s(x) - s(x^*)}{\psi_\alpha(x) - \psi_\alpha(x^*)},$$

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while $\varphi_\alpha(x)$ is an upper bound. We obtain the integral in (24) satisfies

$$b(x)\nu_g(x^*, x) \leq \int_{(x^*,x)} \frac{\varphi_\alpha(x)\psi_\alpha(y) - \psi_\alpha(x^*)\varphi_\alpha(y)}{\psi_\alpha(x) - \psi_\alpha(x^*)} \nu_g(\mathrm{d}y) \leq \varphi_\alpha(x)\nu_g(x^*, x).$$

Taking limits when $x \to x^{*+}$ we obtain $\varphi_\alpha(x) \to \varphi_\alpha(x^*)$, $\nu_g(x, x^*) \to 0$ and

$$\lim_{x \to x^{*+}} b(x) = \frac{\psi_\alpha(x^*)(\partial \varphi_\alpha^+ / \partial s)(x^*)}{(\partial \psi_\alpha^+ / \partial s)(x^*)},$$

concluding that (24) holds. □

As we have seen in Section 3, if the speed measure does not charge x^* neither does the representing measure. This means that if representation (12) holds, then $m(\{x^*\}) = 0$ is enough to guarantee the differentiability of V_α with respect to ψ_α. We also have the following result.

COROLLARY 4.2. Assume the conditions of Theorem 4.1. If the speed measure does not charge x^* there is SSF.

Proof. By the previous theorem we know that V_α is differentiable with respect to ψ_α. Condition $m(\{x^*\}) = 0$ implies that ψ_α has derivative with respect to the scale function. We conclude

$$\frac{\partial V_\alpha}{\partial s}(x^*) = \frac{(\partial V_\alpha / \partial \psi_\alpha)(x^*)}{(\partial \psi_\alpha / \partial s)(x^*)}.$$ □

The previous result, under the additional assumption that ψ_α and φ_α are differentiable with respect to s, could be derived from Corollary 3.7 in [23]. This result states that the representing measure of V_α does not charge x^* if and only if V_α is differentiable with respect to s. Also Theorem 4.1 can be derived from the mentioned result under the additional assumption by using the chain rule.

As a consequence of the previous results, we obtain, by using the chain rule, conditions under which the principle of SF holds.

COROLLARY 4.3. Assume that g is differentiable at x^*. Under the conditions of Theorem 4.1, if ψ_α is differentiable at x^* and $\psi'_\alpha(x^*) \neq 0$ (or under the conditions of Corollary 4.2, if s is differentiable at x^* and $s'(x^*) \neq 0$) then the principle of SF holds.

The previous result is closely related to Theorem 2.3 in [19], which states that, in the non-discounted problem, there is SF if the reward and the scale function are differentiable at x^*. Theorem 2.3 in [25] ensures the validity of the SF principle for the discounted problem under the assumption that g, ψ_α and φ_α are differentiable at x^*. It should be noticed that these results are valid in general, not only in one-sided problems.

5. Examples

In this section we show how to solve some OSPs using the previous results. We present two classical examples (American and Russian options), and also include some new examples in which the SF principle is not useful to find the solution.

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5.1 American call options

Consider a geometric Brownian motion given by $X_t = x\exp(\sigma W_t + (\mu - \sigma^2/2)t)$, where $\{W_t\}$ is a standard Brownian motion, $\mu \in \mathbb{R}$ and $\sigma^2 > 0$. The state space is $I = (0, \infty)$. We refer to [2], p. 132 for the basic characteristics of this process. The infinitesimal generator is $Lf = (1/2)\sigma^2 x^2 f'' + \mu x f'$, with domain

$$\mathcal{D}_L = \{f : f, Lf \in \mathcal{C}_b(\mathcal{I})\}.$$

Consider the pay-off function $g(x) = (x - K)^+ (x \in \mathbb{R})$, where K is a positive constant, and a positive discount factor α satisfying $\alpha > \mu$. The reward function g satisfies the RRC for $x_1 = K$: it is enough to consider $\tilde{g} \in \mathcal{C}^2$, bounded in $(0, K)$ and such that $\tilde{g}(x) = x - K$ for $x \geq K$. This function $\tilde{g}$ satisfies the inversion formula (15) as a consequence of Proposition 3.4. Observe that Equation (22) holds. Equation (23) is in this case

$$\lim_{z\to\infty} \frac{\tilde{g}(z)}{\psi_\alpha(z)} = \lim_{z\to\infty} z^{1-\gamma_1},$$

with

$$\psi_\alpha(x) = x^{\gamma_1} \quad \text{and} \quad \gamma_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\alpha}{\sigma^2}}.$$

The last limit vanishes if $1 - \gamma_1 < 0$, which is equivalent to $\mu < \alpha$. To find x^* we solve Equation (20). After computations, we find

$$x^* = K\left(\frac{\gamma_1}{\gamma_1 - 1}\right).$$

It is not difficult to verify (17) and (18) in order to apply Theorem 3.1. We conclude that the problem is right sided with optimal threshold x^*. Observe that the hypotheses of Theorem 4.1 and Corollaries 4.2 and 4.3 are fulfilled. In consequence all the variants of SF principle hold. This problem was solved by Merton [16].

5.2 Russian options

The Russian option was introduced by Shepp and Shiryaev [26], where the option pricing problem is solved by reduction to an OSP of a two-dimensional Markov process. Later, in [27], the authors give an alternative approach to the same problem solving a one-dimensional OSP. In 2000, Salminen [24], making use of a generalization of Lévy's theorem for Brownian motion with drift, shortened the derivation of the valuation formula in [27] and solved the related OSP.

Consider $\alpha > 0$, $r > 0$ and $\sigma > 0$. Let $\{X_t\}$ be a Brownian motion on $I = [0, \infty)$, with drift $-\delta < 0$, where $\delta = (r + \sigma^2/2)/\sigma$ and reflected at 0. In [24] it is shown that the OSP to be solved has underlying process $\{X_t\}$ and reward function $g(x) = \mathrm{e}^{\sigma x}$. For the basic characteristics of the process we refer to [2], p. 129. The infinitesimal generator is $Lf(x) = f''(x)/2 - \delta f'(x)$ for $x > 0$ and $Lf(0) = \lim_{x\to 0^+} Lf(x)$, with domain

$$\mathcal{D}_L = \left\{f : f, Lf \in \mathcal{C}_b(\mathcal{I}),\ \lim_{x\to 0^+} f'(x) = 0\right\}.$$

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The pay-off function $g(x)$ satisfies the RRC for every $x_1 > 0$: for $x_1 > 0$ it is easy to find a function $\tilde{g}$ with continuous second derivatives such that $\tilde{g} = g$ in $[x_1, \infty)$ and such that the right derivative at 0 is 0. By the application of Proposition 3.4 we obtain that $\tilde{g}$ satisfies the inversion formula (15), then the RRC holds. We obtain

$$(\alpha - L)g(x) = (\alpha - \sigma^2/2 + \delta\sigma)\mathrm{e}^{\sigma x} = (\alpha + r)\mathrm{e}^{\sigma x} > 0.$$

In order to apply Theorem 3.1 we solve Equation (20) which in this case is

$$\mathrm{e}^{\sigma x} = \frac{1}{\gamma - \delta}\left(\frac{\gamma - \delta}{2\gamma}\mathrm{e}^{(\gamma+\delta)x} + \frac{\gamma + \delta}{2\gamma}\mathrm{e}^{-(\gamma-\delta)x}\right)\int_x^\infty 2(\alpha + r)\mathrm{e}^{(-\gamma-\delta+\sigma)y}\mathrm{d}y,$$

with $\gamma = \sqrt{2\alpha + \delta^2}$, obtaining (observe that $-\gamma - \delta + \sigma < 0$)

$$x^* = \frac{1}{2\gamma}\ln\left(\left(\frac{\gamma + \delta}{\gamma - \delta}\right)\left(\frac{\gamma - \delta + \sigma}{\gamma + \delta - \sigma}\right)\right).$$

It is easy to verify conditions (17) and (18), to obtain, by application of Theorem 3.1, that the problem is right sided with threshold x^*, as proved in [27].

5.3 Skew Brownian motion

We consider a SBM at zero. This process behaves like a standard Brownian motion outside the origin, but has an asymmetric behaviour when hitting $x = 0$, modelling a permeable barrier. The behaviour at $x = 0$ is regulated by a parameter $\beta \in (0, 1)$, known as the *skewness parameter*. The state space of this process is $\mathcal{I} = \mathbb{R}$. For details on this process and its basic characteristics we refer to [2], p. 126 or [14]. The infinitesimal generator is $Lf(x) = f''(x)/2$ if $x \neq 0$ and $Lf(0) = \lim_{x\to 0} Lf(x)$, with domain

$$\mathcal{D}_L = \{f : f, Lf \in \mathcal{C}_b(\mathcal{I}), \beta f'(0^+) = (1 - \beta)f'(0^-)\}.$$

Consider the pay-off function $g(x) = x^+$. Function g satisfies the RRC for $x_1 = 0$: to see this it is necessary to construct $\tilde{g}$ such that $\tilde{g} = g$ in $[0, \infty)$ $\tilde{g}$ with second derivative bounded in $(-\infty, 0)$ and such that $\tilde{g}'(0^-) = \beta/(1 - \beta)$ (so that $\tilde{g}$ satisfies the local conditions to belong to $\mathcal{D}_L$). Applying Proposition 3.4 it can be concluded that $\tilde{g}$ satisfies (15), so the RRC holds. We have $(\alpha - L)g(x) = \alpha x (x \geq 0)$. Equation (20) is in this case

$$x^* = \frac{1}{\sqrt{2\alpha}}\left(\frac{1 - 2\beta}{\beta}\sinh(\sqrt{2\alpha}x^*) + \mathrm{e}^{\sqrt{2\alpha}x^*}\right)\int_{x^*}^\infty \mathrm{e}^{-\sqrt{2\alpha}t}\alpha t\, 2\beta\, \mathrm{d}t$$

or equivalently

$$x^* = \frac{1}{2\sqrt{2\alpha}}\left((2\beta - 1)\mathrm{e}^{-\sqrt{2\alpha}x^*}(\sqrt{2\alpha}x^* + 1) + \sqrt{2\alpha}x^* + 1\right). \tag{25}$$

In general, this equation cannot be solved analytically. If we consider the particular case $\beta = 1/2$, in which the process is the ordinary Brownian motion, we obtain the known result $x^* = 1/\sqrt{2\alpha}$ (see [29]). Consider a particular case, in which $\alpha = 1$ and $\beta = 0.9$. Solving numerically Equation (25) we obtain $x^* \simeq 0.82575$. Checking (17) and (18) we conclude that the problem is right sided with optimal threshold x^*.

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5.3.1 An example without SF

Consider again the SBM with parameter $\beta = 1/3$, a pay-off function $g(x) = (x+1)^+$ and a discount $\alpha = 1/8$. We have $(\alpha - L)g(x) = \alpha(x+1)(x \geq 0)$. Observe that $x^* = 0$ is a solution of (16) (with equality). It is easy to see that the assumptions of Theorem 3.1 are fulfilled. We conclude that the problem is right sided with threshold $x^* = 0$. Moreover, we know

$$V_\alpha(x) = \begin{cases} x+1, & x \geq 0, \\ \psi_\alpha(x), & x \leq 0, \end{cases}$$

where

$$\psi_\alpha(x) = \begin{cases} \mathrm{e}^{\sqrt{2\alpha}x}, & x \leq 0, \\ \dfrac{1-2\beta}{\beta}\sinh(x\sqrt{2\alpha}) + \mathrm{e}^{\sqrt{2\alpha}x}, & x \geq 0. \end{cases}$$

Unlike the previous examples, the value function V_α is not differentiable at x^*. As we see in Figure 1, the graph of V_α shows an angle at $x = 0$. By application of Theorem 4.1 and Corollary 4.2 we conclude that V_α is differentiable with respect to ψ_α and SSF holds.

An example considering a regular diffusion with non-differentiable scale function, in which the SF fails to hold, was provided for the first time by Peskir [19].

5.4 Sticky Brownian motion

Consider a Brownian motion, sticky at 0. We refer to [2] p. 123 for the basic characteristics of this process. It behaves like a Brownian motion out of 0, but spends a positive time at $x = 0$; this time depends on a positive parameter that we assume to be 1 in this example. The state space of this process is $\mathcal{I} = \mathbb{R}$. The scale function is $s(x) = x$ and the speed measure is $m(\mathrm{d}x) = 2\,\mathrm{d}x + 2\delta_{\{0\}}(\mathrm{d}x)$. The infinitesimal generator is $Lf(x) = f''(x)/2$ when $x \neq 0$, and $Lf(0) = \lim_{x\to 0} Lf(x)$, with domain

$$\mathcal{D}_L = \{f : f,\ Lf \in \mathcal{C}_b(\mathcal{I}),\ f''(0^+) = f'(0^+) - f'(0^-)\}.$$

Consider the reward function $g(x) = (x+1)^+$ that satisfies the RRC for $x_1 = -1$ (it can be seen with the same arguments considered so far). We discuss the solution of the OSP in

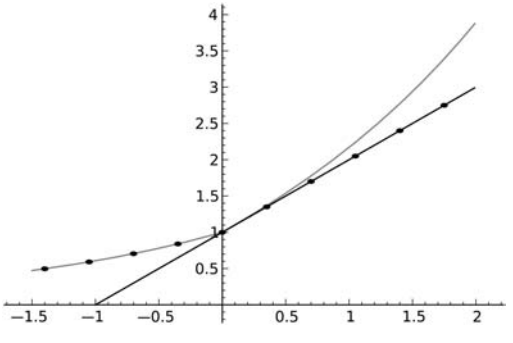


Figure 1. Solution of the OSP for the skew Brownian with $\alpha = 1/8$, $\beta = 1/3$ and reward function $g = (x+1)^+$. Black graph corresponds to g, while the grey graph corresponds to ψ_α. V_α is the dotted line.

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Table 1. Sticky Brownian motion: solution of the OSP depending on α.

α	x^*	(16)	(19)	SF & SSF	$\exists \mathrm{d}V_\alpha/\mathrm{d}\psi_\alpha$	$\exists \mathrm{d}(V_\alpha/\varphi_\alpha)/\mathrm{d}s$	Figure
$\alpha \in (0, \alpha_1)$	$x^* > 0$	$=$	$=$	Yes	Yes	Yes	Figure 2(a)
$\alpha = \alpha_1$	$x^* = 0$	$=$	$<$	No	Yes	Yes	Figure 2(b)
$\alpha \in (\alpha_1, \alpha_2)$	$x^* = 0$	$>$	$<$	No	No	No	Figure 2(c)
$\alpha = \alpha_2$	$x^* = 0$	$>$	$=$	Yes	No	No	Figure 2(d)
$\alpha \in (\alpha_2, +\infty)$	$x^* < 0$	$=$	$=$	Yes	Yes	Yes	Figure 2(e)

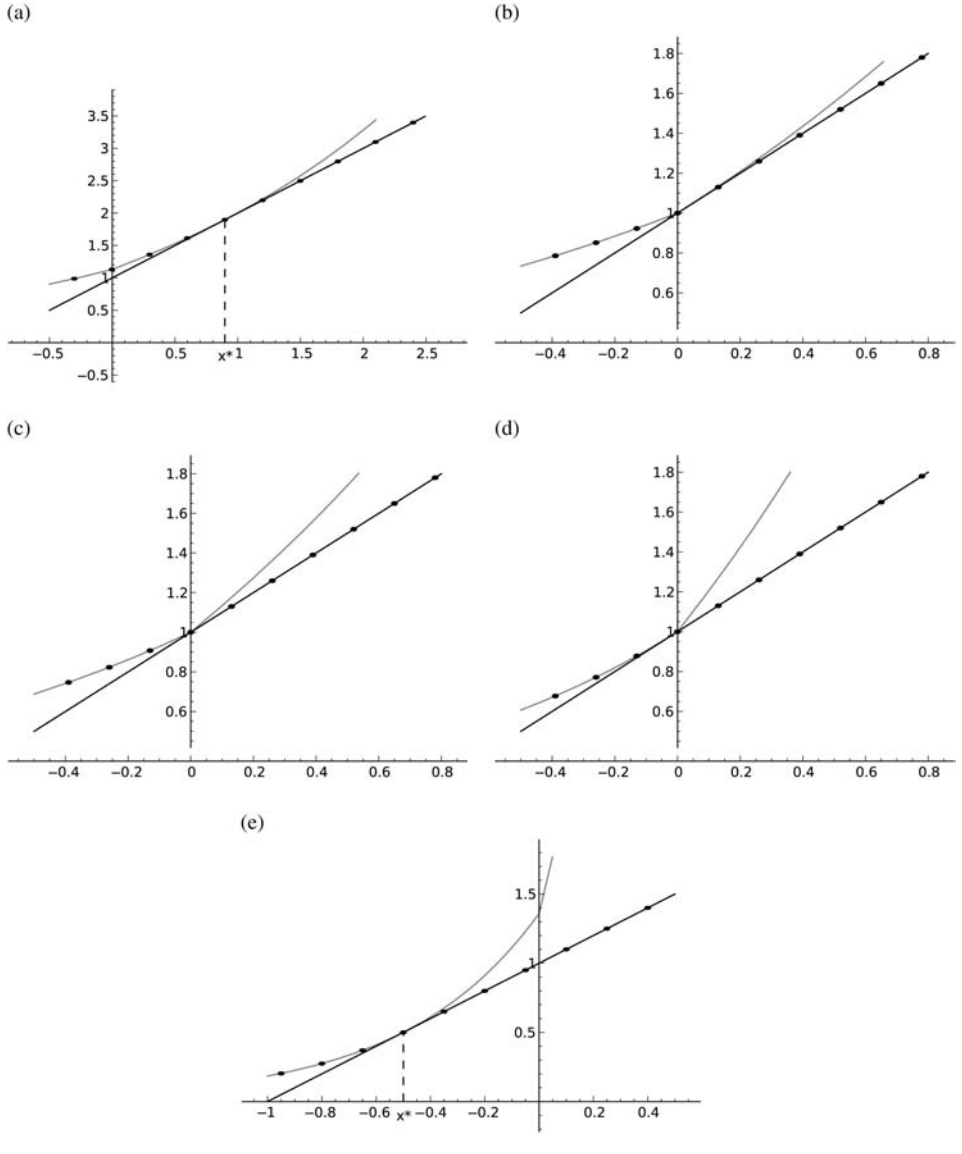


Figure 2. Solution of the OSP for the sticky Brownian motion with different values of α with reward function $(x+1)^+$ (graphic in black). The graphic in grey corresponds to $(cte)\psi_\alpha$. The value function V_α is indicated with dots, it coincides with $(cte)\psi_\alpha$ for $x \leq x^*$ and with g for $x \geq x^*$. (a) $\alpha = 0.1$, (b) $\alpha_1 \backsimeq 0.09$, (c) $\alpha = 0.28$, (d) $\alpha = \alpha_2 = 0.05$ and (e) $\alpha = 2$.

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terms of the discount factor, in particular we are interested in finding values of α such that the optimal threshold is the sticky point. We use (20) in a different way: we fix $x = 0$ and solve the resulting equation in α. We obtain

$$1 = w_\alpha^{-1} \int_{(0,\infty)} \mathrm{e}^{-\sqrt{2\alpha}y} \alpha(y+1) 2\,\mathrm{d}y$$

and we find that $\alpha_1 = ((-1+\sqrt{5})^2/8) \simeq 0.19$ is the unique solution. It can be seen, by application of Theorem 3.1, that if $\alpha = \alpha_1$ the problem is right sided with threshold $x^* = 0$. In this case the representing measure of V_α does not charge x^* despite the speed measure does. Furthermore, Theorem 4.1 can be applied to conclude that V_α is differentiable with respect to ψ_α. It also should be noticed that both SF and SSF fail to hold in this case. This was expectable because the sufficient conditions given in [25] and [23] for the different types of SF are not fulfilled. Another interesting point to remark is that $\mathrm{d}(V_\alpha/\varphi_\alpha)/\mathrm{d}s$ (and also $\mathrm{d}(V_\alpha/\psi_\alpha)/\mathrm{d}s$) exists at 0, which is part of the conclusion of Theorem 2.1 in [25], despite this result is not applicable in this case. This last fact seems to be related with the existence of $\mathrm{d}V_\alpha/\mathrm{d}\psi_\alpha$.

Another approach to obtain $x^* = 0$ is when the strict inequality holds in (16) and also (19) holds. We solve (in α) Equation (19) (with equality), which is

$$g(0) = w_\alpha^{-1} \psi_\alpha(0) \int_{[0,\infty)} \psi_\alpha(y)(\alpha - L)g(y) m(\mathrm{d}y). \tag{26}$$

Since the measure $m(\mathrm{d}y)$ has an atom at $y = 0$, the solution of the previous equation differs from α_1. Solving (26) we find the root $\alpha_2 = 1/2$. It is easy to see that for $\alpha \in [\alpha_1, \alpha_2]$, the minimal x satisfying inequality (16) is 0. Theorem 3.1 can be applied to conclude that the problem is right sided with threshold $x^* = 0$. For $\alpha \in (\alpha_1, \alpha_2]$ we cannot apply any of the results of Section 4, and in fact, for $\alpha \in (\alpha_1, \alpha_2)$ none of the SF principles is fulfilled. With $\alpha = \alpha_2$ there is SF (and also SSF, since $s(x) = x$), but this is not a consequence of (26), it is due to the particular reward function. This example shows that the theorems on SF in Section 4 only give sufficient conditions. In Table 1 we summarize the information about the solution of the OSP. We also give, in Figure 2, some graphics showing the solution for different values of α.

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